Non-extensive Statistical approach for High Energy Collisions

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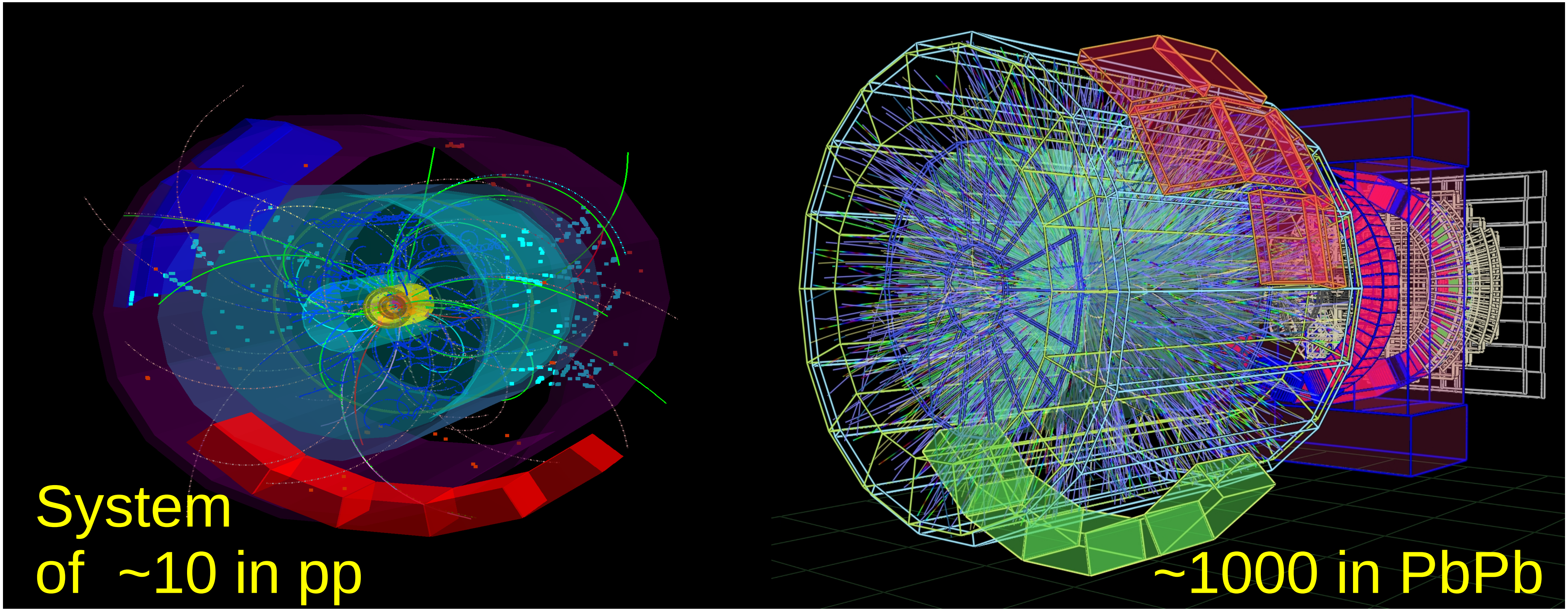
O U T L I N E

- Motivation...
  - This is a QCD session, and a talk withaout QCD at all.
  - But we might understand an OLD experimental parameter, T with the recent physical knowledge.

- Derivation...
  - of the Tsallis/Rényi entropies from the first principles
  - Providing phyiscal meaning of the 'mysterious q'

- Application
  - for Bag model
  - QGP temperature

MOTIVATION



How can we measure the temperature of what?

This is not a system of 10^{23} particles, but 10^{3}

MOTIVATION

Formulated questions from the theory...

– What is responsible for the power low tail measured at high-p_T?

– Can we assume thermodynamical equilibrium for high-p_T particles?

– What is the origin of the ’collectivity’? Is it coming from ’quark level’ or ’hadron level’?

– Is there difference between baryon and meson formation? What is the statistical origin of this (e.g coalescence, fragmentation, etc.)?

The VHMPID LoI (2013)

Why to use Tsallis/Rényi entropy formula?

- It generalizes the Boltzmann-Gibbs-Shanon formula.
- It treats statistical entanglement between subsystem and reservoir (due to conservation).
- It claims to be universal: applicable for whatever material quantity of the reservoir.
- It leads to a cut power law energy distribution in the canonical treatment.

Why NOT to use Tsallis/Rényi formulas?

- They lack 300 years of classical thermo-dynamic foundation
- Tsallis is NOT additive, Rényi is NOT linear
- There is an extra parameter: the mysterious q
- How do different q systems equilibrated?
- Why this and not other?
- It looks pretty formal....

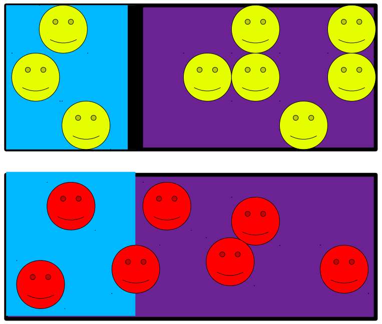
So here is some input to get rid of bad feelings...

The derivation of Tsallis/Rényi Entropy and the Physical Meaning of the q

General derivation as inproved canonical

The story is about...

