Casimir effect as an explanation of dark energy

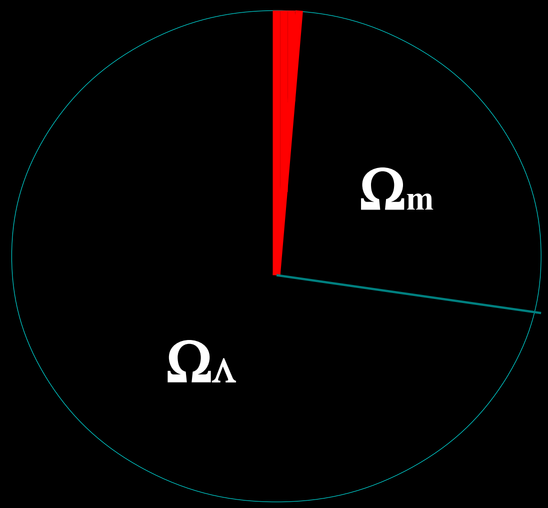
Jiro Matsumoto
Nagoya University

Introduction

Accelerated expansion of the Universe

→ $\Omega_m + \Omega_r + \Omega_K \neq 1$. Ω_Λ is needed.

Energy Fraction of the Universe



Cosmological constant problem

$\rho_\Lambda \sim (10^{-3}eV)^4$

Zero point energy of the quantum field

$\rho_0 \sim m^4$

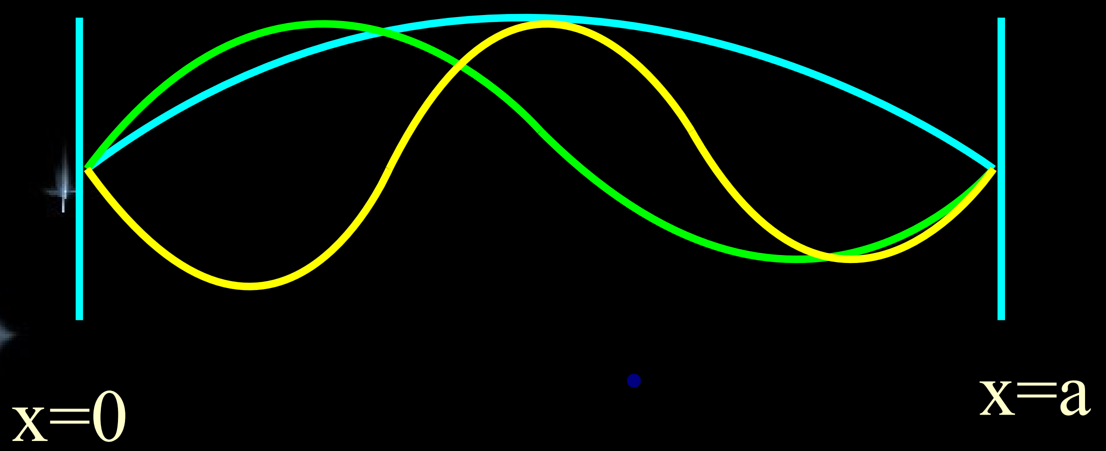
The Casimir effect

The Casimir effect

The Casimir energy is the zero point energy caused by boundary conditions.

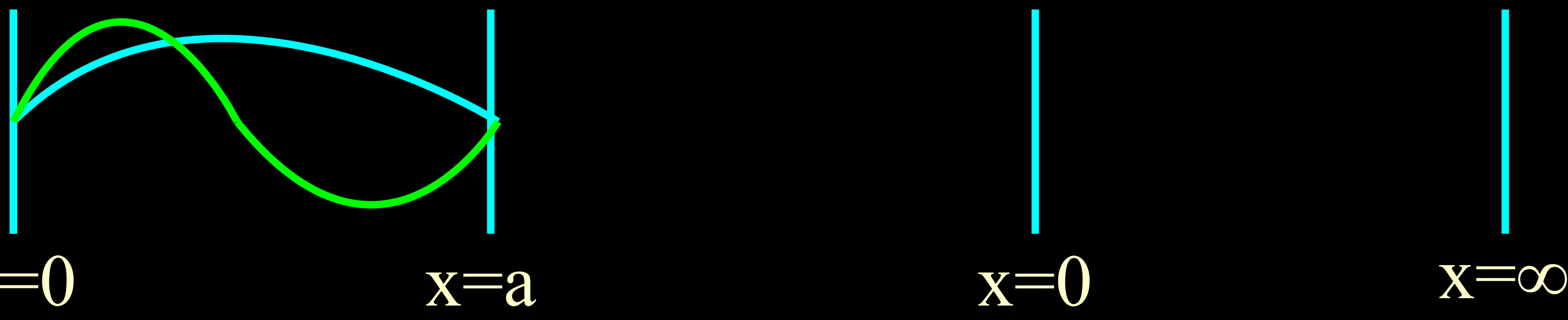
Equation of Motion $$\frac{1}{c^2}\frac{\partial^2\phi(t,x)}{\partial t^2}-\frac{\partial^2\phi(t,x)}{\partial x^2}+\frac{m^2c^2}{\hbar^2}\phi(t,x)=0.$$

