The gradient expansion: a relativistic framework for non-linear cosmology

(and its relation to Lagrangian perturbation theory)

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Sep 5, 2012

CERN workshop: Theoretical methods for non-linear cosmology

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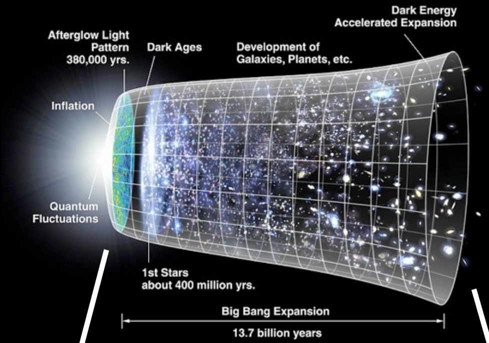
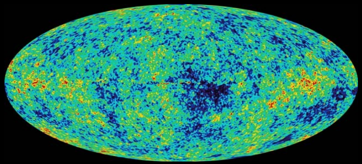
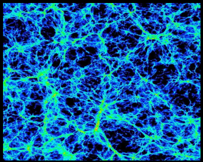




non-linear
evolution



$\Delta T = \frac{T}{T_0} \leq \mathcal{O}(10^{-5})$, with $T_0 = 2.725$K

mass density

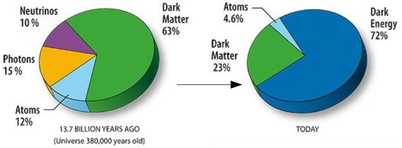
$$\rho(\boldsymbol{x},t) \equiv \overline{\rho}(t)\,[1+\delta(\boldsymbol{x},t)]$$

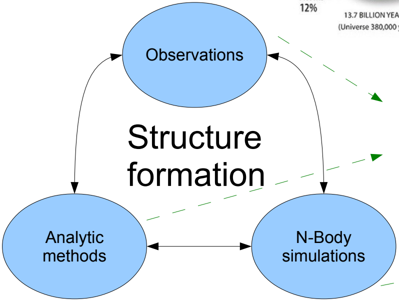
separation of global effects from local effects: mean density

density contrast

Mission:

Verify/disprove ΛCDM





e.g.:

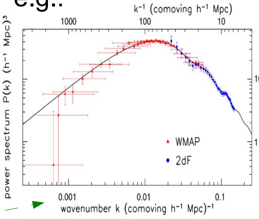


Fig 8.17 (A. Sanchez) 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

$$P(k) \sim \langle \tilde{\delta}^2(k) \rangle$$

The measurement of $P(k)$, etc. lead to tight constraints to ΛCDM

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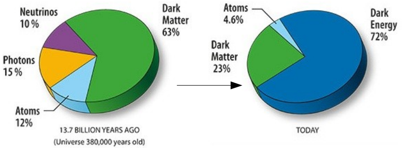

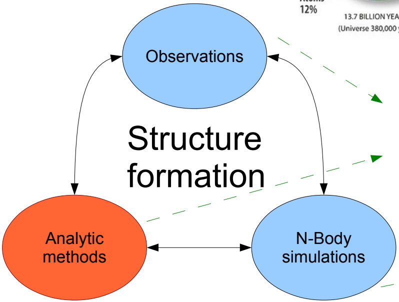

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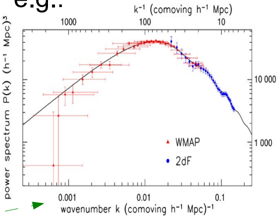


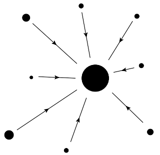
Fig 8.17 (A. Sanchez) 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

- Relativistic corrections (scale, density, velocity)?
- Purely relativistic effects (backreaction, light propagation, . . .)