- Two body thermodynamics: 1 subsystem *(E_1)* + reservoir *($E-E_1$)*
- Finite system, finite energy → microcanonical description
  - microcanonical $\sum_j \epsilon_j = E$
  - canonical $\sum_j \langle \epsilon_j \rangle = E$



- Maximize a monotonic function of the Boltzmann-Gibbs entropy, *L(S)* (0th law of thermodynamics)
- Taylor expansion of the *L(S)* = max, principle beyond $-\beta E$

Description of a system & reservoir

- For generalized entropy function $L(S_{12}) = L(S_1) + L(S_2)$
- In order to exist β of the system $L\left(S(E_1)\right) + L\left(S(E - E_1)\right) = \max$

  TS Biró P. Ván: Phys Rev. E84 19902 (2011)

- Thermal contact between system *(E_1)* & reservoir *(E-E_1)*, requires to eliminate E_1 :

$$\beta_1 = L'\left(S(E_1)\right) \cdot S'(E_1)$$
$$= L'\left(S(E - E_1)\right) \cdot S'(E - E_1)$$

- This is usually handled in canonical limit, but now, we keep higher orders in the Taylor-expansion in E_1/E

$$\beta_1 = L'\left(S(E)\right) \cdot S'(E) - \left[S'(E)^2 L''(S(E)) + S''(E) L'(S(E))\right] E_1 + \ldots$$

Description of a system & reservoir

- Assuming $\beta_1=\beta$, the Lagrange multiplier become familiar for us:

$$\beta = L'\left(S(E)\right)\cdot S'(E) = L'(S)\cdot\frac{1}{T}$$

- To satisfy this, simply solve

$$\frac{L''(S)}{L'(S)} = -\frac{S''(E)}{S'(E)^2}$$

- Universal Thermostat Independence (UTI) Principle: l.h.s. must be as an *S*-independent constant for solving *L(S)*,

$$\frac{L''(S)}{L'(S)} = a$$

- Based on *L(S)* →*S* for small *S*, coming from 3[rd] law of the thermodynamics *L'(0)=1* and *L(0)=0*

$$L(S) = \frac{e^{aS}-1}{a}$$

- EoS derivatives do have physical meaning:

$$S'(E) = 1/T$$
$$S''(E) = -1/CT^2$$

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$$L(S) = \frac{e^{aS} - 1}{a}$$

- Non-additivity parameter is simply the heat capacity of the reservoir:

$$\boxed{a = 1/C}$$

From two system to many...

- Analogue to Gibbs ensamble generalize

$$S = -\sum_i P_i \ln P_i \quad \rightarrow \quad L(S) = \sum_i P_i L(-\ln P_i)$$

- The *L*-additive form of a generally non-additive entropy, given by

$$L(S(E_1)) - \beta E_1 = \frac{1}{a}\left(e^{aS(E_1)} - 1\right) - \beta E_1 = \max.$$

- Introducing $a = 1/C(E) \quad \rightarrow \quad L(S(E_1)) = L(-\ln P_1) = \frac{1}{a}(P_1^{-a} - 1)$

- we need to maximize: $\frac{1}{a}\sum_i (P_i^{1-a} - P_i) - \beta \sum_i P_i E_i - \alpha \sum_i P_i = \max.$

which, results Tsallis: $S_{\mathrm{Tsallis}} := L(S) = \frac{1}{q-1}\sum_i (P_i - P_i^q)$

and its inverse Rényi: $S_{\mathrm{Rényi}} := S = \frac{1}{1-q}\ln \sum_i P_i^q$

The temparature slope

- Taking P_i weights of system, E_i , results cut power law:

$$P_i = \left(Z^{1-q} + (1-q)\frac{\beta}{q} E_i \right)^{\frac{1}{q-1}} = \frac{1}{Z}\left(1 + \frac{Z^{-1/C}\, e^{S/C}}{C-1} \frac{E_i}{T} \right)^{-C}$$

- Partition sum is related to Tsallis entropy, $L(S_1)$ and E_1

$$\ln_q Z := C\left(Z^{1/C} - 1 \right) = L\left(S_1 \right) - \frac{1}{1-1/C}\beta E_1$$

- In $C \to \infty$ limit, the inverse log slope of the energy distribution:

$$T_{\rm slope}(E_i) = \left(-\frac{d}{dE_i} \ln P_i \right)^{-1} = T_0 + E_i/C,$$ with $$T_0 = T e^{-S/C} Z^{1/C} (1 - 1/C)$$

Application:
Quark Gluon Plasma temperature

Experimental data fits by $T_{slope}(E)$

- Taking the $T_{slope}(E)$ fit using $$T_{\rm slope}(E_i) = \left(-\frac{d}{dE_i}\ln P_i\right)^{-1} = T_0 + E_i/C,$$
- Fitted data
  - RHIC@200GeV AuAu: T_0= 48 MeV, C=4.5

    T.S. Biró, K. Ürmössy, Zs. Schram: JPG36 064044 (2009)
    T.S. Biró, K. Ürmössy: JPG37, 0940027 (2010),
  - ALICE@900GeV pp: T_0= 55 MeV, C=8

    J. Cleymans, D. Worku: JPG39, 025006 (2012)

- Findings: *K=2* (mesons) and *K=3* (baryons)

$$P_{\rm hadron}(E) = P_i^K(E/K)$$ and $$T_{\rm slope}^{\rm hadron}(E) = T_{\rm slope}^{\rm quark}(E/K)$$

The obtained values are surprizingly low!!! Why?

