Foundation of Hydrodynamics of Strongly Interacting Systems

Cheuk-Yin Wong
Oak Ridge National Laboratory

- Introduction
  - finite quantum systems
  - quantum mechanics & hydrodynamics
- Schrödinger equation in hydrodynamical form,
- Many sources of pressure in a many-body system
- Klein-Gordon equation in hydrodynamical form,
  - particles and antiparticle degrees of freedom
  - many sources of pressure in a many-body system
- Conclusions and challenges

Hydrodynamical systems of interest

Hydrodyanamics has applications in many areas of physics, for both finite and infinite systems.

We are interested in *finite* quantum systems:
e.g.

- An atomic nucleus as a liquid drop
- A droplet of Bose-Einstein condensate with interactions or a boundary
- An expanding quark-gluon plasma
- Others…

We are interested in finite systems in which particles obey quantum mechanics.

Quantum Mechanics & Hydrodynamics

Quantum mechanics and hydrodynamics have many elements in common:

- density field n(r,t)
- velocity field u(r,t) or current n(r,t)u(r,t)
- equation of motion for n(r,t)
- Euler-type equations for n(r,t)u(r,t)

We would like to examine to what extent quantum mechanics may be part of the foundation for the hydrodynamics of finite quantum systems.

Quantum systems in a hydrodynamical description

Madelung(1926),Bohm,Takabayasi,…
Wong(J.Math.Phys.17,1008 (1976))
Wong(J.Math.Phys.51,122304(2010))

Single-particle Schrödinger equation in external field $V(r,t)$,

$$i\hbar\frac{\partial}{\partial t}\psi(r,t) = -\frac{\hbar^2}{2m}\nabla^2\psi(r,t) + V(r,t)\psi(r,t)$$

Write $\psi(r,t) = \varphi(r,t)\exp\{iS(r,t)/\hbar - i\Omega(t)\}$,

$\varphi(r,t)$, $S(r,t)$, and $\Omega(t)$ are real functions.

Schrödinger equation can be written in hydrodynamcial form:

$$\frac{\partial\varphi^2}{\partial t} + \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\varphi^2\nabla_j S\right) = 0$$

$$\frac{\partial}{\partial t}\left(\varphi^2\nabla_i S\right) + \sum_{j=1}^{3}\nabla_j\left(\varphi^2\nabla_i S\,\nabla_j S/m + p_{ij}^{(q)}\right) = -\varphi^2\nabla_i V$$

where $p_{ij}^{(q)}$ is the quantum stress tensor arising from quantum effects.

$$p_{ij}^{(q)} = -(\hbar^2/2m)\,\varphi\,\nabla_i\nabla_j\varphi + (\hbar^2/2m)\,\nabla_i\varphi\,\nabla_j\varphi$$

or $$p_{ij}^{(q)} = -(\hbar^2/4m)\nabla^2\varphi^2\delta_{ij} + (\hbar^2/m)\,\nabla_i\varphi\,\nabla_j\varphi$$

or $$p_{ij}^{(q)} = +(\hbar^2/4m)\nabla^2\varphi^2\delta_{ij} - (\hbar^2/m)\,\varphi\,\nabla_i\nabla_j\varphi$$

Derivation of Schrödinger equation in hydrodynamical form

$$i\hbar\partial_t\psi(r,t)=\frac{1}{2m}(-i\hbar\nabla)^2\psi(r,t)+V(r,t)\psi(r,t)$$

$\psi(r,t)=\varphi(r,t)\exp\{iS(r,t)/\hbar-i\Omega(t)\}$, $\varphi, S,$ and Ω are real functions

We construct $\psi^* i\hbar\partial_t\psi+\psi\, i\hbar\partial_t\psi^*$, and we get

$$\varphi^2\left(-2\partial_t S\ -4\partial_t\Omega\ \right)=\frac{1}{2m}\left\{-2[\varphi\,\nabla^2\varphi\ -2\varphi^2(\nabla S\ )^2]+2\varphi^2 V\right.$$

Divide by φ^2, take the gradiant ∇_i, and multiple by φ^2, we get

$$\partial_t[\varphi^2\nabla_i S\ ]+\sum_{j=1}^{3}\nabla_j[\frac{\varphi^2}{m}\nabla_i S\nabla_j S]=-\frac{1}{2m}\varphi^2\nabla_i\frac{\nabla^2\varphi}{\varphi}-\varphi^2\nabla_i V.$$

$$\varphi^2\nabla_i\frac{\nabla^2\varphi}{\varphi}=-(\nabla_i\varphi)\sum_j\nabla_j\nabla_j\varphi+\varphi\nabla_i\sum_j\nabla_j\nabla_j\varphi$$

$$=-\sum_j\nabla_j\{(\nabla_i\varphi)(\nabla_j\varphi)-(\nabla_j\nabla_i\varphi)(\nabla_j\varphi)\}+\sum_j\nabla_j\{(\varphi\nabla_i\nabla_j\varphi)-(\nabla_j\varphi)(\nabla_j\nabla_j\varphi)\}$$

$$=\sum_j\nabla_j\{-(\nabla_i\varphi)(\nabla_j\varphi)+(\varphi\nabla_i\nabla_j\varphi)\}$$

