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An exact Jacobi map in the geodesic light-cone (GLC) gauge

G. Fanizza
M. Gasperini
G. Marozzi
G. Veneziano

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What Jacobi map is?

- The Jacobi map concerns the trajectories of photons from the source we are interested in to the observer
- Physical meaning of the Jacobi map:

$$\begin{aligned} d_A^2 &= \det J^A_B(\lambda_s, \lambda_o) \\ d_L &= (1+z)^2 d_A \end{aligned} \tag{1}$$

  where d_A is the angular distance and d_L is the luminosity distance
- Let us recall that, in general:

$$d_A^2 = \frac{dS_s}{d\Omega_o} \tag{2}$$

Definition of the Jacobi map (1)

- Geodesic deviation equation:

$$\nabla_\lambda^2 \xi^\mu = R_{\alpha\beta\nu}{}^\mu k^\alpha k^\nu \xi^\beta \tag{3}$$

where λ is the affine parameter and $\nabla_\lambda \equiv k^\alpha \nabla_\alpha$

- Sachs basis $\{s_A^\mu\}_{A=1,2}$ such that

2-D flat subspace	$g_{\mu\nu} s_A^\mu s_B^\nu = \delta_{AB}$	
Orthogonality conditions	$s_A^\mu u_\mu = 0 \qquad s_A^\mu k_\mu = 0$	(4)
Parallel transport	$\nabla_\lambda s_A^\mu = 0$	

- In such a way, let us project the displacement on the Sachs basis

$$\xi^A \equiv \xi^\mu s_\mu^A \tag{5}$$

that is now a spacetime scalar.

Definition of the Jacobi map (2)

- The geodesic deviation equation can be written as

$$\frac{d^2\xi^A}{d\lambda^2} = R^A_B \xi^B \tag{6}$$

with $R_{AB} \equiv R_{\alpha\beta\nu\mu} k^\alpha k^\nu s^\beta_A s^\mu_B$

- General solution of (5):

$$\xi^A(\lambda_s) = J^A_B(\lambda_s, \lambda_o) \left(\frac{k^\mu \partial_\mu \xi^B}{k^\nu u_\nu} \right)_o \tag{7}$$

where $J^A_B(\lambda_s, \lambda_o)$ is the Jacobi map we are interested in

- Fundamental equation to solve:

$$\frac{d^2}{d\lambda^2} J^A_B(\lambda, \lambda_o) = R^A_C J^C_B \tag{8}$$

with initial conditions

$$J^A_B(\lambda_o, \lambda_o) = 0 \qquad \frac{d}{d\lambda} J^A_B(\lambda_o, \lambda_o) = \delta^A_B (k^\nu u_\nu)_o \tag{9}$$

The GLC gauge

The GLC coordinates consist of a timelike coordinate τ (which can always be identified with the proper time of the synchronous gauge), of a null coordinate w and of two angular coordinates $\tilde{\theta}^a$ ($a = 1, 2$):

$$ds^2 = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab}(d\tilde{\theta}^a - U^a dw)(d\tilde{\theta}^b - U^b dw) \qquad (10)$$

Fundamental properties:

- $w =$ constant defines the past light-cone of the observer (ourselves)
- $u_\mu = -\partial_\mu \tau$ describes a geodesic flow (related to SG)
- $k^\mu = \omega \Upsilon^{-1} \delta^\mu_\tau$ is the quadri-momentum of the photon (constant w and θ^a)
- residual gauge freedom: $\tilde{\theta}^a \to \bar{\theta}^a(w, \tilde{\theta}^a)$

The Jacobi map in the GLC gauge

- In GLC gauge, we get

$$\xi^A(\lambda_s) = J^A_B(\lambda_s, \lambda_o) \left(u_\tau^{-1} \partial_\tau \xi^B \right)_{\lambda_o} \tag{11}$$

- Using the properties of the GLC gauge, we can easily construct the Jacobi map with the following solution

$$J^A_B(\lambda, \lambda_o) = s^A_a(\lambda) \left\{ \left[\left(u_\tau^{-1} \partial_\tau s \right)^{-1} \right]^a_B \right\}_{\lambda=\lambda_o}$$