Thermal model to heavy-ion collisions

- Test of T_0 in physical models, in a finite termostats, small subsystem: $\lim_{C\to\infty} T_0 = T_1$ and $T_1 = 1/\beta_1 = T e^{-S/C}$

- Taking Stefan-Boltzmann in a bag, with a fix volume, V and bag constant, B

$$E/V = \sigma T^4 + B \qquad p = \frac{1}{3}\sigma T^4 - B \qquad S = \frac{4}{3}\sigma V T^3$$

- The heat capacity is: $C = \frac{dE}{dT} = 4\sigma V T^3 + \left(\sigma T^4 + B\right)\frac{dV}{dT}$

Thermal model to heavy-ion collisions

- Let's discuss some specific cases:

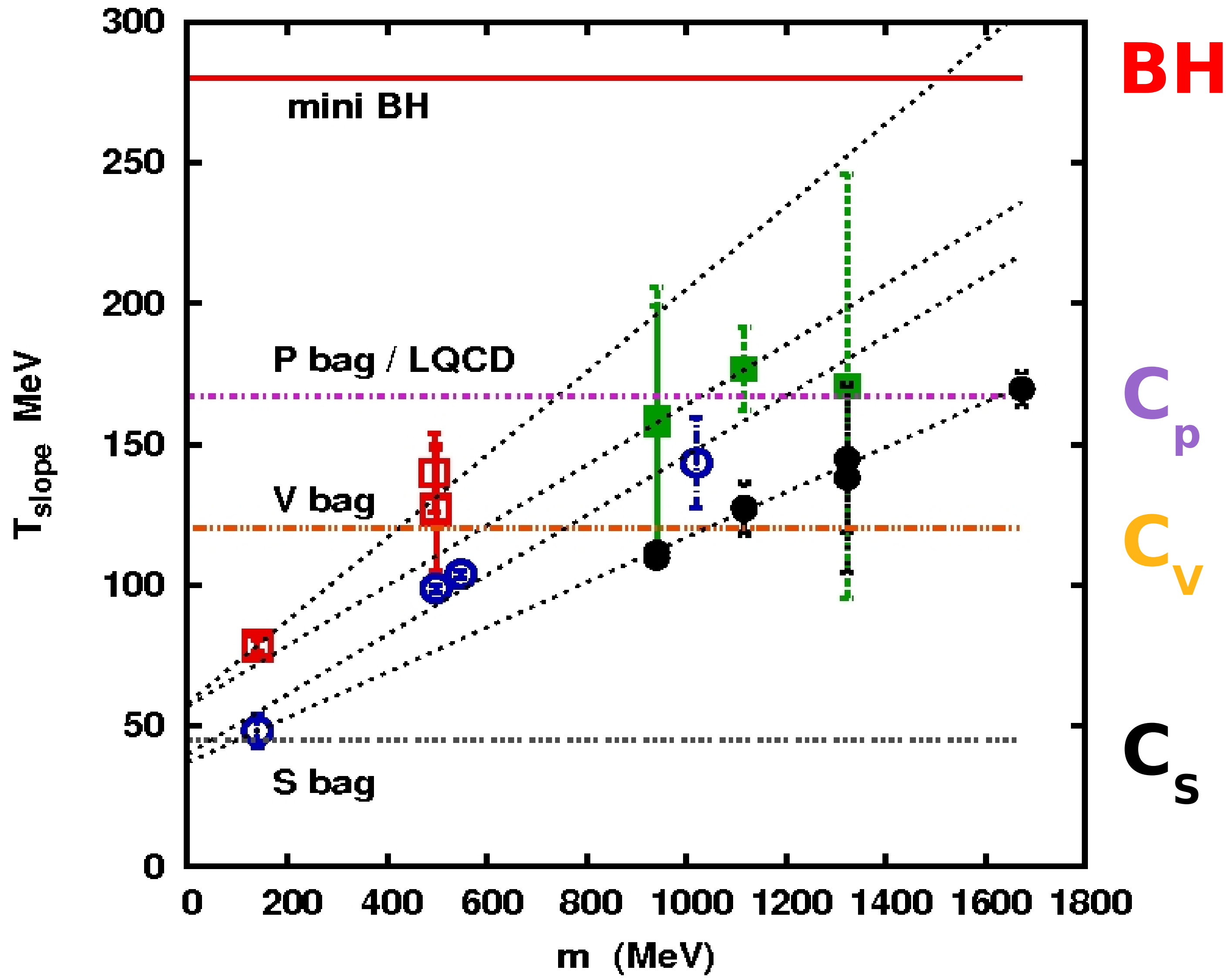
	Heat capacity	Subsystem's T	Note
C_V	$C_V = 4\sigma V T^3 = 3S$	$T_{1V} = T e^{-1/3}$	
C_p	$C_p = \infty$	$T_{1P} = T$	
C_S	$C_S = 3S(1 - T_*^4/T^4)/4$	$T_{1S} \leq T e^{-4/3}$	$C_S \leq 3S/4$
BH	$C = -2S$	$T_1 = T e^{1/2}$	

- Taking the lattice QCD value T=167 MeV, T_{slopes} are:

$$T_{1P} = T = 167 \text{ MeV}, T_{1V} = T e^{-1/3} \approx 120 \text{ MeV and } T_{1S} \leq T e^{-4/3} \approx 45 \text{ MeV}$$