% effect**?**

The gradient expansion for ΛCDM:

A relativistic approximation that can follow the non-linear evolution for generic initial conditions



- Builds up a series solution in numbers of gradients (${}^{(3)}R_{ij}$) around an initial seed metric k_{ij} for the $3-$metric γ_{ij},
  $\mathrm{d}s^2 = -\mathrm{d}t^2 + \gamma_{ij}\mathrm{d}q^i\mathrm{d}q^j$. $\quad i, j, \ldots = 1, 2, 3$
- ${}^{(3)}R_{ij} \doteq \partial_i \Gamma_j + \Gamma^2$ with the Christoffel symbol $\Gamma_j \doteq \partial_j \gamma$, contains single^2 and double spatial derivatives of the metric γ_{ij}. The latter are important for the density evolution.
  $\rightarrow$ The series is basically in powers of double spatial gradients.
- The series holds for $\frac{1}{a}\partial_k \gamma_{ij} \ll \partial_t \gamma_{ij}$ [Comer et al., *PRD* **D49** (1994) 2759]
  (valid on scales beyond causal contact. However, for smooth enough initial data it can be also extended into the non-linear regime).

A formalism for the gradient expansion

The gradient expansion can be directly applied to the Einstein equations.

[Comer, Deruelle, Langlois, Parry (1994); Matarrese, Kolb, Riotto (2006), ...]

$\boxed{\text{We choose a Hamilton Jacobi Theory, constructed from the action of gravity.}}$

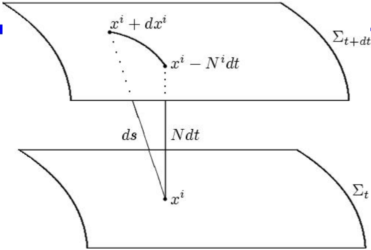
[Salopek et al., *PRD* **49** (1994) 2872, and *MNRAS* **271** (1994) 1005, ...]

next slides involve:

- summarise the set up for the Hamilton Jacobi equation (HJE)
- show how recursive/approximate solutions can be obtained
- from comoving coordinates to Newtonian coordinates

(we shall demand summation over repeated indices)

Set up for the HJE



To obtain the HJE:

1.) Start with the action of gravity with a pressureless ΛCDM component: $\mathcal{S}$.

2.) Use ADM formalism:

- (1+3) split $\to N$, N^i, and metric splits into timelike and spacelike components $h_{\mu\nu}$.
- Choose the velocity potential of the fluid to define the time hypersurfaces. $\to \; \mathrm{d}s^2 = -\mathrm{d}t^2 + \gamma_{ij}\mathrm{d}q^i\mathrm{d}q^j$
- Treat $h_{\mu\nu}$, N, and N^i as conjugate variables of an action principle $\to$ re-express the Lagrangian.
- Conjugate momenta to N and N^i vanish. Treat N and N^i as Lagrange multipliers and vary the action w.r.t. them. $\to$ constrained Hamiltonian H.

4.) The HJE is then: $\boxed{\frac{\partial S}{\partial t} + H = 0.}$

Solutions to the HJE

The evolution equations are then: $8\pi G = 1$

$$\frac{\partial \mathcal{S}}{\partial t} + \int \mathrm{d}^3 q \left[\frac{2}{\sqrt{\gamma}} \frac{\delta \mathcal{S}}{\delta \gamma_{ij}} \frac{\delta \mathcal{S}}{\delta \gamma_{kl}} \left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl} \right) - \frac{\sqrt{\gamma}}{2} (R - 2\Lambda) \right] = 0 \,, \quad (1)$$

$$\frac{\partial \gamma_{ij}}{\partial t} = \frac{2}{\sqrt{\gamma}} \frac{\delta \mathcal{S}}{\delta \gamma_{kl}} \left(2\gamma_{ik}\gamma_{jl} - \gamma_{ij}\gamma_{kl} \right) \,. \quad (2)$$

To solve the above we impose: $\boxed{\mathcal{S} = \mathcal{S}^{(0)} + \mathcal{S}^{(2)} + \mathcal{S}^{(4)} + \ldots}$, with

$$\mathcal{S}^{(0)} = -2 \int \mathrm{d}^3 q \sqrt{\gamma} H(t) \,,$$

$$\mathcal{S}^{(2)} = \int \mathrm{d}^3 q \sqrt{\gamma} J(t) R \,,$$

$$\mathcal{S}^{(4)} = \int \mathrm{d}^3 q \sqrt{\gamma} \left[L_1(t) R^2 + L_2(t) R^{ij} R_{ij} \right]$$

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- plug $\mathcal{S}$ into eq. (1). This gives:

$$\frac{\partial H}{\partial t} + \frac{3}{2} H^2 - \frac{\Lambda}{2} = 0 \,,$$

$$\frac{\partial J}{\partial t} + J H - \frac{1}{2} = 0 \,, \quad \text{etc.}$$

$\rightarrow$ leads to the time evolution factors H, J ...

