The storage ring pEDM and eEDM projects and Leapfrog tracking

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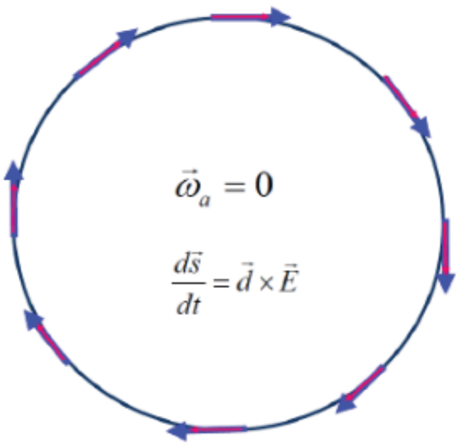
Intensity Fontier - The pEDM Proposal

A Proposal to Measure the Proton

Electric Dipole Moment with 10^{-29} e·cm

Sensitivity

by the Storage Ring EDM Collaboration



The EDM storage ring experiments

We plan a storage ring based experiment to measure with an accuracy of 10^{-29} e-cm the electric dipole moment (EDM) of the proton. In another "precursor" storage ring experiment we will measure the electric dipole moment of the electron. The eEDM ring will be smaller and much less expensive than the pEDM ring and will be constructed first.

In the pEDM we will use polarized protons at the magic momentum of 0.7 Gev/c. In the eEDM, polarized electrons at the magic momentum of 15 MeV/c. The EDM will be measured by spin polarimetry.

Why magic?

The rings will be strictly electrostatic (no magnets). At the magic momentum, in absence of magnetic fields, the spin direction will remain "frozen" in its direction at injection (longitudinal) and the EDM will be measured as proportional to a small vertical component of the spin that will gradually appear.

The magic condition

Spin dynamics is governed by the covariant Thomas-Bargman-Michel-Telegdi T-BMT equation

$$\frac{d\mathbf{s}}{dt} = -\frac{q}{m\gamma}\mathbf{f} \times \mathbf{s}, \tag{1}$$

where $\mathbf{s}$ is the real 3-dimensional spin vector of a 1/2-spin particle, and $\mathbf{f}$ is a function of the position and the momentum of the particle and of the electric and magnetic field encountered by the particle along its trajectory. In a pure electrostatic ring, *e.g.* with no magnets or RF cavities, $\mathbf{f}$ reduces to

$$\mathbf{f} = \left(a\gamma - \frac{\gamma}{\gamma^2 - 1}\right)\frac{\mathbf{E} \times \mathbf{v}}{c^2}, \tag{2}$$

with a the spin anomaly. At the magic momentum $pc = mc^2/\sqrt{a}$ it is exactly $\mathbf{f} = \mathbf{0}$ and the spin remains frozen.

Orbit/Spin Tracking in an electric ring

Simulation of an electrostatic storage ring for the pEDM is important and should be done by more than one method to compare and benchmark.

Tracking should be symplectic for stability in the long range and fast, because ring turns will be many.

There are codes using:

1. Integration of differential equations for orbit (Lorentz) and spin (Thomas-BMT) with Runge-Kutta type routines

2. Map description of machine elements or of the whole lattice

3. Symplectic integration for propagation by discrete kicks.

In this contribution we will describe a Leapfrog code of type 3

Orbit tracking by leapfrog

Consider canonical integration of the Lorentz diff. equation of motion, while for a moment we only have the electric field **E**

$$\frac{d\mathbf{p}}{dt} = e\mathbf{E} + e\mathbf{V} \times \mathbf{v} \tag{3}$$

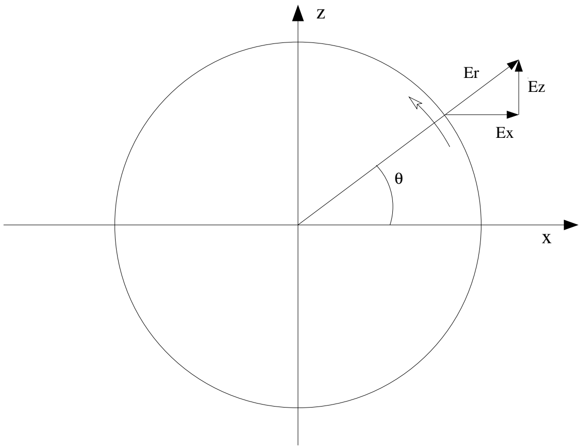
by leapfrog or Verlet[1] kicks, method introduced for accelerators in a seminal 1983 paper by Ronald Ruth[2], that is a kick integration method that interleaves drifts, were only the space coordinates are advanced, with symplectic kick bends where the momentum components are advanced. Leapfrog is an algorithm accurate to 2.nd order in time step.