for Tsallis distribution of valence quarks

The temperature slope for different models



TS Biró, GGB, P. Ván, arXiv:1208.2533

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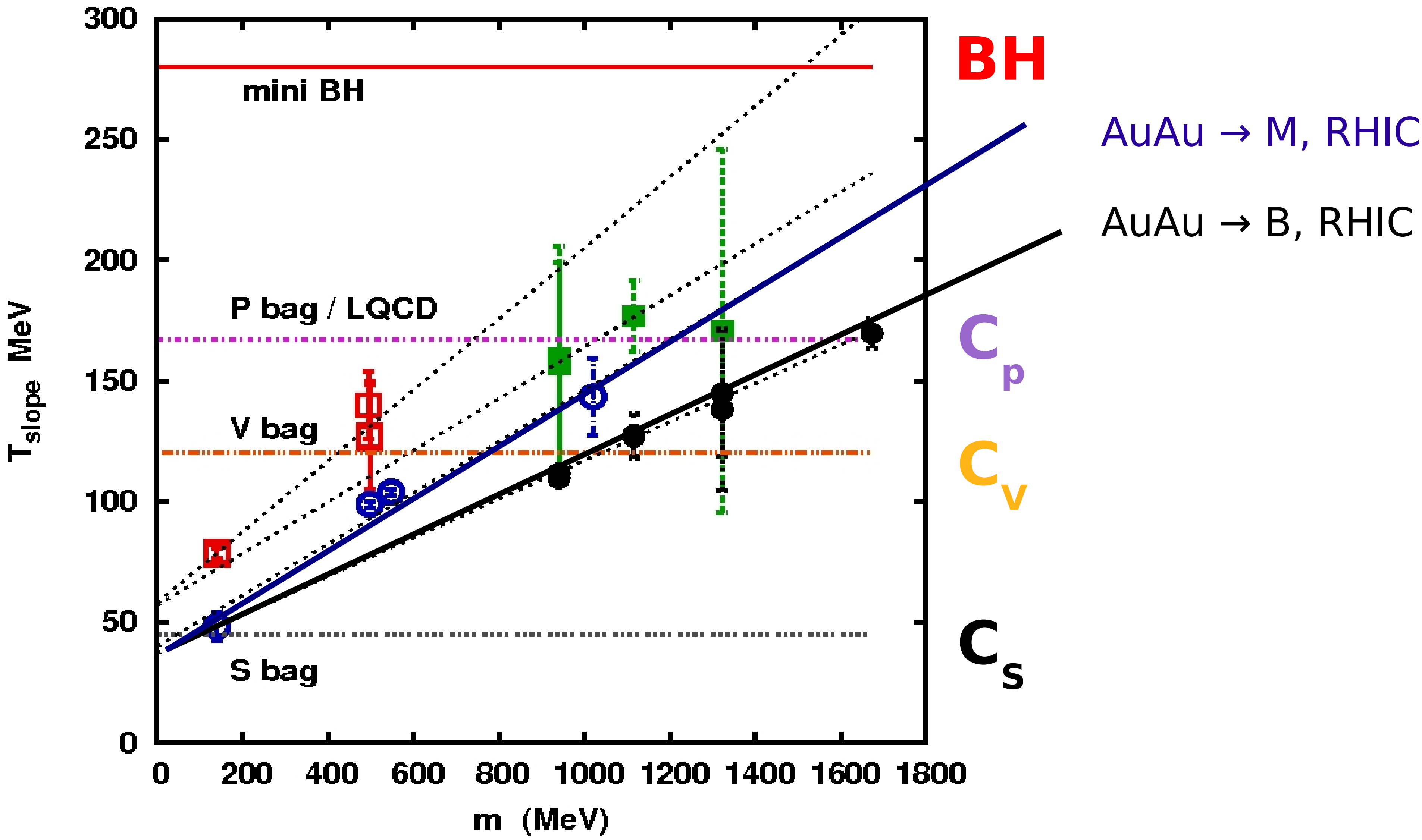

TS Biró, GGB, P. Ván, arXiv:1208.2533

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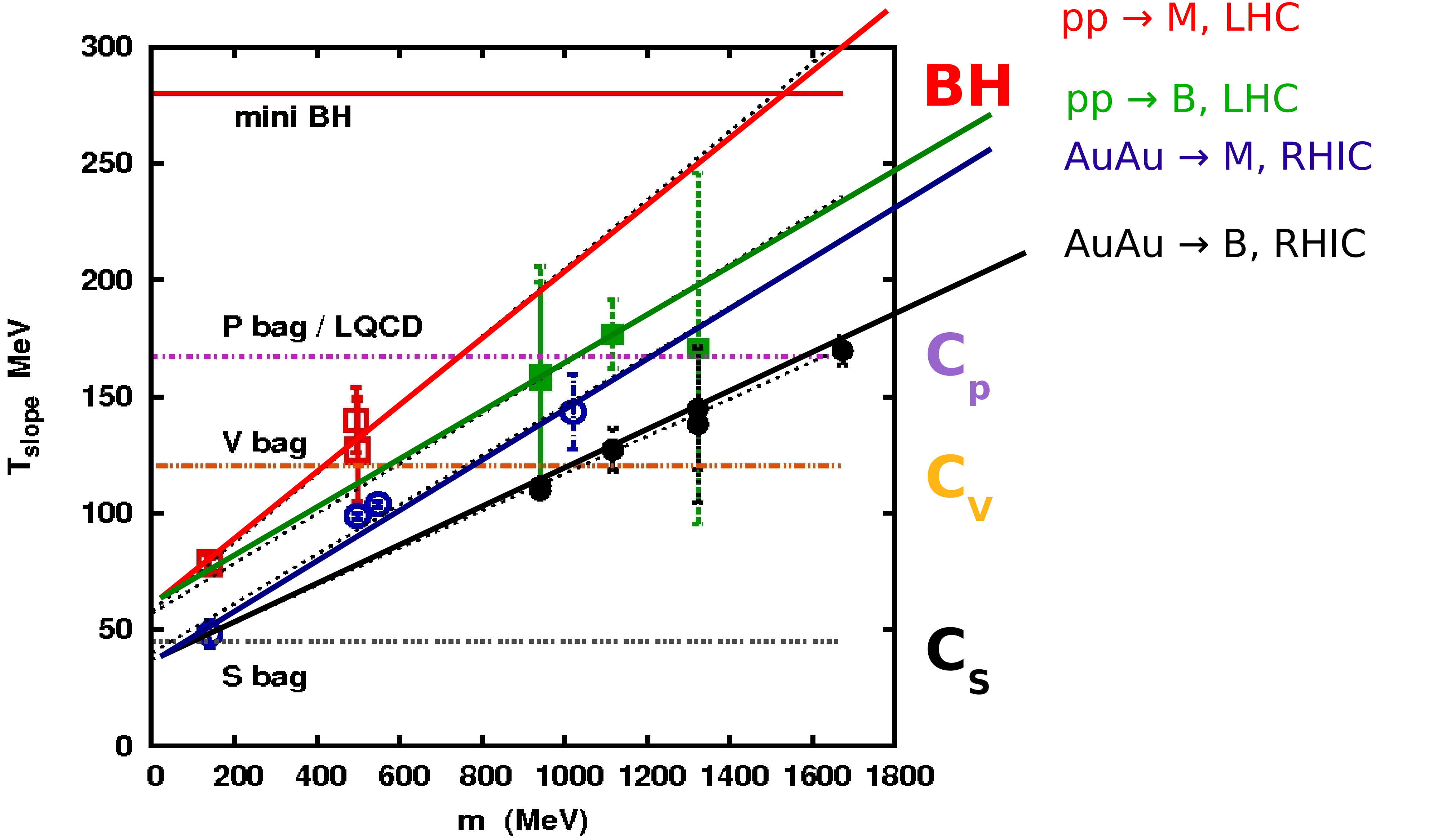