- put $\mathcal{S}$ into eq. (2): $\rightsquigarrow \frac{\partial \gamma_{ij}}{\partial t}$

The time evolution of the metric

Then we obtain:

$$\frac{\partial \gamma_{ij}}{\partial t} = 2H\gamma_{ij} + J\left(R\gamma_{ij} - 4R_{ij}\right) + L_1\left(3\gamma_{ij}R^2 - 8RR_{ij} + 8R_{;ij}\right)$$
$$+ L_2\Big(16R_{ik}R^k{}_j + 4R_{;ij} - 12RR_{ij} - 4\Box R_{ij}$$
$$+ \gamma_{ij}\left[\Box R - 5R^{km}R_{km} + 4R^2\right]\Big) + \ldots$$

a semicolon denotes a covariant derivative with respect to γ_{ij}, and $\Box X = X^{;m}{}_{;m}$

Exact, but impossible to solve exactly, since it contains an infinite number of terms, and $R_{ij} = R_{ij}(\gamma_{kl})$.

Instead, solve the equation recursively in terms of an initial seed metric k_{ij}.

The recursive procedure: at zero gradients

$$\frac{\partial \gamma_{ij}}{\partial t} = 2H\gamma_{ij} + J\left(R\gamma_{ij} - 4R_{ij}\right) + L_1\left(3\gamma_{ij}R^2 - 8RR_{ij} + 8R_{;ij}\right)$$
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$$+ \gamma_{ij}\left[\Box R - 5R^{km}R_{km} + 4R^2\right]\Big) + \ldots$$

Solve iteratively, at zero gradients:

$$\frac{\partial \gamma_{ij}^{(0)}}{\partial t} = 2H\gamma_{ij}^{(0)} \quad \Rightarrow \quad \boxed{\gamma_{ij}^{(0)} = A^2(t)k_{ij}}$$

E.g., in case of Einstein-de Sitter: $H = \frac{2}{3t}$, *and* $\gamma_{ij}^{(0)} = a^2(t)k_{ij}$

$\rightarrow$ $\boxed{\text{This is the seperate universe approximation.}}$

The recursive procedure: at second order in gradients

$$\frac{\partial \gamma_{ij}}{\partial t} = 2H\gamma_{ij} + J\left(R\gamma_{ij} - 4R_{ij}\right) + L_1\left(3\gamma_{ij}R^2 - 8RR_{ij} + 8R_{;ij}\right)$$
$$+ L_2\Big(16R_{ik}R^k{}_j + 4R_{;ij} - 12RR_{ij} - 4\Box R_{ij}$$
$$+ \gamma_{ij}\left[\Box R - 5R^{km}R_{km} + 4R^2\right]\Big) + \dots$$

Solve iteratively, at two gradients:

$$\frac{\partial \gamma^{(2)}_{ij}}{\partial t} = 2H\gamma^{(2)}_{ij} + J\left(R^{(0)}\gamma^{(0)}_{ij} - 4R^{(0)}_{ij}\right) \qquad R^{(0)}_{ij} \equiv R_{ij}(\gamma^{(0)}_{kl})$$

$$\Rightarrow \boxed{\gamma^{(2)}_{ij} = A^2(t)k_{ij} + \lambda(t)\left(\hat{R}k_{ij} - 4\hat{R}_{ij}\right)} \qquad \hat{R}_{ij} \equiv R_{ij}(k_{kl})$$

E.g., in case of Einstein-de Sitter: $\gamma^{(2)}_{ij} = a^2(t)k_{ij} + \frac{9}{20}a^3(t)t_0^2\left(\hat{R}k_{ij} - 4\hat{R}_{ij}\right)$