We define $\quad p_{ij}^{(q)}=-\frac{\hbar^2}{2m}\varphi\,\nabla_i\nabla_j\varphi\ +\frac{\hbar^2}{2m}\nabla_i\varphi\,\nabla_j\varphi$

then $\quad \partial_t[\varphi^2\nabla_i S\ ]+\sum_{j=1}^{3}\nabla_j[\varphi^2\nabla_i S\nabla_j S/m+p_{ij}^{(q)}]=-\varphi^2\nabla_i V.$

Examples of quantum pressure

$$p_{ij}^{(q)} = -(\hbar^2/2m)\varphi\nabla_i\nabla_j\varphi \;\; + (\hbar^2/2m)\;\nabla_i\varphi\nabla_j\varphi$$

Example (1) Plain wave wavefunction,

$$\varphi(x) = A\cos(kx + B)$$

$$p_{xx}^{(q)} = \frac{\hbar^2}{2m}k^2A^2[\cos^2(kx+B) + \sin^2(kx+B)] = \frac{\hbar^2}{2m}k^2A^2,$$

(i) the quantum pressure is spatially constant inside a flat well

(ii) the quantum pressure is perpendicular to the direction of motion

(iii) the quantum pressure increases with k^2

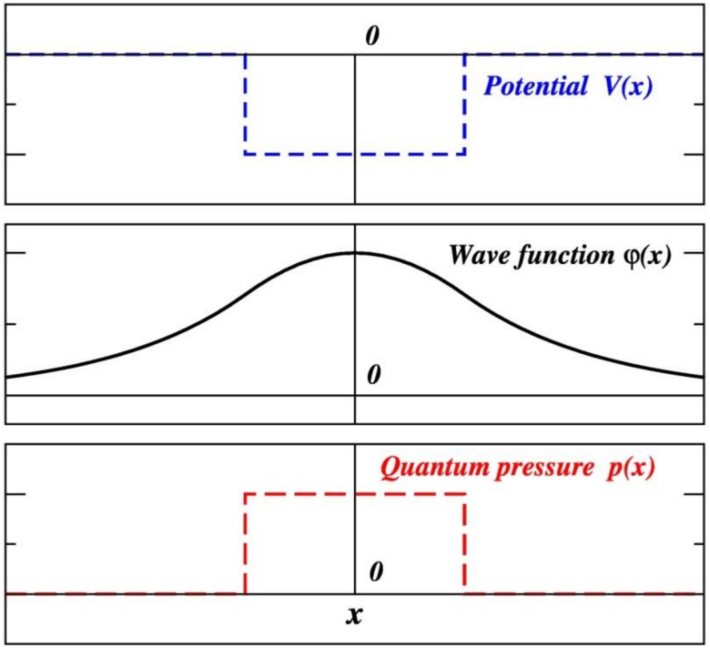
Example (2) Exponential wavefunction

$$\varphi(x) = A\exp\{-\kappa x\}$$

$$p_{xx}^{(q)} = -\frac{\hbar^2}{2m}\kappa^2A^2 + \frac{\hbar^2}{2m}\kappa^2A^2 = 0$$

the quantum pressure is zero outside a flat potential well.

Example (3) A single particle in a square well potential



A droplet of Bose-Eistein condensate will have the above features.
If the potential is suddenly removed, the quantum pressure will tend to explode the droplet.

Example (4) : A single particle in a harmonic oscillator

$$V(x) = \frac{1}{2} m\omega^2 x^2$$

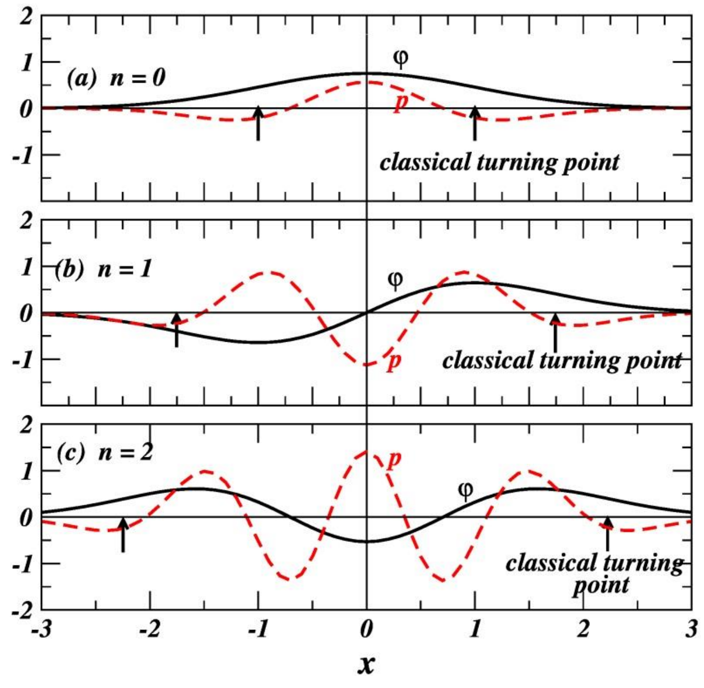
$$\varphi(x) = N_n \exp\{-\xi^2/2\} H_n(\xi), \qquad \xi = (m\omega/\hbar)^{1/2} x$$