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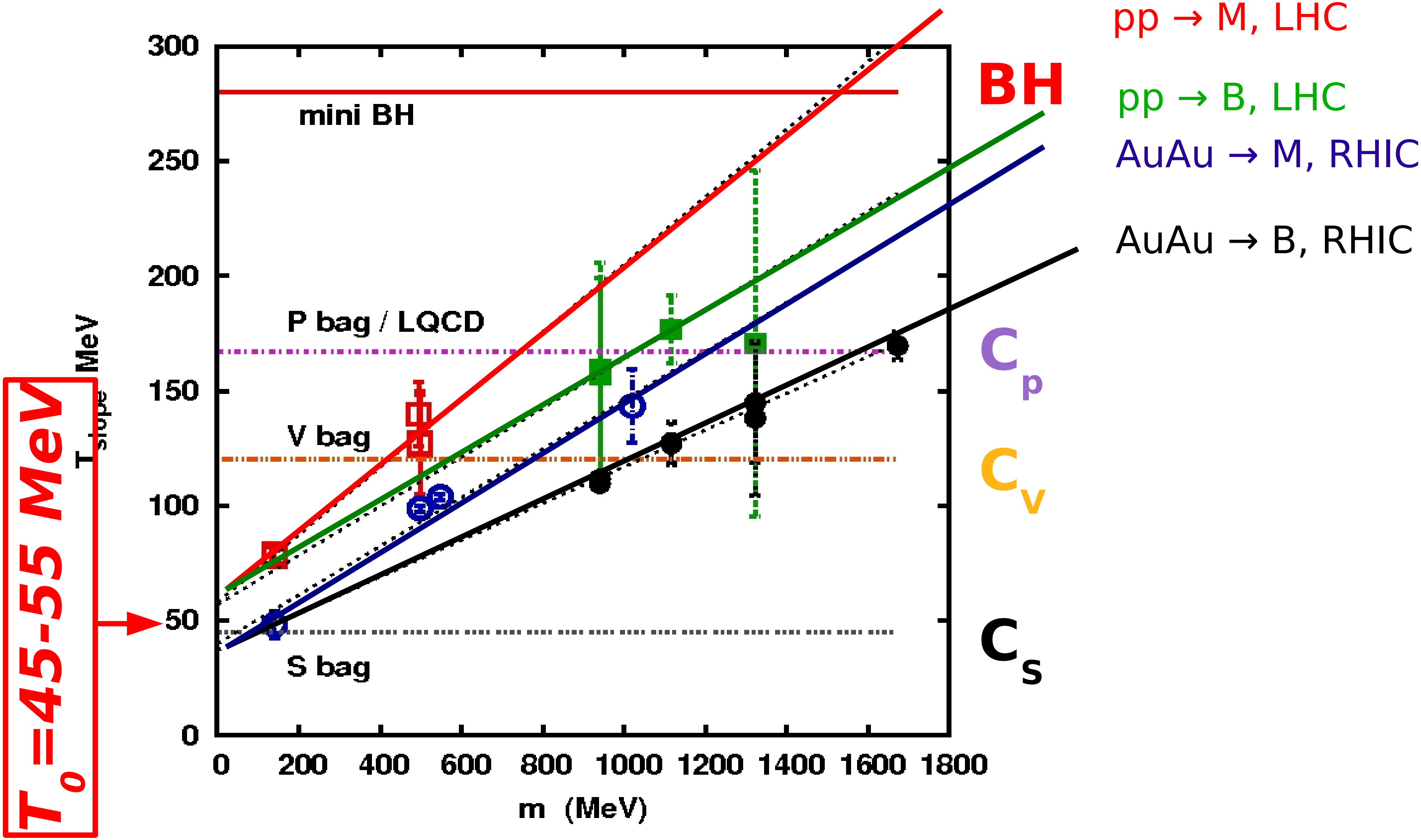