- So the angular distance is immediately given by:

$$d_A^2 = \det \left(J^A_B(\lambda_s, \lambda_o) \right) = \frac{\sqrt{\gamma(\lambda_s)}}{\frac{1}{4} \left[\det \left(u_\tau^{-1} \partial_\tau \gamma^{ab} \right) \gamma^{3/2} \right]_{\lambda=\lambda_o}} \quad , \quad \gamma \equiv \det \gamma_{ab} \tag{12}$$

First-order Jacobi map in the synchronous gauge (1)

- In carthesian coordinates:

$$ds^2_{SG} = -dt^2 + a^2(t)[(1-2\bar{\psi})\delta_{ij} + D_{ij}\bar{E}]dx^i dx^j \tag{13}$$

with $D_{ij} = \partial_i\partial_j - \frac{1}{3}\Delta_3$

- In polar coordinates:

$$ds^2_{SG} = -dt^2 + a^2(t)[(1-2Z)dr^2 - 2S_a d\theta^a dr + h_{ab}d\theta^a d\theta^b] \tag{14}$$

with

$$Z = \bar{\psi} - \frac{1}{2}\left(\partial_r^2 - \frac{1}{3}\Delta_3\right)\bar{E} \qquad S_a = -\left(\partial_r - \frac{1}{r}\right)\bar{E}$$

$$h_{ab} = \gamma^0_{ab}\left[1 - 2\bar{\psi} - \left(\frac{1}{3}\Delta_3 - \frac{1}{r}\partial_r\right)\right]\bar{E} + \nabla_a\partial_b\bar{E} \tag{15}$$

First-order Jacobi map in the synchronous gauge (2)

- The term at the observer is given by

$$\frac{\sin\theta_o}{\det(\partial_\tau s^B_b(\lambda_o))} = 1 - \frac{1}{2}\left(\frac{\Delta_2}{r^2} - 2\partial_r^2 + \frac{2}{r}\partial_r\right)\bar{E} \tag{16}$$

  It is possible to recover the pure $\sin\theta_o$ by fixing the residual gauge degrees of freedom of the SG

- The angular distance is given by

$$\left(d_A^2\right)_{SG} = \frac{a_s^2 r_s^2 \sin\theta_s}{\sin\theta_o}\left\{1 + \left(\frac{3}{2}\partial_r^2 - \frac{1}{2}\Delta_3\right)\bar{E}_o - 2\bar{\psi}_s\right.$$
$$+ \left(\frac{1}{6}\Delta_3 - \frac{1}{2}\partial_r^2\right)\bar{E}_s + \frac{1}{2}\int_{\eta_s^+}^{\eta_s^-} dx\gamma_0^{ab}\partial_a\partial_b\left(\partial_r - \frac{1}{r}\right)\bar{E}$$
$$\left. - \frac{1}{4}\int_{\eta_s^+}^{\eta_s^-} dx\gamma_0^{ab}\int_{\eta_s^+}^{x} dy\partial_a\partial_b\left[\bar{\psi} + \left(\frac{1}{6}\Delta_3 - \frac{1}{2}\partial_r^2\right)\bar{E}\right]\right\} \tag{17}$$

Jacobi map in the Poisson gauge (1)

Starting from the second-order Poisson gauge

$$
\begin{aligned}
ds^2_{PG} &= a^2(\eta)\left[-(1+2\Phi)d\eta^2 + (1-2\Psi)\delta_{ij}dx^i dx^j\right] \\
&= a^2(\eta)\left[-(1+2\Phi)d\eta^2 + (1-2\Psi)(dr^2 + r^2 d^2\Omega)\right]
\end{aligned}
\tag{18}
$$

with $\Phi = \phi + \frac{1}{2}\phi^{(2)}$ and $\Psi = \psi + \frac{1}{2}\psi^{(2)}$, we define

$$
P(\eta, r, \theta^a) = \int_{\eta_{in}}^{\eta} d\eta' \frac{a(\eta')}{a(\eta)} \phi(\eta', r, \theta^a) \tag{19}
$$

which is directly connected to the peculiar velocity

Jacobi map in the Poisson gauge (2)

- Up to first-order, we get

$$\frac{\sin\theta_o}{\det\left(\partial_\tau s^B_b(\lambda_o)\right)} = 1 - 2\partial_r P_o \tag{20}$$

where the correction at the pure $\sin\theta_o$ is given by the peculiar velocity term.