$$p_{xx} = \frac{\hbar^2}{2m} \kappa^2 N_n \exp\{-\kappa^2 x^2/2\}\left\{(2n+1-\xi^2) H_n^2(\xi) - [\xi H_n(\xi) + 2n H_{n-1}(\xi)]^2\right\}$$

$$m \frac{\partial}{\partial t}\left(\varphi^2 \nabla_i S\right) + \sum_j \nabla_j \left(m\varphi^2 \nabla_i S \, \nabla_j S + p_{ij}^{(q)}\right) = -\varphi^2 \nabla_i V$$

A hydrostatic equlibrium is attained, when the force due to the quantum pressure balances the external force $\nabla_x V$ acts on the density φ^2 :

$$\nabla_x p_{xx}^{(q)} = -\varphi^2(x) \nabla_x V(x)$$



Example (5):

A wavepacket without a boundary or without an interaction will expand due to the presence of quantum pressure.

Many-body system in mean-field description

In the time-dependent mean-field approximation, we describe the system by a set of single-particle states $\psi_\alpha(r,t)$ and occupation numbers $n_\alpha(t)$

$$\text{(I)} \qquad i\hbar \frac{\partial}{\partial t}\psi_\alpha(r,t) = -\frac{\hbar^2}{2m}\nabla^2\psi_\alpha(r,t) + V(r,t)\psi_\alpha(r,t)$$

$$\text{where } \; V(r,t) = \int d^3r' \sum_\beta n_\beta(t)\psi_\beta(r_2,t)\; \psi^*_\beta(r_2,t)\; v(r,r_2),$$

$$\text{(II)} \quad \frac{\partial n_1(t)}{\partial t} = \frac{\pi}{\hbar}\sum_{234}\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)$$

$$\times[(1\pm n_1)(1\pm n_2)n_3n_4 - n_1n_2(1\pm n_3)(1\pm n_4)]|\langle 12|v'|34\rangle|^2$$

$+$ sign for bosons

$-$ sign for fermion

$v(r_1,r)$ is the two-body interaction, and $v'(r_1,r)$ is the residual interaction after taking into account the mean-field interactions.

Thermal equilibrium is reached when the gain and loss terms are equal.
The occupation number is then a Fermi-Dirac or a Bose-Einstein distribution characterised by a temperature.

Many-body system in mean-field description

Now consider the dynamcail motions in which the time scale for the dynamical motion is much greater than the time cale for relaxtion into the equalibrium value of n_α.

We can then assume the presence of thermal equalibrium and consider n_α to be given by the eqilbrium occupation numbers.

The density field of the system is given by

$$n(r,t)=\sum_{\beta} n_\beta(t)\psi_\beta(r,t)\,\psi^*_\beta(r,t)=\sum_{\beta} n_\beta(t)|\,\varphi_\beta(r,t)\,|^2$$

The current field of the system is given by

$$n(r,t)u_i(r,t)=\sum_{\beta} n_\beta(t)|\,\varphi_\beta(r,t)\,|^2\nabla_i S_\beta(r,t)$$

Many-body system in hydrodynamical description

Tme-dependent mean-field theory leads to the hydrodynamical equation

$$\frac{\partial n(r,t)}{\partial t}+\sum_{j=1}^{3}\frac{\partial}{\partial x_j}nu_j=0$$

$$\frac{\partial}{\partial t}nu_i+\sum_{j=1}^{3}\frac{\partial}{\partial x_j}nu_iu_j=-\frac{1}{m}\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(p_{ij}^{(q)}+p_{ij}^{(t)}+p_{ij}^{(v)}\right)$$

$$p_{ij}^{(q)}=-(\hbar^2/2m)\sum_{\beta} n_\beta\,\varphi_\beta\,\nabla_i\nabla_j\varphi_\beta+(\hbar^2/2m)\sum_{\beta}n_\beta\,\nabla_i\varphi_\beta\,\nabla_j\varphi_\beta$$

$$p_{ij}^{(t)}=-(\hbar^2/2m)\sum_{\beta}n_\beta\,\varphi_\beta^2(\nabla_iS_\beta-u_i)(\nabla_jS_\beta-u_j)$$

$$\frac{\partial}{\partial x_j}p_{ij}^{(v)}(r,t)=n\frac{\partial}{\partial x_j}\int d^3r_2\;\;n(r_2,t)\;\;v(r,r_2)=n\frac{\partial}{\partial x_j}\left(\frac{\partial(W^{(v)}n)}{\partial n}\right)$$

$W^{(v)}=$ energy per particle arising from mean-field interaction

$$p_{ij}^{(v)}=n\frac{\partial(W^{(v)}n)}{\partial n}-W^{(v)}n$$

Many different sources of pressure in hydrodynamics

- Quantum stress tensor $p_{ij}^{(q)}$
- Thermal stress tensor $p_{ij}^{(t)}$, deviation of velocity fields of individual particles (occupied states) from the average velocity fields.
- Mean field interaction $p_{ij}^{(v)}$, that is nearly instantaneous, for a fluid element at rest.