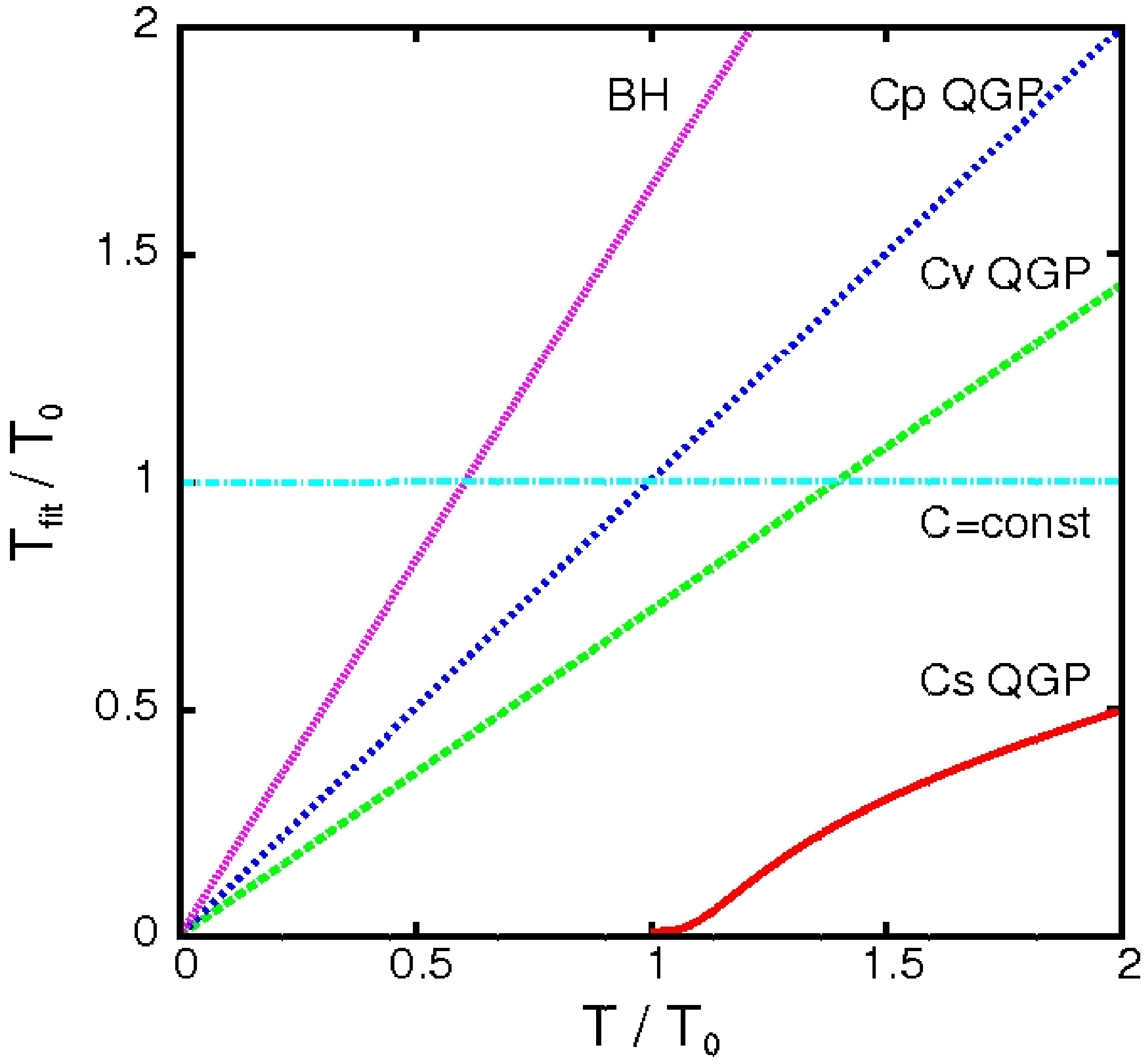
TS Biró, GGB, P. Ván, arXiv:1208.2533

S U M M A R Y

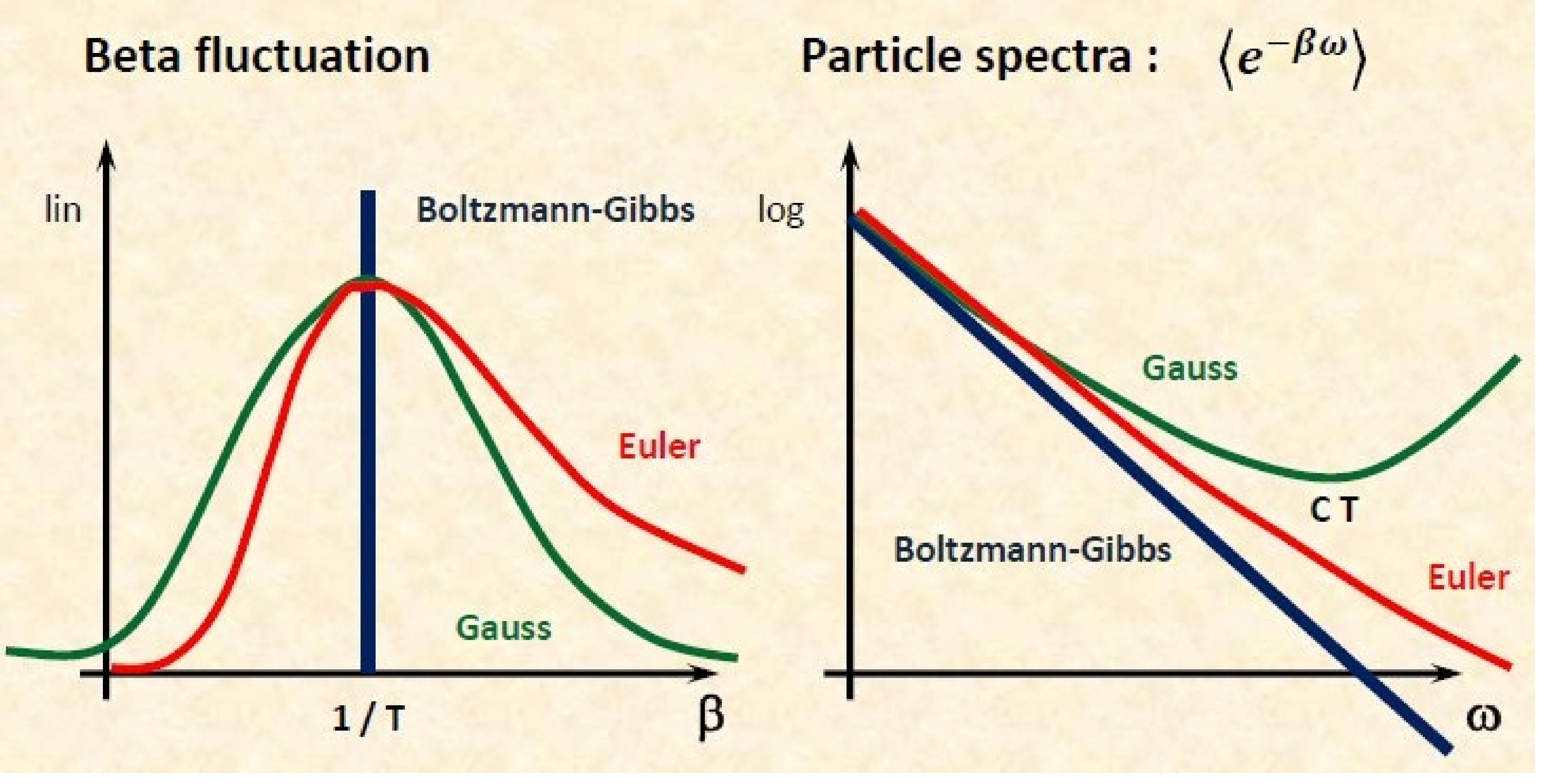
- Derivation
  - Microcanonical treatment
  - Obtained Tsallis/Rényi entropies from the first principles.
  - Not only assumption, but rather a recipe.
  - Providing phyiscal meaning of the 'mysterious q',
  - *q=1-1/C=1-a*
  - *Boltzmann Gibbs limit C → ꝏ, a → 0 (q → 1), L(S) → S*
- Application
  - for Bag model the QGP temperature
    TSB, GGB, PV: arXiv: 1208.2533 EPJ A Letter (2013)
  - Ideal gas TSB Physica A392 (2013) 3132
- See more applications
  - TSB, GGB, K. Ürmössy: microcanonical Tsalls in ee/pp