Other integration algorithms, like Runge-Kutta are accurate to 4.th order in time. However they were written with *mathematical* accuracy in mind, while the 2.nd order Leapfrog is exactly symplectic, *i.e.* was written with *physical* accuracy in mind.

Making Runge-Kutta symplectic has been discussed[3], but it makes a computer code slower to run, which defies our goal of short computer time for tracking the pEDM ring.

Orbit coordinates

Use Cartesian "laboratory" coordinates (x, z, y),with $\widehat{y}$ vertical axis, and time as the independent variable. Vertical electric field component is calculated by a power expansion out of the "horizontal" x, z plane of the ring



The circular ring lattice shown is obtained by tracking a "reference particle" *i.e* at nominal energy injected tangentially.

Orbit Leapfrog formalism basics

Ménagerie of quantities for the game

$r_o[m]$	$=$	radius of curvature
a	$=$	magnetic anomaly
$\wp \equiv pc[GeV]$	$=$	$U_o/\sqrt{a}$ moment
$U_T[GeV]$	$=$	$\sqrt{(pc)^2 + U_o^2}$, total energy
γ	$=$	$U_T/mc^2, \quad \beta = \sqrt{1 - 1/\gamma^2}$
$B\rho[V \cdot s/m]$	$=$	$10^9 \wp/c$, rigidity
$eE[eV/m]$	$=$	$(= \wp/r_0)\beta c$ Electric bend field

Leapfrog tracking conserves the value of the Hamiltonian, that is being continuously recalculated during runs.

$$\mathcal{H} = \sqrt{(\wp - eA)^2 + (mc^2)^2} + e\phi. \tag{4}$$

Momentum kick are done by integration of the Lorentz equation, for an electric or magnetic bend, respectively

$$\frac{d\mathbf{p}}{dt} = e\mathbf{E}, \quad \text{or :} \quad = e\mathbf{B} \times \mathbf{v} \qquad (5)$$
$$\mathbf{E} = -\nabla\mathbf{\Phi}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

The potentials Φ and **A**, needed for the Hamiltonian, in static fields, should obey the equations

$$\nabla^2\Phi \equiv \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \qquad (6)$$
$$\nabla \cdot \mathbf{A} = 0, \quad \nabla \times \mathbf{A} = 0$$

(solutions for $\Phi(x, y, z)$ and $\mathbf{A}(x, y, z)$ will be found by power expansion.)

Will work on 4 examples (1) circular ring, (2) race-track structure, (3) 8-super-period structure with 8 bends, 8 drifts and 8 electrostatic quadrupoles, (4) simple magnetic structure (helix)

Basic leapfrog cell is a sequence

$$\boxed{\text{drift} + \text{momentum kick} + \text{drift}}$$

Momentum kick follows Lorentz equation (electric case)

$$\frac{d\mathbf{p}}{dt} = q\mathbf{E}, \ \mathbf{E} = \nabla\phi \tag{7}$$

The potential, needed for the Hamiltonian, should obey the Laplace equation

$$\nabla^2\phi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{8}$$

(an explicit solution is found by power expansion.)

The reference particle, around which the whole beam dances, is the magic particle whose spin would remain frozen in position during the propagation.