C.Y.Wong et al.,Nucl.Phys.A253,469(1975)
C.Y.Wong et al.,Phys.Rev.C15,1558(1977)
C.Y.Wong, Phys.Rev.C17,1832(1978)

Example: Finite nucleus as a liquid drop

$$p_{ij}^{(q)} = \frac{\hbar^2}{5m}\left(\frac{6\pi^2}{4}\right)^{2/3} n^{5/3}\delta_{ij}$$

$$p_{ij}^{(t)} = \frac{\hbar^2}{5m}\left(\frac{6\pi^2}{4}\right)^{2/3} n^{5/3}\left[\frac{2mkT}{\hbar^2 m(6\pi^2 n/4)^{2/3}}\right]^2 \delta_{ij}$$

$$p_{ij}^{(v)} = \frac{3}{8}(t_0 + \frac{1}{3}t_3 n)\, n^2\delta_{ij}$$

e.g. SkyrmeI nucleon-nucleon interaction

$$v(\vec{r}_i - \vec{r}_j) = \left[t_0 + \frac{1}{6}t_3 n\left(\frac{\vec{r}_i + \vec{r}_j}{2}\right)\right]\delta(\vec{r}_i - \vec{r}_j)$$

$$t_0 = -1057\,\mathrm{MeV/fm}^3 \text{ (two-body force)}$$

$$t_3 = +14463\,\mathrm{MeV/fm}^6 \text{(three-body force)}$$

A conclusion

Quantum mechanics is an important foundation for
the hydrodynamics of an atomic nucleus as a liquid drop
and other non-relativistic finite quantum systems.

Klein-Gordon equation as a wave mechanical equation (1)

Consider Klein-Gordon equation for a single-particle in an external scalar field $\mathbb{S}(r,t)$,

$$\left[p_0{}^2 - \vec{p}^{\,2} - (\underbrace{m + \mathbb{S}(r,t)}_{M(r,t)})^2 \right] \Psi = 0$$

We wish to cast the Klein-Gordon equation in hydrodynamical form.

Difficulty: the density field $n(r,t) = 2\,\mathrm{Im}\,(\Psi^* \partial_t \Psi)$ is not necessarily a positive quantity

Difficulty resolved by Dirac (Dirac eqiation),

Weisskopf and Pauli (introduction of field theory concepts)

Wave mechanical description [Feshbach-Villars (Rev. Mod. Phys. 30, 24 (1958))]:

introduces particle and anitparticle components, with postive probability densities,

and $\pm$ve single-particle energies

Recent wave mechanical description [Wong (J. Math. Phys. 51, 122304 (2010)]:

introduces particle and anitparticle components, with postive probability densities,

and only $+$ve single-particle energies

The wave mechanical description facilitates the transition to a hydrodynamical descripiton.

Klein-Gordon equation as a wave mechanical equation (2)

We decompose wavefunction Ψ into two components in the usual way

$$\Psi = \psi_+ + \psi_-^*$$

We introduce Ψ_4 as $\quad \Psi_4 = \psi_+ - \psi_-^*$

We introduce $E > 0$ as the positive solution of the quadartic equation in E

$$\int \mathrm{d}^3 r\, \Psi^*(r,t) \left[E^2 + (i\hbar\partial_t E) - \vec{p}^{\,2} - M^2(r)\right] \Psi^*(r,t) = 0$$

We can rewrite the Klein - Gordon equation as

$$\mathrm{i}\hbar\partial_t \Psi = (E\,\Psi_4) \qquad \text{(Feshbach \& Villar uses } m \text{ instead of } E\text{)}$$

$$\mathrm{i}\hbar\partial_t (E\,\Psi_4) = \left[\vec{p}^{\,2} + M^2(r,t)\right] \Psi$$

We construct ψ_+ and ψ_- components in terms of Ψ and Ψ_4

$$\psi_+ = \frac{1}{2}\left(\Psi + \Psi_4\right), \quad \text{and} \quad \psi_- = \frac{1}{2}\left(\Psi^* - \Psi_4^*\right),$$

The Klein - Gordon equation can be reduced into two coupled

Schrödinger equations for ψ_+ and ψ_- :

$$i\hbar\partial_t \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix} = \frac{1}{2E}\Big\{[(-i\hbar\nabla)^2 + M^2(r,t) + [E^2 - i\hbar\partial_t E]\begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix}$$

$$+[(-i\hbar\nabla)^2 + M^2(r,t) - [E^2 - i\hbar\partial_t E]\begin{pmatrix}\psi_-^* \\ \psi_+^*\end{pmatrix}\Big\}$$