B A C K U P

The temperature slope for different models



What do we measure as temperature?



ee: Basic model assumptions for e^+e^-

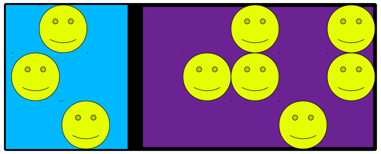
In case of a high-energy collisions (at high-p_T) we can expect:

- Consider jets: narrow objects
- Narrow momentum distribution

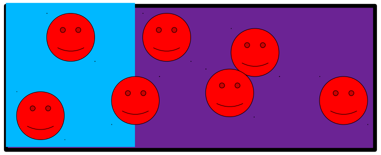
} 1-dimensional object

- Events at $O(10^6)$ → Statistics
- At $\sqrt{s} \geq M_Z$ and ~90% of the events are 2-jet-events: $\sqrt{s}/2 = E$

- Energy-momentum conservation with $m_j = 0$ thus: $\epsilon_j = |\vec{p}|$
  - Micro-canonical: $\sum_j \epsilon_j = E$ (E conserv.)



  - Canonical case: $\sum_j < \epsilon_j > = E$



Ref: K Ürmössy, GGB, TS Biró, arXiv:1101.3023 (2011)

ee: Canonical & microcanonical ensambles

Canonical case for TP:

- One-particle distribution (with fix multiplicity *N*): $f_N(\epsilon) = A_c\, e^{-\beta_N \epsilon}$

- Gamma distribution for multiplicity: $p(N) = A_m\, N^{\alpha-1} e^{-\beta N},$

→ Momentum distribution (CTP): $\frac{\mathrm{d}\ \sigma}{\mathrm{d}\ ^D p} = \sum p(N)\, N\, f_N(\epsilon) \approx \frac{\kappa_{D,E}}{\left(1 + \frac{D}{\beta} x\right)^{\alpha+D+1}}$

Microcanonical generalization of TP

- One-particle distrib. (with fix multiplicity *N*): $f_N(\epsilon) = A_{mc}\, (1-x)^{D(N-1)-1}$

- Shifted Gamma distribution for multiplicity (no to violate the KNO scaling, $N_0 = 1 + 2/D$): $p(N) = A_m\, (N - N_0)^{\alpha-1} e^{-\beta(N-N_0)}$

→ Momentum distribution (μCTP): $\frac{d\sigma}{d^D p} \propto \frac{1-x}{\left(1 - \frac{D}{\beta}\ln(1-x)\right)^{\alpha+D+1}}$

Ref: K Ürmössy, GGB, TS Biró, arXiv:1101.3023 (2011)

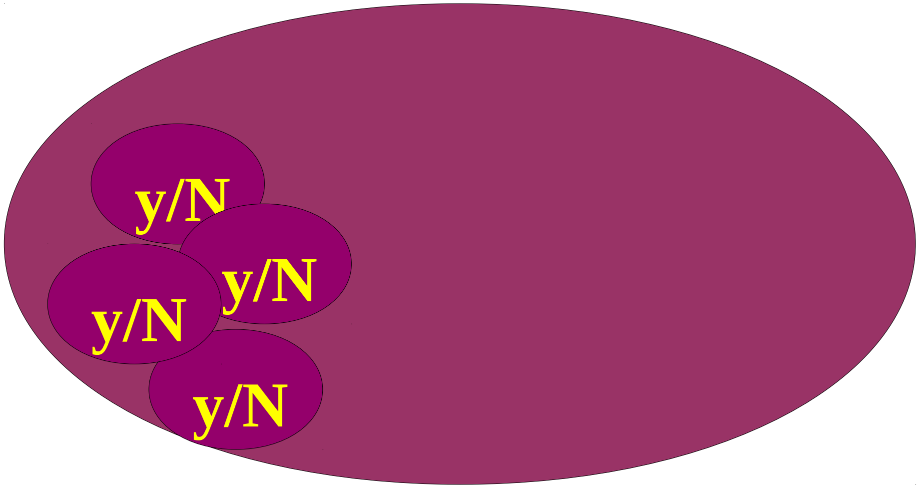
Associative composition ⇉ evolution eq.