- The angular distance is given by

$$\left(d_A^2\right)_{PG} = \frac{a_s^2 r_s^2 \sin\theta_s}{\sin\theta_o}\left\{1 - 2\int_{\eta_{in}}^{\eta_o} d\eta' \, \frac{a(\eta')}{a(\eta_o)} \partial_r\phi\left(\eta', 0, \theta^a\right) - 2\psi_s \right.$$
$$\left. -\frac{1}{2}\int_{\eta_o}^{\eta_s^-} dx \gamma_0^{ab} \int_{\eta_o}^{x} dy \frac{1}{2}\partial_a\partial_b\left[\psi(\eta_+, y, \theta^a) + \phi(\eta_+, y, \theta^a)\right]\right\} \tag{21}$$

The second-order peculiar velocity

- The $\sin\theta_o$ correction in the PG derives from the Lorentz transformation of the solid angle at the observer $d\Omega_o$ which appears in the definition of d_A^2
- In fact, the second-order calculation too gives us

$$\frac{\sin\theta_o}{\det\left(\partial_\tau s_b^B(\lambda_o)\right)} = \left[(1-\partial_r P)^2 + \nabla_i P \nabla^i P - 2\psi_o \partial_r P\right]_o - \int_{\eta_{in}}^{\eta_o} d\eta' \frac{a(\eta')}{a(\eta_o)} \partial_r \left(\phi^{(2)} - \psi^2 + \nabla_i P \nabla^i P\right) \tag{22}$$

- Let us define $v_i = -\partial_i \tau^{(1)} - \partial_i \tau^{(2)}$ and $n_\mu = (0, -a(1-\psi), 0, 0)$
- These exactly reproduce the Lorentz boost correction

$$\frac{1}{d\Omega_o} = \frac{1}{d\tilde{\Omega}_o}\left[\frac{1-\vec{v}\cdot\vec{n}}{1-v^2}\right]_o \approx \frac{1}{d\Omega_o}\left[1 + v^2 - 2\vec{v}\cdot\vec{n} + (\vec{v}\cdot\vec{n})^2\right]_o \tag{23}$$

Comparison between the two gauges (1)

- In order to compare the expressions that we found, let us consider the gauge transformation rule for the bi-scalar object d_A^2
- We get the perfect equivalence between the two expressions if we expand $\bar{E}$ around the observer position

$$\bar{E} \approx \left(E + E_i x^i + E' \eta + E_i' x^i \eta + E_{ij} x^i x^j + E'' \right)_o \qquad (24)$$

and fix the residual gauge freedom of the SG in the following way

Finiteness requirement	$(E_i)_o = 0$
Isotropy around the observer	$(E_{ij})_o = 0$

Comparison between the two gauges (2)

- It is remarkable that the peculiar velocity term (gauge dependent quantity) vanishes in the SG
- The gauge fixing allows us to re-express the observer term in the SG as a pure $\sin\theta_o$
- The main physical effect at the observer is due to his/her peculiar velocity that transforms the solid angle as a Lorentz boost predicts

Conclusion

- We solved exactly the geodesic deviation equation, using the GLC gauge
- We obtained a non-perturbative expression for the luminosity distance without any requirement of symmetries (isotropy and homogeneity)
- Our result nicely appears as the product of a pure source term times a pure observer one. This is true only in the GLC gauge!!!
- We re-expressed our result in two well-known perturbative gauges and we show that the two result perfectly agrees between themselves

Future applications

- Non-perturbative approach to backreaction problem
- Understanding how inhomogeneities may affect cosmological observables
  - redshift drift
  - micro-lensing
  - power spectrum of CMB and large-scale structure