Leapfrog cell-Electric bends

Discuss what happens to a reference particle confined to the horizontal plane

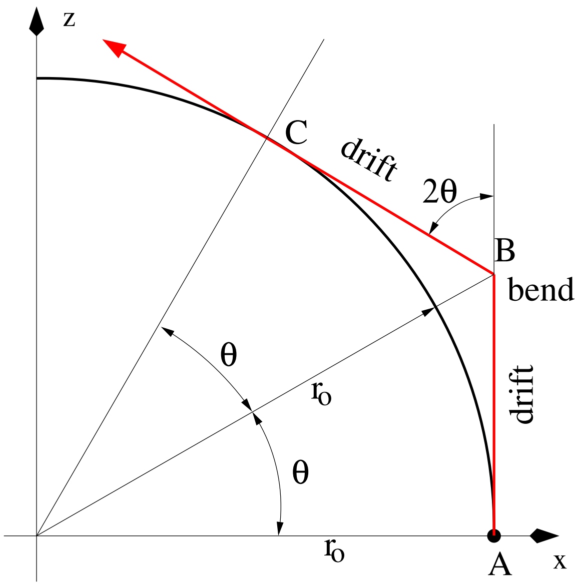


Fig.1

A →B →C,

drift, kick-bend, drift

drift A-B

Start in **A** with Initial coordinates

$$(\mathbf{A})\quad x = r_o,\ z = 0,\ \wp_x = 0,\ \wp_z = \wp.$$

Eq's for the drift, with a time step dt for the drift **A**→**B**:

$$\begin{aligned} \frac{dx}{dt} &= \frac{\wp_x}{U_o\gamma}c, & \frac{dz}{dt} &= \frac{\wp_z}{U_o\gamma}c, \text{ or} \\ x &:= x + \wp_x/(U_o\gamma)c\,dt, & z &:= z + \wp_z/(U_o\gamma)c\,dt \end{aligned} \tag{9}$$

using the identity $\wp = U_o\beta\gamma$, we obtain at the kick bend **B** the new position

$$(\mathbf{B})\quad x = r_o,\ z = \beta c\,dt,\ \wp_x = 0,\ \wp_z = \wp.$$

kick in B

In **B** a kick is imparted to the momentum **p**c, using the Lorentz Equation, with a time step δt, different from the dt of the drift.

$$\wp_x := \wp_x - eE_x c\,\delta t, \qquad \wp_z := \wp_z - eE_z c\,\delta t \tag{10}$$

For cylindrical bend the field E is purely radial, with components

$$eE_x = -eE\, r_o/r \cos\theta \qquad eE_z = eE\, r_o/r \sin\theta. \tag{11}$$

Now find the relation between dt and δt for leapfrog *i.e*:

1. Through the bend the value of the total momentum pc must be conserved

2. The trajectory in **C** should return tangent to the circle, as in the figure. Namely:

$$\text{arccos}\left[(\mathbf{p}(A)\cdot\mathbf{p}(C))/p^2\right] = 2\theta \tag{12}$$

If both conditions hold, the basic trajectory will be a polygon circumscribed to the circle. Other particles in the beam will dance around it in betatron oscillations.

For condition (1): moment conservation, combining the preceding equations

$$\wp_x = -\wp/r\cos\theta\,\beta c\,\delta t,\;\; \wp_z = \wp\,(1-(1/r)\sin\theta\,\beta c\,\delta t) \qquad (13)$$

then after kick (**C**):

$$\wp_x^2 + \wp_z^2 = (pc)^2\left[1 + ((\beta c/r)\delta t)^2 - (2/r)\sin\theta\,\beta c\,\delta t\right]. \qquad (14)$$

Since: $\cos\theta = z/r$, $\sin\theta = x/r$, taking the value of x from Eq.(9), the term in [ ] in Eq.(14) above reduces to 1 when

$$\boxed{\delta t = 2\,dt}$$

For condition (2): new angle, it is calculated from the scalar product of the momentum before and after the kick

- (**A**) before kick: $\wp_x = 0,\;\; \wp_z = \wp$
- (**C**) after kick: $\wp_x = -(\wp/r)\cos\theta\,\beta c\,\delta t,\;\; \wp_z = \wp\left(1 - 2\sin^2\theta\right)$

$$\text{angle} = \arccos\frac{\wp(A)\cdot\wp(B)}{(pc)^2} = \arccos\left(1-2\sin^2\theta\right) = \boxed{2\theta}\;\; \textit{q.e.d.}$$

Reference Trajectory

Let us produce a reference trajectory on the horizontal plane by Leapfrog tracking along a polygonal pattern tangent to a structure made of straights (drifts) and circular arcs (bends). So, The leap-frog orbit is slightly longer than the reference orbit. The more kicks we put in a bend the lesser this difference is.