Klein-Gordon equation as a wave mechanical equation (4)

In a more general case for a single-particle in an external scalar $S(r,t)$ and a vector potential $A(r,t)=(A_0,\vec{A})$, the of Klein-Gordon equation is

$$\left[(i\hbar\partial_t - eA_0)^2 - (-i\hbar\nabla - e\vec{A})^2 - M^2\right]\Psi = 0.$$

We consider the state of either a particle or antiparticle state with an definite n_ν number,

$$n_{\text{particle}} = n_\nu \equiv \int d^3r(\psi_+^*\psi_+ - \psi_-^*\psi_-)$$

$$n_{\text{particle}} = n_\nu = \begin{cases} +1, & \text{particle state with } \psi_+ \text{ dominant} \\ -1, & \text{particle state with } \psi_- \text{ dominant} \end{cases}$$

We introduce $E>0$ as the positive solution of the quadartic equation in E

$$\int d^3r\,\Psi^*(r,t)\left[(E-e_\nu A_0)^2 + i\hbar\partial_t(E-e_\nu A_0) - (-i\hbar\nabla - e_\pm\vec{A})^2 - M^2\right]\Psi^*(r,t) = 0$$

The coupled Schrodinger equation becomes [Wong(J.Math.Phys.51,122304(2010)]

$$(i\hbar\partial_t - e_\pm A_0)\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} = \frac{1}{2(E-e_\nu A_0)}\left\{[(-i\hbar\nabla - e_\pm\vec{A})^2 + M^2(r) + [(E-e_\nu A_0)^2 - i\hbar\partial_t(E-e_\nu A_0)]\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}\right.$$
$$\left. + [(-i\hbar\nabla - e_\pm\vec{A})^2 + M^2(r) - [(E-e_\tau A_0)^2 - i\hbar\partial_t(E-e_\tau A_0)]\begin{pmatrix}\psi_-^*\\ \psi_+^*\end{pmatrix}\right\}$$

where $e_\pm = \begin{cases} +e & \text{is for } \psi_+ \\ -e & \text{is for } \psi_- \end{cases}$ and $e_\nu = n_{\text{particle}}\, e = n_\nu\, e.$

Klein-Gordon equation in hydrodynamical form (1)

$$i\hbar\partial_t\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}=\frac{1}{2E}\left\{[(-i\hbar\nabla)^2+M^2(r,t)+[E^2-i\hbar\partial_t E]\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}\right.$$

$$\left.+[(-i\hbar\nabla)^2+M^2(r,t)-[E^2-i\hbar\partial_t E]\begin{pmatrix}\psi_-^*\\ \psi_+^*\end{pmatrix}\right\}$$

We write $\quad \psi_\pm(r,t)=\varphi_\pm(r,t)\exp\{iS_\pm(r,t)/\hbar-i\Omega_\pm(t)\}.$

$\varphi_\pm(r,t)$, $S_\pm(r,t)$, and $\Omega_\pm(t)$ are real functions

We construct $\quad \psi_\pm^* i\hbar\partial_t\psi_\pm-\psi_\pm i\hbar\partial_t\psi_\pm^*$, and we get

$$\partial_t[E\varphi_\pm^2]+\nabla\cdot[\varphi_\pm^2\nabla S_\pm]=X_\pm$$

$$2X_\pm=\left\{\psi_\pm^*(-i\hbar\nabla)^2\psi_\pm^*-\psi_\pm(i\hbar\nabla)^2\psi_\pm\right\}$$

$$+[M^2(r)+E^2]\left\{\psi_\pm^*\psi_\pm^*-\psi_\pm\psi_\pm\right\}$$

$$+[i\hbar\partial_t E]\left\{\psi_\pm^*\psi_\pm^*+\psi_\pm\psi_\pm\right\}$$

Remark (1): the currents of the two components are not separately conserved due to the production of particle-antiparticle pairs

Klein-Gordon equation in hydrodynamical form (2)

Remarks (2): There is a conservation of the net particle number n_{particle}

$$\partial_t[E\varphi_\pm^2]+\nabla\cdot[\varphi_\pm^2\nabla S_\pm] = X_\pm$$

$$\partial_t[E\varphi_+^2]+\nabla\cdot[\varphi_+^2\nabla S_+] - \quad \partial_t[E\varphi_-^2]+\nabla\cdot[\varphi_-^2\nabla S_-] = X_+ - X_-$$

$$\partial_t[E(\varphi_+^2-\varphi_-^2)]+\nabla\cdot[\varphi_+^2\nabla S_+ - \varphi_-^2\nabla S_-] = X_+ - X_-$$

$$= -\nabla\cdot[(\psi_+^*\nabla\psi_-^* - \psi_-^*\nabla\psi_+^*) + \nabla\cdot(\psi_+\nabla\psi_- - \psi_-\nabla\psi_+)$$

$X_+ - X_-$ is a complete divergence

$$\therefore \quad n_{\text{particle}} = \int d^3r\,\frac{E}{m}(\varphi_+^2-\varphi_-^2) \text{ is conserved.}$$