Dirichlet boundary conditions: $\phi(0)=0$, $\phi(a)=0$.



$$\phi_n^{(\pm)}(t,x)=\left(\frac{c}{a\omega_n}\right)^{\frac{1}{2}}\mathrm{e}^{\pm i\omega_n t}\sin k_n x. \qquad \omega_n=\sqrt{\frac{m^2c^4}{\hbar^2}+c^2k_n^2}, \quad \boxed{k_n=\frac{\pi n}{a}}, \quad n\in\mathbb{N}^+$$

The Casimir energy of the free scalar field



$$k_n = \frac{\pi n}{a}, \quad n \in \mathbb{N}^+$$

$$\omega_n = \sqrt{\frac{m^2c^4}{\hbar^2} + c^2k_n^2},$$

$$E_0(a,m) = \frac{\hbar}{2}\sum_{n=1}^{\infty}\omega_n.$$

Compare ⟺

k: continue valued

$$\omega_k = \sqrt{\frac{m^2c^4}{\hbar^2} + c^2k^2},$$

$$E_{0M}(m) = \frac{\hbar}{2}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\omega_k L.$$

The Casimir Energy ≡
Zeropoint Energy under the Boundary Conditions
- Zeropoint Energy under the ``No'' Boundary Conditions

$E \equiv E_0(a,m) - E_{0M}(m)$

The Abel-Plana Formula

$$\sum_{n=0}^{\infty} F(n) - \int_0^{\infty} F(t)dt = \frac{1}{2}F(0) + i\int_0^{\infty} \frac{dt}{e^{2\pi t}-1}\left[F(it) - F(-it)\right].$$

$$E_0(a,m) = \frac{\hbar}{2}\sum_{n=1}^{\infty}\omega_n. \quad E_{0M}(m) = \frac{\hbar}{2}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\omega_k L.$$

$$E(a,m) = -\frac{mc^2}{4} - \frac{\hbar c}{4\pi a}\int_{\frac{2mca}{\hbar}}^{\infty}\frac{\sqrt{y^2 - \frac{4m^2c^2a^2}{\hbar^2}}}{e^y - 1}dy.$$

Massless

$$E(a,0) = -\frac{\hbar c}{4\pi a}\zeta(2) = -\frac{\pi\hbar c}{24a}.$$

Massive

$$E(a,m) \approx -\frac{mc^2}{4} - \frac{\hbar c\sqrt{mca/\hbar}}{4\sqrt{\pi}a}e^{-\frac{2mca}{\hbar}}.$$

About this work

$R^3 \times (S^1)^n \times R$ space-time

$H_0 \sim 10^{-33}\text{eV} << 10^{-3}\text{eV} \sim (\rho_\Lambda)^{1/4}$

The finite temperature Casimir effect of the graviton and fermions

The Casimir energy density

Massive

$$\rho_0 = N\left(\frac{m}{2\pi}\right)^{\frac{4+n}{2}} \sum_{\substack{(l_4,l_5,\cdots,l_{n+3})\in\mathbb{Z}^n,\\ (l_4,l_5,\cdots,l_{n+3})\neq 0}} \frac{K_{\frac{4+n}{2}}\left[md\sqrt{\sum_{i=4}^{3+n} l_i^2}\right]}{d^{\frac{4+n}{2}}\sqrt{\sum_{i=4}^{3+n} l_i^2}^{\frac{4+n}{2}}}.$$

$$K_n(z) = \frac{\left(\frac{z}{2}\right)^n \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)} \int_1^{\infty} \mathrm{e}^{-zt}(t^2-1)^{\frac{2n-1}{2}}\, dt.$$

Massless

$$\rho_0 = N\frac{\Gamma\left(\frac{4+n}{2}\right)}{2\pi^{\frac{4+n}{2}} d^{4+n}} \sum_{\substack{(l_4,l_5,\cdots,l_{n+3})\in\mathbb{Z}^n,\\ (l_4,l_5,\cdots,l_{n+3})\neq 0}} \left(\sum_{i=4}^{3+n} l_i^2\right)^{-\frac{4+n}{2}}.$$

E. Elizalde, Commun. Math. Phys. 198, 83 (1998) [arXiv:hep-th/9707257].
J. Ambjorn and S. Wolfram, Annals of Physics 147, 1 (1983).