The recursive procedure: at fourth order in gradients

$$\frac{\partial \gamma_{ij}}{\partial t} = 2H\gamma_{ij} + J\left(R\gamma_{ij} - 4R_{ij}\right) + L_1\left(3\gamma_{ij}R^2 - 8RR_{ij} + 8R_{;ij}\right)$$
$$+ L_2\Big(16R_{ik}R^k{}_j + 4R_{;ij} - 12RR_{ij} - 4\Box R_{ij}$$
$$+ \gamma_{ij}\left[\Box R - 5R^{km}R_{km} + 4R^2\right]\Big) + \ldots$$

At four gradients:

$$\frac{\partial \gamma_{ij}^{(4)}}{\partial t} = 2H\gamma_{ij}^{(4)} + J\left(\hat{R}k_{ij} - 4\hat{R}_{ij}\right)$$
$$+ C_1\hat{R}^2 k_{ij} + C_2\hat{R}^{km}\hat{R}_{km}k_{ij} + C_3\hat{R}\hat{R}_{ij} + C_4\hat{R}_{ik}\hat{R}^k{}_j$$
$$+ D_1\Box\hat{R}k_{ij} + D_2\hat{R}_{|ij} + D_3\Box\hat{R}_{ij}\,,$$

covariant derivatives $|$ and $\Box$ are w.r.t. k_{ij}

E.g.: $C_1 = 8\frac{\lambda J}{A^2} - \frac{23}{4}\frac{L}{A^2}$, with $L = \frac{4}{3}L_1 = -\frac{1}{2}L_2$.

$\Rightarrow$ $\boxed{\gamma_{ij}^{(4)} = \ldots, \text{ see next slide}}$

The final metric up to four gradients (for simplicity: $\Lambda = 0$)

$$\boxed{\begin{aligned}\gamma_{ij} \simeq a^2 k_{ij} + \frac{9}{20} a^3 t_0^2 \left(\hat{R} k_{ij} - 4\hat{R}_{ij}\right) + \frac{81}{350} a^4 t_0^4 \Bigg[& \left(-4\hat{R}^{km}\hat{R}_{km} + \frac{5}{8}\Box\hat{R} + \frac{89}{32}\hat{R}^2\right) k_{ij} \\ & - 10\hat{R}\hat{R}_{ij} + 17\hat{R}_{ik}\hat{R}^k{}_j - \frac{5}{2}\Box\hat{R}_{ij} + \frac{5}{8}\hat{R}_{|ij}\Bigg] .\end{aligned}}$$

This is valid up to

$$t_{\mathsf{con}} \sim \mathcal{O}(1-3)\frac{1}{t_0^2 \hat{R}^{3/2}} .$$

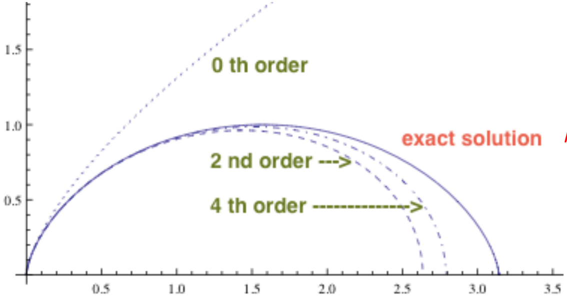
Corresponds roughly to the collapse time of regions with curvature $\hat{R}$.

Performance? Example: spherical collapse of a dust ball in a FRW universe

$$\mathrm{d}\boldsymbol{x}^2 = a^2(\tau)\left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\mathrm{d}\Omega^2\right], \qquad \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\frac{f_0}{a^3} - \frac{\kappa}{a^2}$$

$$\Rightarrow a(u) = \frac{1}{2}(1-\cos u), \quad \sqrt{|\kappa|}\tau(u) = \frac{1}{2}(u-\sin u)$$

$(8\pi G = 3,\ f_0 = \kappa)$



Parametric plot taken from [Enqvist et al., *JCAP* **1203** (2012) 026]