In an example of a structure with 8 bends and 8 drifts of circa 270 m of total length, using 32 kicks in each bend of 36 m of radius, the difference in effective radius between the geometrical base line and the polygon is about 1 mm.

The step is much larger than the required step of a solution by integration for similar accuracy, with a very large gain in computing speed.

Tracking a reference particle will create a reference trajectory. An example is shown in the following picture.

Reference Trajectory by tracking

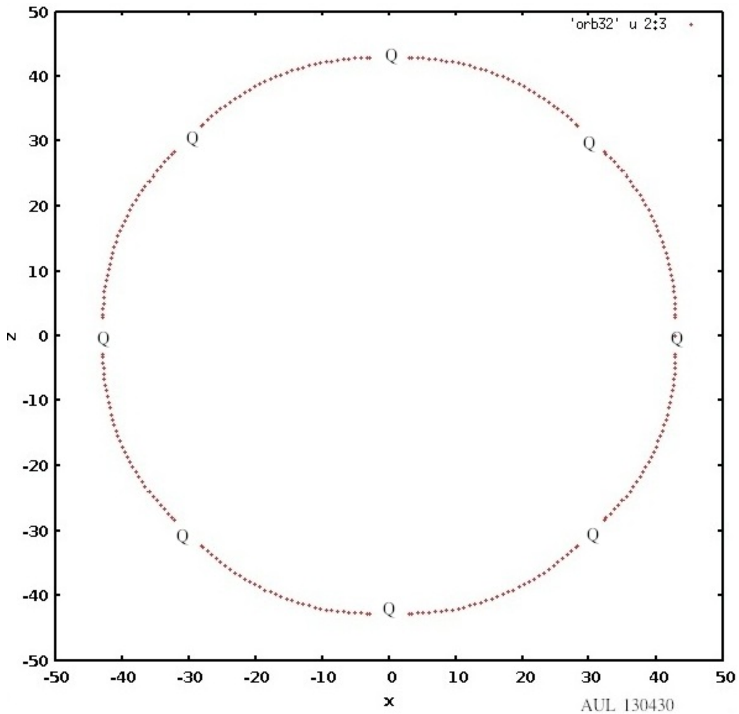


Fig.2

32 kicks per bend
bend length=28.276 m
drift length 2×2.83 m
intra bend drift length = 0.44 m
nominal curvature radius = 36 m
$E_{cyl} = -1.164745510^{7} V/m$

Evaluation of the electric field

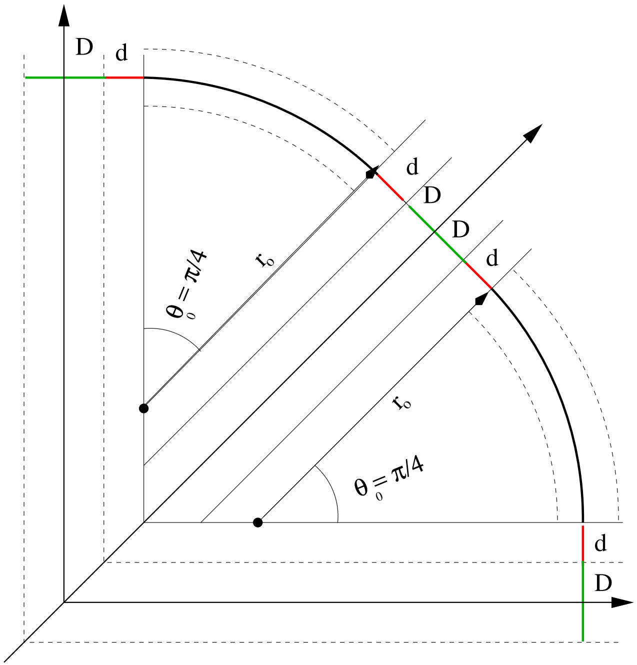


Fig.3

In a general lattice the center of curvature for the calculation of the electric field continuously changes and has to be re-evaluated every time

The sketch (for the preceding lattice) suggests how

'D' is any added drift space
'd' is a leapfrog inner-bend drift

General tracking

The Leapfrog formalism extends to 3 dimensions and applies unchanged to particles that don't have a magic energy or are injected in the lattice on a finite transverse emittance.

