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Superpolynomial complex quantum states

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Polynomial vs superpolynomial

- Origin: computational complexity.

- "Simple" problems: can be solved by a polynomial time algorithm. Example: naive multiplication of two $n \times n$ matrices requires $O(n^3)$ arithmetic operations.

- "Hard" problems: there is no known polynomial time algorithm. Example: Factoring a binary integer of n bit requires $O(\exp(n^{\frac{1}{3}} \ln(n)^{\frac{2}{3}})$ operations (best classical algorithm).

- Quantum factorization algorithm requires only $O(n^3)$ operations (P. Shor, 1994).

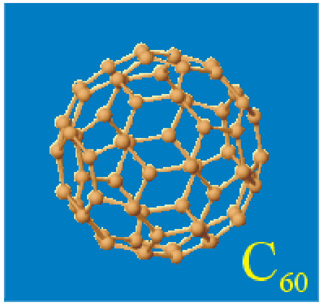
- *Superpolynomial implies high complexity.*

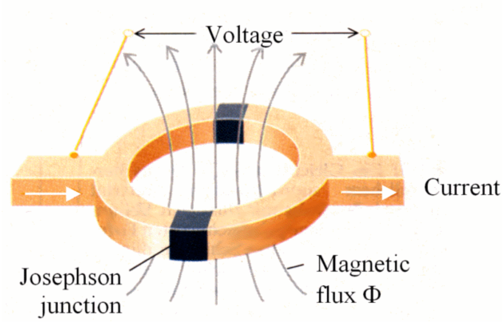
Motivation: Schrödinger's cat

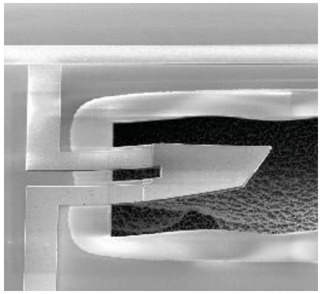
- Cat states: "coherent superposition of two macroscopically distinguishable states."

- Experimental realization: $|\psi_1\rangle + |\psi_2\rangle$ for massive objects.

- Double-slit experiments with heavy molecules, coherent superposition of supercurrent in superconducting interference device (SQUID), micro mechanical oscillators (30 micrometers long).







- Goal: probing quantum mechanics at the macroscopic scale.

Debate on Quantum Computing

- Quantum computers need to be built on large scale to be useful.
- Skeptics: Quantum computing is impossible as quantum mechanics breaks down at large scale.
- Counterargument: Quantum computers require only 1000 qubits while macroscopic coherent superposition of supercurrent involving 10^9 electrons was already realized in SQUID experiments.

Debate on Quantum Computing

- In quantum information the cat state is written as $|\mathbf{0}\rangle^{\otimes \boldsymbol{n}} + |\mathbf{1}\rangle^{\otimes \boldsymbol{n}}$ (GHZ).

Many degrees of freedom, lots of entanglement… but very easy to describe (low complexity).

SQUID coherent superposition can be described by 2×10^9 "terms."

- A generic quantum state of n qubits must be described by 2^n amplitudes: exponential complexity!

A generic state of 1000 qubits requires $2^{1000} \sim 10^{300}$ "terms."

- A quantum state must be sufficiently complex to be useful for quantum computing, otherwise we could simulate it.
- Critics of quantum computing base their arguments on the very possibility of creating complex states.

Motivation

- S. Aaronson (2004), D. Aharonov and U. Varizani (2012): test quantum mechanics at the high complexity limit by creating complex “Schrodinger’s cats” in the lab.

- Is it possible to create complex states?
  - Yes: give strong evidence for the possibility of large scale quantum computers (creating complex states is much easier than doing quantum computation).
  - No: hints for a better understanding of QM.

- Links between complexity of quantum states and universal quantum computing.

Complexity 101

- Creating complex states is not feasible with current technologies: True, but we should take it as an interesting experimental challenge.

- Major obstacle: There are many measures of complexity and they are not consistent.

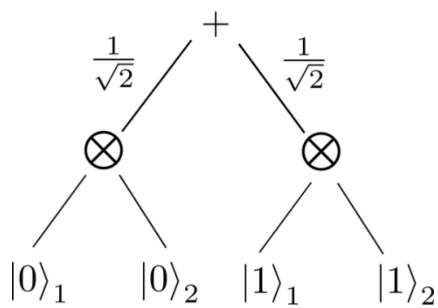
- Most states in the Hilbert space are superpolynomial complex (for every known complexity measures), but there have been no explicit example.

- As a first step, we search for *an explicit example of superpolynomial complex multi-qubit states.*

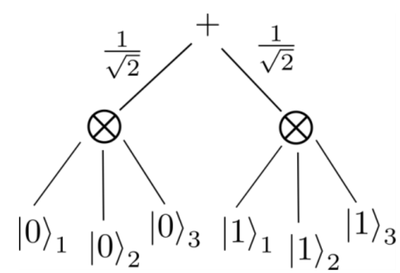
Tree size of quantum states

- S. Aaronson STOC '04: Any quantum state can be described by a rooted tree of $\otimes$ and + gates. Each leaf is labeled with a single-qubit state $\alpha|0\rangle+\beta|1\rangle$.