The finite temperature correction for the Casimir energy

$E_0^{ren} = Vd^n\rho_0$ $\quad F_0 = E_0^{\text{ren}} + \Delta_T F_0,$

$$\Delta_T F_0 \equiv k_B T \sum_J \ln\left(1 \mp e^{-\beta\omega_J}\right),$$

- for bosons, and + for fermions

$$\frac{\Delta_T F_0}{(V d^n)} = \frac{1}{d^n \beta} \int \frac{d^3 k}{(2\pi)^3} \sum_{(l_4, l_5, \cdots, l_{n+3}) \in \mathbb{Z}^n} \ln\left(1 \mp e^{-\beta\sqrt{m^2 + \boldsymbol{k}^2 + (2\pi/d)^2 \sum_{i=4}^{n+3} l_i^2}}\right)$$

$$\rho_T = \frac{|N|}{d^n \beta} \int \frac{d^3 k}{(2\pi)^3} \sum_{(l_4, l_5, \cdots, l_{n+3}) \in \mathbb{Z}^n} \ln\left(1 \mp e^{-\beta\sqrt{m^2 + \boldsymbol{k}^2 + (2\pi/d)^2 \sum_{i=4}^{n+3} l_i^2}}\right)$$

$$- \frac{|N|}{\beta} \int \frac{d^{n+3} k}{(2\pi)^{n+3}} \ln\left(1 \mp e^{-\beta\sqrt{m^2 + \boldsymbol{k}^2}}\right)$$

The finite temperature correction for the Casimir energy II

The Abel-Plana formula

$$\sum_{n=0}^{\infty} F(n) - \int_0^{\infty} dt F(t) = \frac{1}{2}F(0) + i\int_0^{\infty} \frac{dt}{e^{2\pi t}-1}\left[F(it) - F(-it)\right],$$

$$\rho_T = -4|N|d^{j-n+1}\sum_{s=1}^{\infty}(\pm 1)^s \sum_{j=0}^{n-1} \sum_{(l_{5+j},\cdots,l_{n+3})\in\mathbb{Z}^{n-j-1}} \sum_{u=1}^{\infty} \left\{\frac{\sqrt{m^2+(2\pi/d)^2\sum_{i=5+j}^{n+3} l_i^2}}{2\pi}\right\}^{\frac{5+j}{2}}$$

$$\times \frac{K_{\frac{5+j}{2}}\left[\sqrt{m^2+(2\pi/d)^2\sum_{i=5+j}^{n+3} l_i^2}\sqrt{s^2\beta^2+u^2d^2}\right]}{\left[\sqrt{s^2\beta^2+u^2d^2}\right]^{\frac{5+j}{2}}}.$$

$$\rho_T = -\frac{2|N|m^{4+n}}{(2\pi)^{\frac{4+n}{2}}}\sum_{s=1}^{\infty}(\pm 1)^s \sum_{(l_4,\cdots,l_{n+3})\in\mathbb{Z}^n} \frac{K_{\frac{4+n}{2}}\left[m\sqrt{\beta^2 s^2+d^2\sum_{i=4}^{3+n} l_i^2}\right]}{m^{\frac{4+n}{2}}\sqrt{\beta^2 s^2+d^2\sum_{i=4}^{3+n} l_i^2}^{\frac{4+n}{2}}}.$$

The finite temperature Casimir energy in $R^3 \times (S^1)^n \times R$ space-time

$$\rho_{casimir} = -\frac{|N| m^{4+n}}{(2\pi)^{\frac{4+n}{2}}} \sum_{\substack{s \in \mathbb{Z}, \\ (l_4, \cdots, l_{n+3}) \in \mathbb{Z}^n, \\ (s, l_4, \cdots, l_{n+3}) \neq 0}} (1 - 2\delta_{s0})^{\alpha} (-1)^{\alpha s} \frac{K_{\frac{4+n}{2}}\left[m \sqrt{\beta^2 s^2 + d^2 \sum_{i=4}^{3+n} l_i^2} \right]}{m^{\frac{4+n}{2}} \sqrt{\beta^2 s^2 + d^2 \sum_{i=4}^{3+n} l_i^2}^{\frac{4+n}{2}}},$$

$(-1)^{\alpha} = 1$ for bosons and $(-1)^{\alpha} = -1$ for fermions

Friedmann-Lemaitre equations in 3+n+1 dimensions

FLRW metric $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j + b^2(t)\delta_{\diamondsuit\heartsuit}dx^\diamondsuit dx^\heartsuit$