Eqs.(9) and .(10) in 3 dimensions are

$$\left\{\begin{array}{l} x := x + \wp_x/(U_o\gamma)c\,dt \\ y := y + \wp_y/(U_o\gamma)c\,dt \\ z := z + \wp_z/(U_o\gamma)c\,dt \end{array}\right. , \quad \begin{array}{l} \wp_x := \wp_x - eE_x 2c\,dt \\ \wp_y := \wp_y - eE_y 2c\,dt \\ \wp_z := \wp_z - eE_z 2c\,dt \end{array} . \qquad (15)$$

However, In a general case the leapfrog conditions (1) for momentum and angle are not fully satisfied in a bend because, due to transverse oscillations, the particle sees a tangential component of the electric field that modulates the energy.

During tracking the Hamiltonian is continuously calculated. It conserves its initial valiue.

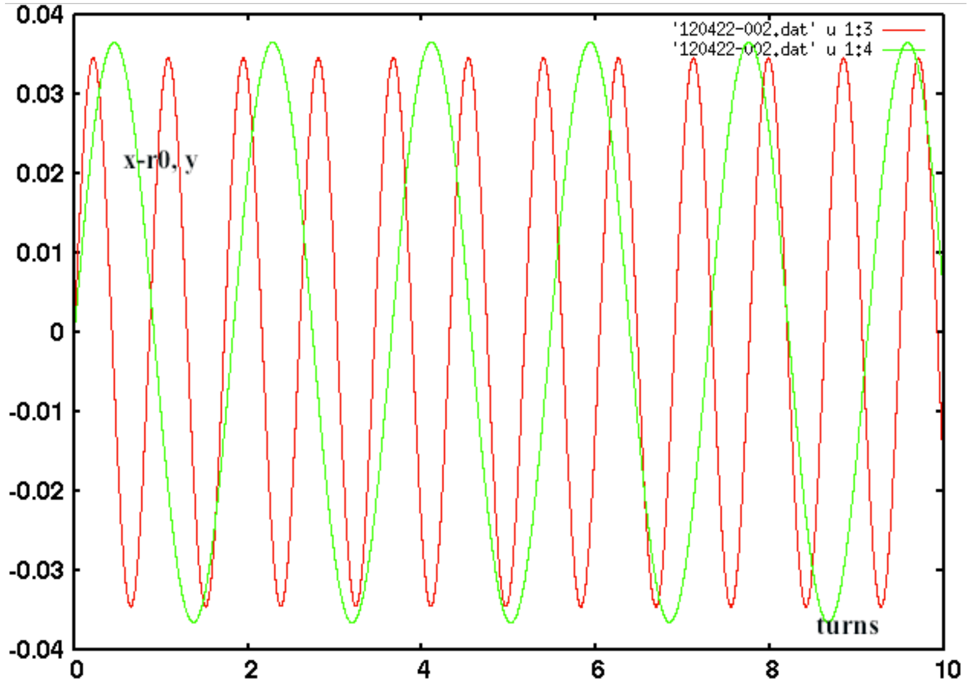


Fig.4 x,y betatron oscillations vs. turn #

Add a RF - Example of RF bucket

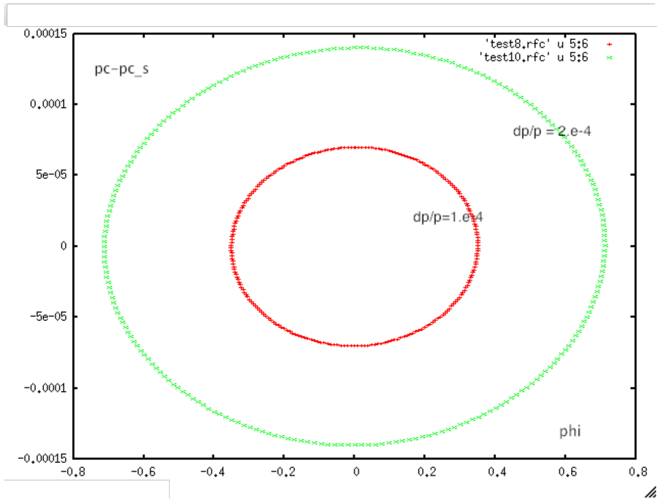


Fig.5 - Phase space of $\Delta \times pc$ for two particles, with $dp/p = 1.e^{-4}$ and $2.e^{-4}$, with $V_{RF} = 1000V$ and $h = 24$. Number of turns for a complete oscillations is 335, corresponding synchrotron frequency $\nu_s = 0.002985$ oscillations per turn