Non-extensive Gibbs, generalised

logarithm: $f(x) = \frac{1}{Z} e^{-\beta X(x)}.$

Composition rule for sub-systems:

$$x_N(y) := \underbrace{h \circ \ldots \circ h}_{N-1} \left(\frac{y}{N}, \ldots, \frac{y}{N} \right)$$



Meanwhile satisfy: $\lim_{N \to \infty} x_N(y) < \infty.$

Assimptotically, if $N_1, N_2 \to \infty$:

$$x_{N_1+N_2} = \varphi(x_{N_1}, x_{N_2})$$

recursive equation can be given:

$$x_n = h\left(x_{n-1}, \frac{y}{N}\right)$$, where $h(x,0) = x.$

$$x_n - x_{n-1} = h\left(x_{n-1}, \frac{y}{N}\right) - h(x_{n-1}, 0).$$

Evolution equation can carry out:

$$\frac{dx}{dt} = \frac{y}{t_f} h_2'(x, 0^+)$$

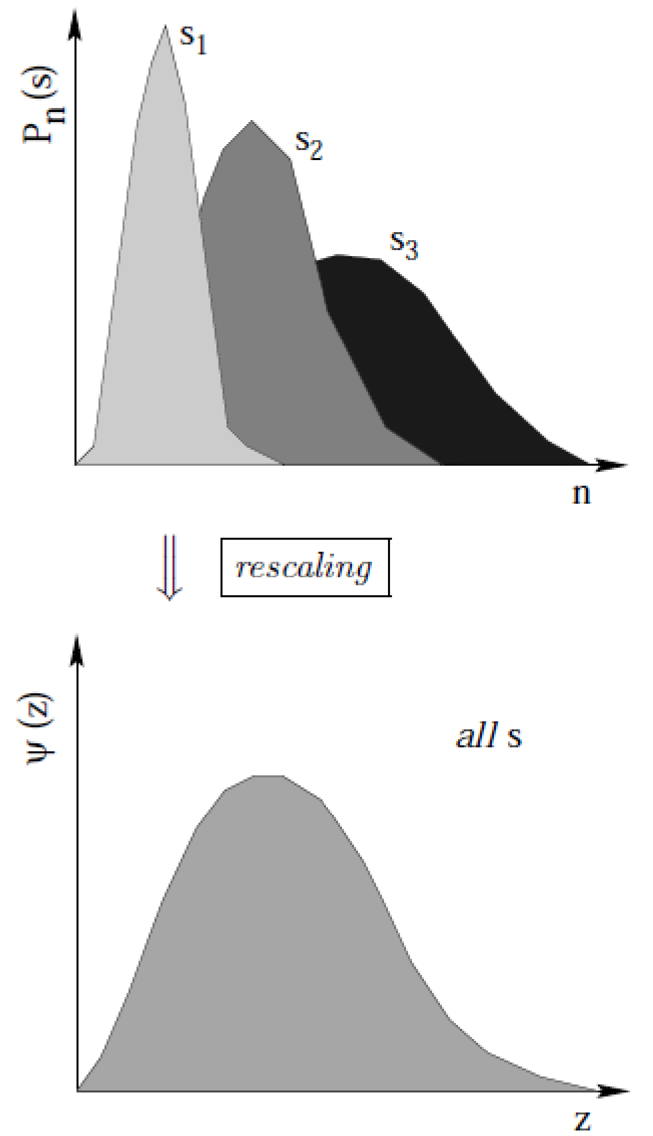
$$L(x) = \int_0^x \frac{dz}{h_2'(z, 0^+)} = y \frac{t}{t_f}.$$

Koba–Nielsen–Olesen (KNO) scaling

Refs:S Hegyi: Nucl. Phys. B 40 (1972) 317, and arXiv:0011301

Hypothesis by Polyakov and Koba–Nielsen–Olesen at very high collision energies, the probability distributions $P_n(s)$ for detecting n final state particles exhibit a scaling (homogeneity) relation:

$$P_n(s) = \frac{1}{\langle n(s) \rangle} \psi\left(\frac{n}{\langle n(s) \rangle} \right)$$



As $s \rightarrow \infty$ with $<n(s)>$ being the average multiplicity of secondaries measured at collision energy s.

KNO: Simple rescaled multiplicity distributions are only a copy of an universal one, $\Psi(z)$ depending on scale $z=n/<n(s)>$ only,

Basics of non-extensive thermodynamics

Non-extensive thermodynamics (Based on: T.S. Biró: EPL84, 56003,2008) associative composition rule, (non-additive) :

$$h(h(x,y),z) = h(x,h(y,z))$$

Then should exist a strict monotonic function, X(x) 'generalised logarithm' (an entropy-like quantity), for which:

$$h(x,y) = X^{-1}\left(X(x) + X(y)\right) \qquad X(h(x,y)) = X(x) + X(y).$$

Examples: (i) Classical Boltzmann–Gibbs thermodynamics:

$$f(E) = e^{-\beta E}/Z \qquad h(x,y) = x + y.$$

(ii) Tsallis–Pareto-like distribution with $a = q - 1$:

$$f(E) = \frac{1}{Z} e^{-\frac{\beta}{a}\ln(1+aE)} = \frac{1}{Z}\left(1 + aE\right)^{-\beta/a} \qquad h(x,y) = x + y + axy$$

$$S = \int f \, \frac{e^{-a\ln(f)} - 1}{a} = \frac{1}{a}\int \left(f^{1-a} - f\right).$$