FL equations

$$3H^2 + 3nHH_b + \frac{1}{2}n(n-1)H_b^2 = \kappa^2 d^n(t)\rho,$$

$$-3H^2 - 2\dot{H} - 2nHH_b - \frac{1}{2}n(n+1)H_b^2 - n\dot{H}_b = \kappa^2 d^n(t)p,$$

$$-6H^2 - 3\dot{H} + 3(1-n)HH_b - \frac{1}{2}n(n-1)H_b^2 + (1-n)\dot{H}_b = \kappa^2 d^n(t)p_b.$$

$$d(t) = d_0 b(t)$$

$$H(t) \equiv \dot{a}(t)/a(t) \quad H_b(t) \equiv \dot{b}(t)/b(t)$$

$$\rho = -T_0^0,\ p = T_i^i/3 \text{ and } p_b = T_\diamondsuit^\diamondsuit/n,$$

The equation of continuity

$$\dot{\rho} + 3H(\rho + p) + nH_b(\rho + p_b) = 0.$$

$$\rho_{casimir} = -\frac{|N|m^{4+n}}{(2\pi)^{\frac{4+n}{2}}} \sum_{\substack{s\in\mathbb{Z},\\ (l_4,\cdots,l_{n+3})\in\mathbb{Z}^n,\\ (s,l_4,\cdots,l_{n+3})\neq 0}} (1-2\delta_{s0})^{\alpha}(-1)^{\alpha s} \frac{K_{\frac{4+n}{2}}\left[m\sqrt{\beta^2 s^2 + d^2\sum_{i=4}^{3+n} l_i^2}\right]}{m^{\frac{4+n}{2}}\sqrt{\beta^2 s^2 + d^2\sum_{i=4}^{3+n} l_i^2}^{\frac{4+n}{2}}},$$

$$p_{casimir} = -\rho_{casimir} - \frac{a(t)}{3}\frac{\partial \rho_{casimir}}{\partial \beta}\frac{\partial \beta}{\partial a(t)},$$

$$p_{b,casimir} = -\rho_{casimir} - \frac{b(t)}{n}\left(\frac{\partial \rho_{casimir}}{\partial b(t)} + \frac{\partial \rho_{casimir}}{\partial \beta}\frac{\partial \beta}{\partial b(t)}\right),$$

These definitions of p and p_b are consistent with those in zero temperature.

Temperature

$$\rho_{casimir} = -\frac{|N|m^{4+n}}{(2\pi)^{\frac{4+n}{2}}} \sum_{\substack{s\in\mathbb{Z},\\ (l_4,\cdots,l_{n+3})\in\mathbb{Z}^n,\\ (s,l_4,\cdots,l_{n+3})\neq 0}} (1-2\delta_{s0})^{\alpha}(-1)^{\alpha s} \frac{K_{\frac{4+n}{2}}\left[m\sqrt{\beta^2 s^2 + d^2\sum_{i=4}^{3+n} l_i^2}\right]}{m^{\frac{4+n}{2}}\sqrt{\beta^2 s^2 + d^2\sum_{i=4}^{3+n} l_i^2}^{\frac{4+n}{2}}},$$

$$\rho_{c,casimir}(t) \propto \beta^{-4-n}(t). \quad \longrightarrow \quad \beta(t) \propto a^{\frac{3}{3+n}}(t) b^{\frac{n}{3+n}}(t).$$

Neutrinos

$$T_{\nu 0} = \left(\frac{4}{11}\right)^{\frac{1}{3}} \left(\frac{a_0}{b_0}\right)^{\frac{n}{3+n}} T_{\gamma 0}, \qquad \frac{1}{a^3 b^n}\frac{d}{dt}\left\{a^3 b^n (H - H_b)\right\} = \kappa^2 d^n (p - p_b).$$

$T_{\nu 0}$ of this scenario can be larger than that of ΛCDM model,

$T_{\nu 0 \Lambda} = 1.945$ K.