We can normalize the single-particle state as

$$n_{\text{particle}} = \int d^3r\,\frac{E}{m}(\varphi_+^2-\varphi_-^2) = \begin{cases} +1, & \text{particle state with } \varphi_+ \text{ dominant} \\ -1, & \text{particle state with } \varphi_- \text{ dominant} \end{cases}$$

Klein-Gordon equation in hydrodynamical form (3)

Remark (3): In the equation of continuity

$$\partial_t[E\varphi_\pm^2]+\nabla\cdot[\varphi_\pm^2\nabla S_\pm]=X_\pm,$$

quanity $X_\pm$ contain terms of the type

$$\psi_\pm^*\nabla\psi_\pm^*,\quad \psi_\pm\nabla\psi_\pm,\quad \psi_\pm^*\psi_\pm^* \text{ and } \psi_\pm\psi_\pm.$$

which leads to the couplings between ψ_+ and ψ_-.

But $\psi_\pm(r,t)=\varphi_\pm(r,t)\exp\{iS_\pm(r,t)/\hbar-i\Omega_\pm(t)\}$.

these terms have the time factor $e^{\pm 2i\Omega(t)}$, representing zitterbewegung motion of frequancy greater than $2m/\hbar$.

Zitterbewegung motion leads to pair production, but the time average of these $e^{\pm 2i\Omega(t)}$ contributions over a longer period of time gives

$$\left\langle\psi_\pm^*\psi_\pm^*+\psi_\pm\psi_\pm\right\rangle_T=\frac{1}{T}\int_0^T dt\,(\psi_\pm^*\psi_\pm^*+\psi_\pm\psi_\pm)\propto\frac{1}{T}\int_0^T dt\,\left(e^{2i\Omega(t)}+e^{-2i\Omega(t)}\right)\sim\frac{1}{2mT}$$

If dynamical time scale $T>>\dfrac{1}{2m}$, then $\dfrac{1}{2mT}<<1$ and terms of the type

$$\psi_\pm^*\nabla\psi_\pm^*,\quad \psi_\pm\nabla\psi_\pm,\quad \psi_\pm^*\psi_\pm^* \text{ and } \psi_\pm\psi_\pm$$

becomes negligible when averaged over the time period T.

Klein-Gordon equation in hydrodynamical form (4)

$$i\hbar\partial_t\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}=\frac{1}{2E}\Big\{[(-i\hbar\nabla)^2+M^2(r,t)+[E^2-i\hbar\partial_t E]\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}$$

$$+[(-i\hbar\nabla)^2+M^2(r,t)-[E^2-i\hbar\partial_t E]\begin{pmatrix}\psi_-^*\\ \psi_+^*\end{pmatrix}\Big\}$$

We write $\psi_\pm(r,t)=\varphi_\pm(r,t)\exp\{iS_\pm(r,t)/\hbar-i\Omega_\pm(t)\}.$

We construct $\psi_\pm^* i\hbar\partial_t\psi_\pm+\psi_\pm i\hbar\partial_t\psi_\pm^*$, and we get

$$\varphi_\pm^2\left(-2\partial_t S_\pm-4\partial_t\Omega_\pm\right)$$

$$=\frac{1}{2E}\Big\{-2[\varphi_\pm\nabla^2\varphi_\pm-2\varphi_\pm^2(\nabla S_\pm)^2]+[M^2(r,t)+E^2]2\varphi_\pm^2$$

$$+\psi_\pm^*(-i\hbar\nabla)^2\psi_\pm^*+\psi_\pm(i\hbar\nabla)^2\psi_\pm+[M^2(r,t)-E^2][\psi_\pm^*\psi_\pm^*+\psi_\pm\psi_\pm]$$

$$+[i\hbar\partial_t E][\psi_\pm^*\psi_\pm^*-\psi_\pm\psi_\pm]\ \Big\}$$

The last four terms involve

$$\psi_\pm^*\nabla^2\psi_\pm^*,\psi_\pm\nabla^2\psi_\pm\ \psi_\pm^*\psi_\pm^*\quad\psi_\pm\psi_\pm$$

and represent zitterbeweging motion and pair production.

Klein-Gordon equation in hydrodynamical form (5)

For a moderately slowly varying mean-field (such as in the expansion phase of a hot system after a nuclear collision), the contributions from pair-production and zittersbewegung motion averaged over the time scale for mean-field motion are small.

Klein-Gordon equation in hydrodynamical form (6)

We shall assume that the hydrodynamical-type motion has time scale T much greater than $\frac{1}{2m}$.

In that case, we can neglect pair production and zitterbegung motion.

That is, we can ignore terms of

$$\psi_{\pm}^{*}\nabla^{2}\psi_{\pm}^{*},\ \ \psi_{\pm}\nabla^{2}\psi_{\pm},\ \ \psi_{\pm}^{*}\psi_{\pm}^{*}\ \text{and}\ \ \psi_{\pm}\psi_{\pm}.$$

We get uncoupled equations of motion for two kinds of particles

Klein-Gordon equation in hydrodynamical form (4)

We write $\psi_\pm(r,t) = \varphi_\pm(r,t)\exp\{iS_\pm(r,t)/\hbar - i\Omega_\pm(t)\}$.