Leapfrog - Magnetic bends for kick in B

Still use Lorentz equation. In a magnetic bend in the simple case of a uniform magnetic field parallel to $\hat{y}$, of value *i.e.* $B \equiv B_y$.

It is (see the *ménagerie.*) : $\boxed{eB = \dfrac{\wp}{r_0 c}}$

The momentum kick equations are in (**B**)

$$\begin{cases} \dfrac{dp_x}{dt} = eB\, v_z & \dfrac{dp_z}{dt} = -eB\, v_x \\ \wp_x = \wp \dfrac{v_z}{r}\delta t, & \wp_z = \wp\left(1 - \dfrac{v_x}{r}\delta t\right) \end{cases} . \tag{16}$$

The components of the velocity in (**B**) are

$$v_x = -v \sin\theta, \quad v_z = v \cos\theta. \tag{17}$$

Comparing (17) with Eq.(11) of the electric case, note the anti-symmetric role of sine and cosine, because the electric bend field is radial, while the velocity is tangential.

At the very beginning of tracking, at **B**, after the drift, it is

$$v_x = 0, \quad v_z = v, \tag{18}$$

so, according to (16) only p_x would receive a kick, which is paradoxical -but a toll one pays using kick formalism- because a magnetic force cannot change the total momentum. After the kick we may therefore write

$$\wp_x = \wp \frac{v}{r_0} \delta t, \quad \wp_z = \wp \sqrt{1 - \frac{\wp_x^2}{\wp^2}} \tag{19}$$

For a magnetic kick the **1.st leapfrog condition** should be satisfied for free by default. The **2.nd leapfrog condition** for the angle still calls for a relation between dt, the time step for a drift, and δt, the time step for a bend.

We see from Fig.**??**, that at the beginning, the polygonal condition for the kick bend angle requires:

$$\frac{\wp_x}{\wp_z} = \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}. \tag{20}$$

It is (see Fig.6)

$$\tan\theta = \frac{v\,dt}{r_0} = \frac{\ell}{r_0}, \tag{21}$$

with ℓ the drift length.

For subsequent points and in general, as shown from Fig.**??**, Eq.(20 should be replaced by

$$\frac{\wp_r}{\wp_\theta} = \frac{2\tan\theta}{1-\tan^2\theta}. \tag{22}$$

where $\wp_r$ and $\wp_\theta$ are the radial and tangential component of the momentum after the kick, respectively

$$\wp_r = \wp_x\cos\theta + \wp_z\sin\theta, \quad \wp_\theta = -\wp_x\sin\theta + \wp_z\cos\theta \tag{23}$$

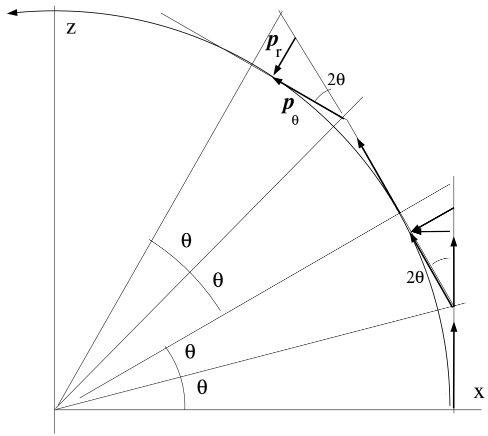


Fig.6 - Geometry for condition leapfrog2

with some algebra, find the relation between δt and dt that sets the angle condition for a magnetic bend

$$\boxed{\delta t = \frac{2\,dt}{1 + (\ell/r_0)^2}}, \tag{24}$$

very close to what found for the electric bend, Eq.(**??**).