Expansion history of the Universe

$$\frac{1}{a^3 b^n}\frac{d}{dt}\left(a^3 b^n H_b\right) = \frac{\kappa^2}{n+2} d^n (\rho + 2p_b - 3p).$$

$$\rho_{casimir} = -\frac{|N| m^{4+n}}{(2\pi)^{\frac{4+n}{2}}} \sum_{\substack{s\in\mathbb{Z},\\ (l_4,\cdots,l_{n+3})\in\mathbb{Z}^n,\\ (s,l_4,\cdots,l_{n+3})\neq 0}} (1-2\delta_{s0})^{\alpha}(-1)^{\alpha s} \frac{K_{\frac{4+n}{2}}\left[m\sqrt{\beta^2 s^2 + d^2 \sum_{i=4}^{3+n} l_i^2}\right]}{m^{\frac{4+n}{2}}\sqrt{\beta^2 s^2 + d^2 \sum_{i=4}^{3+n} l_i^2}^{\frac{4+n}{2}}},$$

As the Universe expands, the matter density decreases, and the right hand side of the equation becomes small and it can vanish at a certain time.

$$\dot{b}(t) \approx 0 \text{ and } \rho + 2p_b - 3p \approx 0.$$

Expansion history of the Universe II

$$\dot{b}(t) \approx 0 \text{ and } \rho + 2p_b - 3p \approx 0.$$

The typical four dimensional energy density of zero temperature

$$\text{constant} \sim b^{-4}(t) \sim a^{\frac{12}{n}}(t)$$

The typical four dimensional energy density of finite temperature

$$a^{-3\frac{4+n}{3+n}}(t) \sim \beta^{-n-4}(t) b^n(t) \sim a^{-3}(t)$$

$$(\rho + 2p_b - 3p)_{casimir} \sim const. \times \beta^{-4-n} \qquad \rho_{\text{matter}} \propto a^{-3}$$

Stability

$$\rho_{casimir} = -\frac{1}{4\pi^2} \sum_{\substack{s,l_4\in\mathbb{Z},\\ (s,l_4)\neq(0,0)}} \left[\frac{15}{2}\frac{1}{(s^2\beta^2+l_4^2d^2)^{\frac{5}{2}}} \right.$$

$$\left. + \frac{N_\psi}{2}(1-2\delta_{s0})(-1)^s \frac{m^2 e^{-m\sqrt{s^2\beta^2+l_4^2d^2}}}{(s^2\beta^2+l_4^2d^2)^{\frac{3}{2}}} \left\{ 1 + \frac{3}{m\sqrt{s^2\beta^2+l_4^2d^2}} + \frac{3}{m^2\left(s^2\beta^2+l_4^2d^2\right)} \right\} \right].$$

$$\rho_{casimir} \sim -\frac{15}{4\pi^2}\left(\frac{1}{d^5}+\frac{1}{\beta^5}\right)$$

$$+ \frac{N_\psi}{4\pi^2} m^2 \left\{ e^{-md}\left(\frac{1}{d^3}+\frac{3}{md^4}+\frac{3}{m^2d^5}\right) + e^{-m\beta}\left(\frac{1}{\beta^3}+\frac{3}{m\beta^4}+\frac{3}{m^2\beta^5}\right)\right\}.$$

$$\rho + 2p_b - 3p = 2\rho_{casimir} - 2d\frac{\partial\rho_{casimir}}{\partial d} + \frac{\beta}{4}\frac{\partial\rho_{casimir}}{\partial\beta} + \rho_{\mathrm{matter}}$$

$$\sim -\frac{15}{4\pi^2}\left(\frac{12}{d^5}+\frac{3}{4\beta^5}\right) + \frac{N_\psi}{4\pi^2}m^2 \left\{ e^{-md}\left(\frac{2m}{d^2}+\frac{14}{d^3}+\frac{36}{md^4}+\frac{36}{m^2d^5}\right)\right.$$

$$\left. + e^{-m\beta}\left(-\frac{m}{4\beta^2}+\frac{1}{2\beta^3}+\frac{9}{4m\beta^4}+\frac{9}{4m^2\beta^5}\right)\right\} + \rho_{\mathrm{matter}}.$$

Neutrinos as the fermions which cause the Casimir effect

$m_1 < 0.1$meV, $n = 8$, $d = 66\mu$m and $T = 30$K

→ Dark Energy and Dark Matter can be explained!!

However, . . .

decay width of Z boson is enhanced by the factor:

$$\sum_{(l_4,\cdots,l_{n+3})\in\mathbb{Z}^n} \left(1 - \frac{\left(\frac{2\pi}{d}\right)^2 \sum_{i=4}^{n+3} l_i^2}{M_Z^2 - m^2}\right) \sqrt{1 - 4\frac{\left(\frac{2\pi}{d}\right)^2 \sum_{i=4}^{n+3} l_i^2}{M_Z^2 - 4m^2}},$$

Conclusions

It has been investigated whether or not Casimir effect from the fermions and the graviton can explain dark energy if they are the only massless particles which can go through the compacted extra dimensions.

Zero temperature Casimir energy and the finite temperature Casimir energy can explain dark energy and dark matter, respectively.

Neutrinos cannot be the fermions which cause accelerating expansion of the Universe.