It is then resonable to neglect the zittersbewegung terms and get

$$\varphi_\pm^2\left(-2\partial_t S_\pm - 4\partial_t \Omega_\pm\right)$$

$$= \frac{1}{2E}\left\{-2[\varphi_\pm \nabla^2 \varphi_\pm - 2\varphi_\pm^2 (\nabla S_\pm)^2] + [M^2(r,t) + E^2]2\varphi_\pm^2\right\}$$

Divide by $\varphi_\pm^2$ and take the gradiant ∇_i. We get the equation

$$\partial_t[E\varphi_\pm^2 \nabla_i S_\pm] + \sum_{j=1}^{3} \nabla_j \varphi_\pm^2 \nabla_i S_\pm \nabla_j S_\pm = -\frac{1}{2}\varphi_\pm^2 \nabla_i \frac{\nabla^2 \varphi_\pm}{\varphi_\pm} - \varphi_\pm^2 \nabla_i \frac{M^2(r,t)}{2}.$$

We show previously

$$-\frac{1}{2}\varphi_\pm^2 \nabla_i \frac{\nabla^2 \varphi_\pm}{\varphi_\pm} = -m\sum_j p_{ij}^{(q)}$$

where

$$p_{ij}^{(q)} = -\frac{\hbar^2}{2m}\varphi \nabla_i \nabla_j \varphi + \frac{\hbar^2}{2m}\nabla_i \varphi \nabla_j \varphi.$$

The Euler equation from Klein - Gordon equation in the no - zitter approximation is

$$\partial_t[E\varphi_\pm^2 \nabla_i S_\pm] + \sum_{j=1}^{3} \nabla_j [\varphi_\pm^2 \nabla_i S_\pm \nabla_j S_\pm + m p_{ij}^{(q)}] = -\varphi_\pm^2 \nabla_i \frac{M^2(r)}{2}$$

Many-body system in mean-field description

For a system of particles interacting in the time-dependent mean-field approximation, we describe the system by a set of single-particle states

$\psi_{\alpha\pm}(r,t)$ and occupation numbers $n_{\alpha\pm}(t)$.

Equations govering $\psi_{\alpha\pm}$ and $n_{\alpha\pm}$ are

$$\text{(I)}\quad i\hbar\partial_t\begin{pmatrix}\psi_{\alpha+}\\ \psi_{\alpha-}\end{pmatrix} = \frac{1}{2E_{\alpha\pm}}\Big\{[(-i\hbar\nabla)^2 + M^2(r,t) + [E_{\alpha\pm}{}^2 - i\hbar\partial_t E_{\alpha\pm}]\begin{pmatrix}\psi_{\alpha+}\\ \psi_{\alpha-}\end{pmatrix}$$

$$+[(-i\hbar\nabla)^2 + M^2(r,t) - [E_{\alpha\pm}{}^2 - i\hbar\partial_t E_\alpha]\begin{pmatrix}\psi^*_{\alpha-}\\ \psi^*_{\alpha-}\end{pmatrix}\Big\}$$

where $\quad M(r,t) = m + \mathbb{S}(r,t),$

$$\mathbb{S}(r,t) = \int d^3r' \sum_{\beta\pm} n_{\beta\pm}(t)\psi_{\beta\pm}(r_2,t)\ \psi^*_{\beta\pm}(r_2,t)\ v(r,r_2),$$

$$\text{(II)}\quad \frac{\partial n_1(t)}{\partial t} = \frac{\pi}{\hbar}\sum_{234}\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)[(1\pm n_1)(1\pm n_2)n_3n_4 - n_1n_2(1\pm n_3)(1\pm n_4)]|\langle 12|v'|34\rangle|^2$$

+ sign for bosons, − sign for fermion

Many-body system in mean-field description (2)

(I) We cosnider dynamical motion whose time scale is long compared to $\hbar/2m$, we can then make the no-pair and no-zitter approximation. This amounts to neglecting the $\begin{pmatrix} \psi^*_{\alpha-} \\ \psi^*_{\alpha+} \end{pmatrix}$ term in the explicit coupling between particles and antiparticles.

(II) We cosnider dynamical motion whose time scale is long compared to the relaxation of the single-particle occupation numbers, we can then make the thermal equlibrium approximation to consider $n_{\alpha\pm}$ to be given by the equalibrium occupation numbers.