$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$



$|GHZ\rangle$



- Size of a tree = **number of leaves.**
- **Tree size of a state (TS) = size of the *minimal* tree = most compact way of writing the state**

$$\underbrace{|0\rangle|0\rangle + |0\rangle\,|1\rangle + |1\rangle\,|0\rangle + |1\rangle\,|1\rangle}_{\text{8 leaves}} = \underbrace{|+\rangle\,|+\rangle}_{\text{2 leaves}}$$

$$|+\rangle = |0\rangle + |1\rangle$$

Compact decompositions: three qubits

Naïve basis expansion: $\sum_{x_i=0,1} c_{x_1x_2x_3}|x_1\rangle|x_2\rangle|x_3\rangle$ **24 leaves**

Using Acin et al. 2001:

$$a|0\rangle|0\rangle|0\rangle + b|1\rangle(c|0'\rangle|0''\rangle + d|1'\rangle|1''\rangle)$$

8 leaves

- Computing tree size is exceedingly difficult, but it is possible to find non trivial upper and lower bound.
- “Simple” states: polynomial upper bound.
- “Complex” states: superpolynomial lower bound.

"Simple" states

- Tree size of some well known states in quantum information:

$$TS(|0\rangle^n) = n$$
$$TS(GHZ) = O(2n)$$
$$TS(W) = O(n^2)$$
$$TS(1D\ cluster) = O(n^2)$$

- Any matrix-product state whose tensors are of dimension $d \times d$ has polynomial complexity $\mathrm{TS} = n^{\log_2 2d}$.

$$|MPS_d^n\rangle = \prod_{j=1}^{n} \left(A_0{}^j|0\rangle + A_1{}^j|1\rangle\right)$$

dxd

- Ground states of 1D spin chains have low complexity.

Links with multilinear formula

- **Superpolynomial lower bound** is proved through a relation between a quantum state and a multilinear formula.

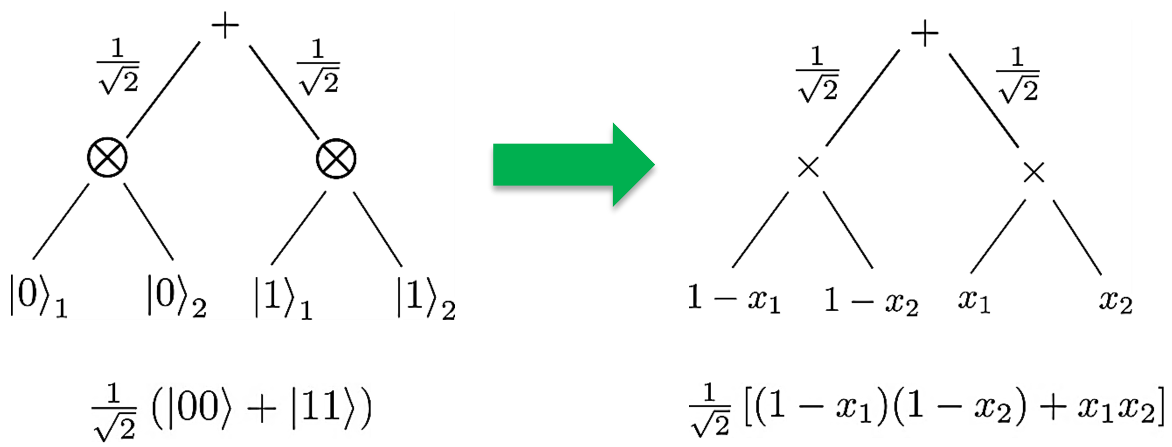
- Expand the state in a basis, then find a multilinear formula that computes the coefficients.

$$|\Psi\rangle = \sum_{x_j=0,1} f(x_1, \ldots, x_n)|x_1, \ldots, x_n\rangle$$

We want a multilinear formula to compute the coefficients

Links with multilinear formula

- To get the multilinear formula from the minimal tree of a state: Replace $|0\rangle_i$ by $1 - x_i$ and $|1\rangle_i$ by x_i, $\otimes$ by $\times$



- Multilinear formula size = number of leaves of the RHS tree.

- Tree size is bounded below by multilinear formula size:

$$\mathrm{TS}(|\psi\rangle) \geq \mathrm{MFS}(f_\psi)$$

- Raz, STOC'04: any multilinear formula that computes the determinant or permanent of a matrix is super-polynomial $(MFS \geq m^{\log m})$

Superpolynomial complex states

- When $n = m^2$, we label the qubits by $x_{11}, x_{12}, \dots, x_{mm}$, then arrange the bit variables to a matrix

$$\{x\} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mm} \end{pmatrix}.$$

- These states have superpolynomial tree size:

$$|\mathrm{det}\rangle = \sum_{x_{ij}=0,1} \det(\{x\}) \, |x_{11}, \dots, x_{mm}\rangle$$

$$|\mathrm{per}\rangle = \sum_{x_{ij}=0,1} \mathrm{per}(\{x\}) \, |x_{11}, \dots, x_{mm}\rangle$$

Raz

Our best decomposition

$$n^{(\log n)/4} \leq TS \leq (\sqrt{n})!$$

Current progress

❖ Most complex few qubit states:

3 qubits

- Biseparable: TS = 5
- GHZ class: TS = 6
- W class: TS = 8

Most complex, but "unstable"

$|W\rangle = |001\rangle + |010\rangle + |100\rangle$

4 qubits

- The most complex class can be written as $|0\rangle|GHZ\rangle + |1\rangle|GHZ'\rangle$ up to SLOCC; TS = 14.
- The most complex class is "stable"
- Example: Dicke state with two excitations

$$|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle$$

❖ Recursive procedure to find most complex states for higher number of qubits.

Open problems

- Prove a stronger lower bound than $n^{\log n}$. An exponential lower bound ($\sim 2^n$)?
- How to create the determinant and permanent states?
- Most important: find a state that is superpolynomial complex with respect to all known complexity measures.
- Compute tree-size complexity of ground states of many-body systems with local interaction.

THANK YOU!