For the magnetic case it is immediate to see that, directly using Eqs.(16) the **1.st leapfrog condition**, *i.e* the conservation of the momentum in a kick is NOT satisfied, differently than for an electric kick, Eq.(**??**). We have to resort to combine the two 1.st order ODE's equations into two 2.nd order Eqs.

Let us re-write Eqs.(16) in the more compact form

$$\frac{dp_x}{dt} = -\omega\, p_z, \quad \frac{dp_z}{dt} = \omega\, p_x, \quad \text{with} \quad \omega = \frac{v}{r_0}. \tag{25}$$

Differentiate both, obtain two formally identical 2.nd order ODE's

$$\frac{d^2p_x}{dt^2} + \omega^2 p_x, \quad \frac{d^2p_z}{dt^2} + \omega^2 p_z \tag{26}$$

with integral

$$p_{x,z} = A_{x,z}\sin(\omega\delta t) + B_{x,z}\cos(\omega\delta t), \tag{27}$$

or

$$\left\{ \begin{array}{lcl} p_{x,n+1} & := & p_{x,n}\cos(\omega\delta t) - p_{z,n}\sin(\omega\delta t) \\ p_{z,n+1} & := & p_{x,n}\sin(\omega\delta t) + p_{z,n}\cos(\omega\delta t) \end{array} \right. . \tag{28}$$

it is immediate to verify that

$$p^2_{x,n+1} + p^2_{z,n+1} = p^2_{x,n} + p^2_{z.n} = p^2. \tag{29}$$

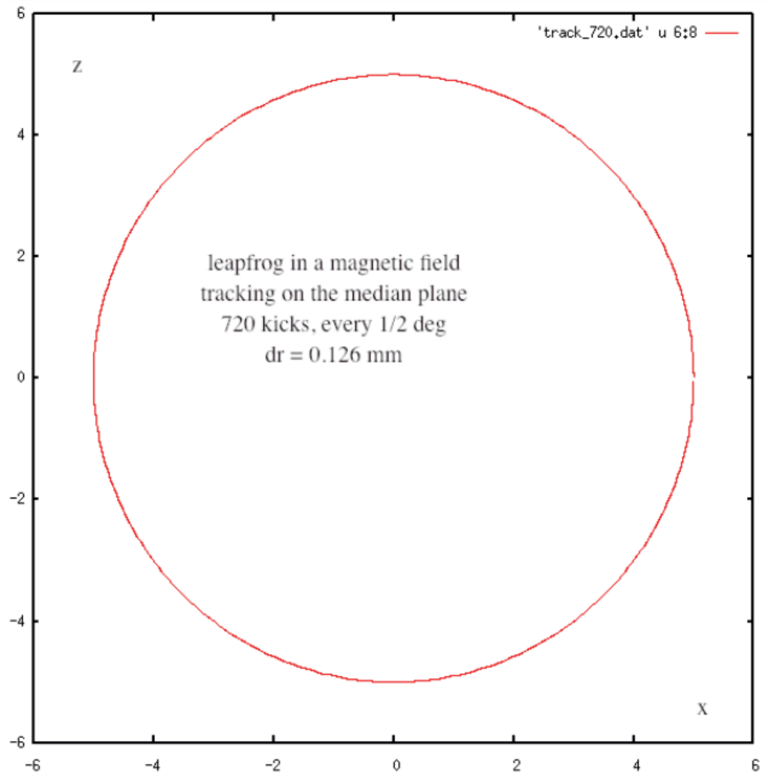


Fig.6 - Circular trajectory on the $x - z$ plane

Fig.6 shows the leapfrog trajectory in the $x - z$ plane. Parameters, averages and values of various quantities for this track example are given in the following table

particle	=	proton
$nsub$	=	720 0.5deg per bend
r_o[m]	=	5
$\wp_0 = pc_0$[GeV]	=	0.70074037
$< pc >$	=	$0.70074037 \pm 7.52471755\, 10^{-8}$
β, γ	=	0.59837911, 1.24810739
$B[V.s/m^2]$	=	0.46748365
$<$ drift length $>$	=	$4.36343389\, 10^{-2} \pm 3.2927225\, 10^{-9}$
$<$ angle $>$	=	$5.06268607\, 10^{-5}$
dr [mm]	=	0.12692779