Gradient expansion (for positive and negative curvature):

$$a(\tau) = \left(\frac{\tau}{\tau_0}\right)^{2/3}\sqrt{1 \mp \frac{9}{10}|\kappa|\tau_0^2\left(\frac{\tau}{\tau_0}\right)^{2/3} + \frac{81}{350}\frac{1}{8}\kappa^2\tau_0^4\left(\frac{\tau}{\tau_0}\right)^{4/3} + \ldots}$$

The gradient metric for our universe, formally valid up to $\frac{t_{\rm con}}{t_0} \simeq 3.4 \frac{H_0^3}{(\nabla^2\Phi)^{3/2}}$

Assuming standard inflationary conditions, the initial metric is

$$k_{ij} = \delta_{ij} \left[1 + \frac{10}{3} \Phi(t_0, \boldsymbol{q}) \right] ,$$

Φ is the primordial Newtonian potential

and the metric takes the form

$$\boxed{\gamma_{ij}(t, \boldsymbol{q}) = a^2(t) \left[\delta_{ij} + 3a(t)\, t_0^2\, \Phi_{,ij} + a^2(t)\, t_0^4\, \hat{C}_{ij}(\boldsymbol{q}) + \hat{D}_{ij}(t, \boldsymbol{q}) + \mathcal{O}(\Phi^3) \right] ,}$$

“$, i$” denotes a differentiation with respect to q_i

with

$$\hat{C}_{ij} = \frac{9}{28} \left[19\Phi_{,il}\Phi_{,lj} - 12\Phi_{,ij}\Phi_{,ll} + 3\delta_{ij} \left\{ \Phi_{,ll}\Phi_{,mm} - \Phi_{,lm}\Phi_{,lm} \right\} \right] ,$$

$$\left[\hat{D}_{ij} = \delta_{ij} \left(\frac{10}{3}\Phi + \frac{5}{2} a(t) t_0^2 \Phi_{,l}\Phi_{,l} \right) - 15 a(t) t_0^2 \Phi_{,i}\Phi_{,j} - 10 a(t) t_0^2 \Phi\Phi_{,ij} \, . \right]$$

This term is neglected in the current literature. It will affect the latter coordinate transformation.

$\to$ (numerical) treatment of

- backreaction, e.g. the evolution of the domain averaged scale factor [Matarrese, Kolb, Riotto (2006); [Enqvist, Hotchkiss, Rigopoulos (2012)]

$$a_D^3(t) = \int_D \mathrm{d}^3 q \sqrt{\gamma}\,, \quad 3\frac{\ddot{a}_D}{a_D} = -4\pi G \langle\rho\rangle_D + Q_D\,, \qquad \gamma \equiv \det[\gamma_{ij}]$$

$$Q_D = \frac{1}{4}\left\langle \left(\gamma^{ij}\dot{\gamma}_{ij}\right)^2 \right\rangle_D - \frac{1}{6}\left\langle \gamma^{ij}\dot{\gamma}_{ij}\right\rangle_D^2 - \frac{1}{4}\left\langle \gamma^{ki}\dot{\gamma}_{ij}\gamma^{lj}\dot{\gamma}_{kl}\right\rangle_D,$$

- light propagation (geodesic equation),

- f_{NL}?

- evolution of density inhomogeneities:

$$\delta(t,\boldsymbol{q}) = (1+\delta_0)\sqrt{\frac{\gamma^0}{\tilde{\gamma}}} - 1\,. \qquad \tilde{\gamma}_{ij} = \gamma_{ij}/a^2, \;\; \gamma_{ij}^0 = k_{ij}$$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So what does that mean?

- The above coordinate transformation brings spacetime to an almost FRW form.
- To leading order we obtain the Newtonian metric and the Zel'dovich displacement field.
- (Note: due to the transformation we have lost the non-perturbative power of the formalism.)

So what does that mean?

- The above coordinate transformation brings spacetime to an almost FRW form.
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- (Note: due to the transformation we have lost the non-perturbative power of the formalism.)

Next steps to do:

- Calculate the gradient metric up to third order, i.e., six spatial gradients. [CR,Rigopoulos; in progress]
- Then, repeat the coordinate transformation up to the very order.