Many-body system in mean-field description (2)

Then each single - particle states satifies

$$\partial_t[E_{\alpha\pm}\varphi^2_{\alpha\pm}]+\nabla\cdot[\varphi^2_{\alpha\pm}\nabla S_{\alpha\pm}]\approx 0$$

$$\partial_t[E_{\alpha\pm}\varphi^2_{\alpha\pm}\nabla_i S_{\alpha\pm}]+\sum_{j=1}^{3}\nabla_j[\varphi^2_{\alpha\pm}\nabla_i S_{\alpha\pm}\nabla_j S_{\alpha\pm}+p^{(q)}_{ij}]=-\varphi^2_{\alpha\pm}\nabla_i\frac{M^2(r,t)}{2}$$

Including the occupation numbers degrees of freedom

$$\partial_t\sum_{\alpha}n_{\alpha\pm}E_{\alpha\pm}\varphi^2_{\alpha\pm}+\nabla\cdot\sum_{\alpha}n_{\alpha\pm}\varphi^2_{\alpha\pm}\nabla S_{\alpha\pm}=0$$

$$\partial_t\sum_{\alpha}n_{\alpha\pm}E_{\alpha\pm}\varphi^2_{\alpha\pm}\nabla_i S_{\alpha\pm}+\sum_{j=1}^{3}\nabla_j\sum_{\alpha}n_{\alpha\pm}\varphi^2_{\alpha\pm}\nabla_i S_{\alpha\pm}\nabla_j S_{\alpha\pm}+p^{(q)}_{ij}]=-\sum_{\alpha}n_{\alpha\pm}\varphi^2_{\alpha\pm}\nabla_i\frac{M^2(r,t)}{2}$$

Many-body system in mean-field description (2)

The define $\varepsilon_{\pm}(r,t) = \sum_{\alpha\pm} n_{\alpha\pm}(t) m \varphi^2_{\alpha\pm}(r,t)$

$$\varepsilon_{\pm}(r,t) u_{\pm}{}^{0}(r,t) = \sum_{\alpha\pm} n_{\alpha\pm}(t) m \varphi^2_{\alpha\pm}(r,t) \frac{E_{\alpha\pm}(r,t)}{m}$$

$$\varepsilon_{\pm}(r,t) u_{\pm}{}^{i}(r,t) = \sum_{\alpha\pm} n_{\alpha\pm}(t) m \varphi^2_{\alpha\pm}(r,t) \frac{\nabla_i S_{\alpha\pm}(r,t)}{m}, \qquad i = 1,2,3$$

$\therefore$ We have

$$\sum_{\alpha\pm} n_{\alpha\pm}(t) \varphi^2_{\alpha\pm} \nabla_i S_{\alpha\pm} \nabla_j S_{\alpha\pm} = \varepsilon_{\pm}(r,t) u^i(r,t) u^j(r,t) + p^{(t)}_{ij}$$

$$p^{(t)}_{ij} = \sum_{\alpha\pm} n_{\alpha\pm}(t) \varphi^2_{\alpha\pm} (\nabla_i S_{\alpha\pm} - m u^i)(\nabla_j S_{\alpha\pm} - m u^j)$$

Equation of motion for the many-body system is

$$\frac{\partial\, \varepsilon_{\pm}\, u_{\pm}{}^{0}}{\partial t} + \sum_{j=1}^{3} \frac{\partial\, \varepsilon_{\pm}\, u_{\pm}{}^{i}}{\partial x_i} = 0$$

$$\frac{\partial\, \varepsilon_{\pm}\, u_{\pm}{}^{i}}{\partial t} + \sum_{j=1}^{3} \frac{\partial\,(\varepsilon_{\pm}\, u_{\pm}{}^{i} u_{\pm}{}^{j})}{\partial x_i} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left(p^{(q)}_{ij} + p^{(t)}_{ij}\right) = -n \nabla_i \frac{M^2(r,t)}{2}$$

Sources of pressure can arise from many sources

- Quantum stress tensor $p_{ij}^{(q)}$

- Thermal stress tensor $p_{ij}^{(t)}$, deviation of velocity fields of individual particles from the average velocity fields.

- Mean field interaction $p_{ij}^{(v)}$, that is nearly instantaneous, for a fluid element at rest.

Dirac equation can be reduced into a Klein-Gordon type equation

For a scalar external interaction, Dirac equation

$$\{i\partial - M(r)\}\psi = 0$$

Multiply by $\{i\partial + M(r)\}$, the Dirac equation becomes

$$\{(i\partial)^2 - M^2(r) - [i\partial M(r)]\}\psi = 0$$

The Dirac equation can be reduced into a Klein - Gordon equation.

with additonal terms.

Following the same procedure as for thw Klein - Gordon equation,

the Dirac equation can be reduced into two Schrödinger equations for

ψ_+ and ψ_- and can be cast in a hydrodynamical form for the particle

fluid and the antiparticle fluid.

Conclusions

Quantum mechanics and hydrodynamics have many elements in common:

- density field n(r,t)
- velocity field u(r,t)
- equation of continuity for n(r,t) and u(r,t)
- Euler-type equations for n(r,t)u(r,t)

Quantum mechanics may be an important part of the foundation for hydrodynamics of finite quantum systems with strong interactions or a boundary.

Challenges

- Relativistic viscous hydrodynamics
- Questions on the degree of thermalization
- Treatment of coupling between particles and antiparticles
- Effects of zitterbewegung on thermalization and on hydrodynamical motion