Magnetic example: 10-turn Helix

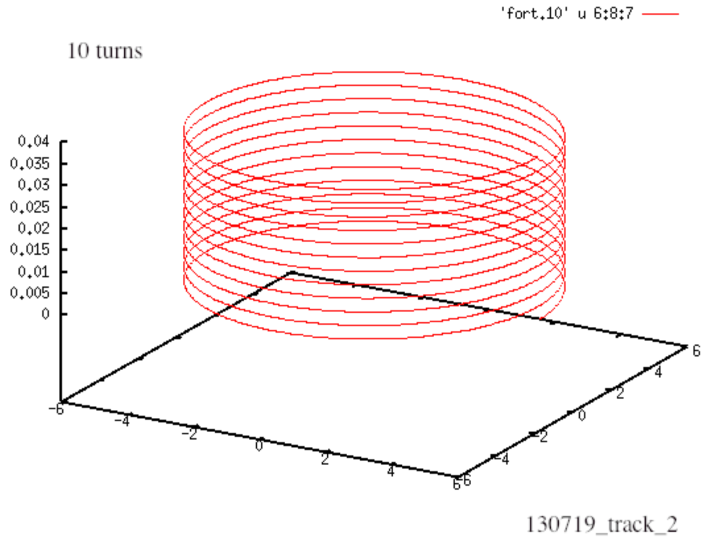


Fig.7 - 10-turn Helix created with Leapfrog

Briefly on Spin Dynamics: EDM

Spin kicks, applied at each Bend and Quad, follow the leapfrog pattern of the orbit.

At the magic energy it is $\mathbf{F} = \mathbf{0}$ and the spin remains frozen If the proton has an EDM, the spin is Not completely frozen: in the rest frame of the particle, the electric field appears as a magnetic field **B'** $\perp$ to **E** and another small term is added to **f** in Eq.(2)

$$\mathbf{B}' = -\gamma\vec{\beta} \times \mathbf{E}. \quad \mathbf{f} := \mathbf{f} + \eta\mathbf{B}' \times \mathbf{v}. \tag{30}$$

The spin will make a precession around this magnetic field and a spin vertical component will appear, that can be measured. For a magic proton this is the only non vanishing additional spin component.

Spin dynamics of a frozen spin

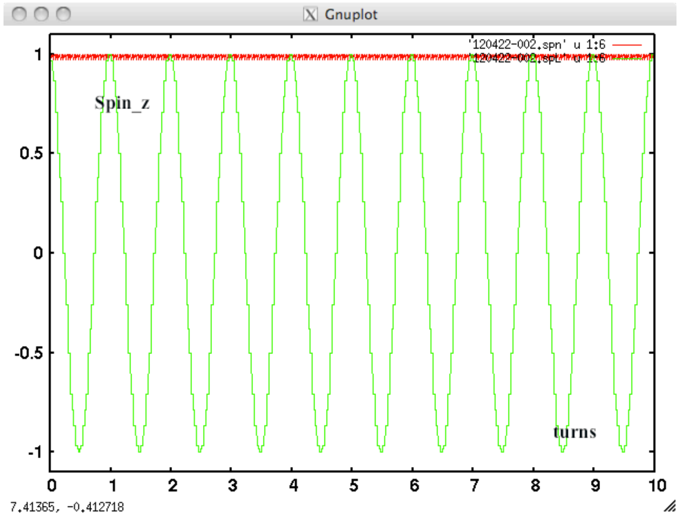


Fig.8 - Longit. component of the frozen spin: red line in accelerator coordinates, green line, in laboratory coordinates. The red line shows little wiggles because the responsible proton is on purpose not perfectly magic and there are betatron oscillations.

References

1 L.Verlet *Computer experiments in classical fluids*
Phys.Rev. **159** 98-103
2 R.D.Ruth *A Canonical Integration Technique*
IEEE Trans. on Nuclear Sciences **NS-3**, No.4, August 1983
3 J.M.Sanz-Serna
Symplectic integrators for Hamiltonian problems: an overview
Acta Numerica (1991), 243-286 QA 297,A21 1992

Thank You !