Time and Fermions: General Covariance *vs.* Ockham's Razor for Spinors

J. Brian Pitts

University of Cambridge, jbp25@cam.ac.uk

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Spinors and Arbitrary Coordinates?

- Use of arbitrary coordinates traditional in GR.
- Conventional wisdom (Kretschmann): arbitrary coordinates admissible in any theory with some ingenuity (Norton, 1993).
- In fact it's easy: just use tensor calculus (Earman, 2006).
- Space-time philosophy usually ignores fermions/spinors due to following general relativists.
- What happens if one remembers spinors?
- Einstein, 1916: can't adapt coordinates to simplify laws in GR, so allow arbitrary coordinates.
- Spinors postdate Einstein's reflection. Do they exist in arbitrary coordinates?

- One meets another piece of conventional wisdom, that spinors in coordinates don't exist (Weyl, 1929b; Weyl, 1929a; Fock, 1929; Infeld and van der Waerden, 1933) (Cartan, 1966, French 1937).
- Use orthonormal basis (tetrad) e^{μ}_{A}: $g_{\mu\nu} e^{\mu}_{A} e^{\nu}_{B} = \eta_{AB} = diag(-1, 1, 1, 1)$ and extra local $O(1, 3)$: 6 extra components, cancelled by 6 pieces of gauge.
- Spinors as coordinate scalars, local $O(1, 3)$ spinors.
- Weyl: impossibility of spinors in coordinates and consequent necessity of orthonormal basis recognized by its inventors as *conceptual innovation* (Scholz, 2005).
- Conventional wisdom conflict: no-spinors-in-coordinates *vs.* Kretschmann easy-with-tensors. Which fails? Both!

- Weyl's conceptual innovation was wrong, artifact of incomplete technical progress.
- Cartan milder: "*insurmountable*" difficulties in coord. $\nabla\psi$.

"Certain physicists regard spinors as entities which are, in a sense, unaffected by the rotations which classical geometric entities (vectors etc.) can undergo, and of which the components in a given reference system are susceptible of undergoing linear transformations which are in a sense autonomous. See for example L. Infeld and B. L. van der Waerden . . . . It is clear that this impossibility . . . provides an explanation of the point of view of L. Infeld and van der Waerden . . . , *which is however geometrically and even physically so startling*." (Cartan, 1966, pp. 150, 151, emphasis added)

- Cartan right to be surprised at detaching spinorhood from spatial rotations: it *isn't* necessary because there are coordinate spinors (Ogievetskiĭ and Polubarinov, 1965; Isham et al., 1971)! (at least for small coord. trans.)
- Drop Cartan's implicit assumption of linear metric-free transformation law for ψ (Ogievetskiĭ and Polubarinov, 1965).
- Clue in DeWitts (DeWitt and DeWitt, 1952).
- Pauli suggested binomial series square root of metric (Belinfante, 1985)?
- Finite-component but spinor coord. trans. law depends on (conformal part of) $g_{\mu\nu}$.

Geometric Objects: Tensor Calculus Completed?

- Are OP metric-spinors $\langle g_{\mu\nu}, \psi \rangle$ geometric objects (up to sign for ψ)?
- Geometric Object: for each space-time point and local coordinate system about it, components and transformation law to other coordinates (Nijenhuis, 1952; Kucharzewski and Kuczma, 1964; Trautman, 1965). E.g., $v^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\nu}} v^{\nu}$.
- Jet $\langle g_{\mu\nu}, g_{\mu\nu,\alpha} \rangle$. Modern view: natural bundles (Nijenhuis, 1972; Fatibene and Francaviglia, 2003; Giachetta, 1999).
- Coord. trans. near 1 gives Lie derivative (Szybiak, 1966), (if defined) covariant derivative (Szybiak, 1963).
- Tensors, tensor densities linear homogeneous: $v' \sim v$.

- Connections affine: $\Gamma' \sim \Gamma + \partial^2$.
- Pert. $g^{\mu\nu} = \eta^{\mu\nu} + \sqrt{32\pi G}\gamma^{\mu\nu}$, $\eta^{\mu\nu} = diag(-1, 1, 1, 1)$?
- $\gamma^{\alpha\beta\prime} = \gamma^{\mu\nu} \frac{\partial x^{\alpha\prime}}{\partial x^{\mu}} \frac{\partial x^{\beta\prime}}{\partial x^{\nu}} + \frac{1}{\sqrt{32\pi G}} \left(\frac{\partial x^{\alpha\prime}}{\partial x^{\mu}} \frac{\partial x^{\beta\prime}}{\partial x^{\nu}} - \delta^{\alpha}_{\mu} \delta^{\beta}_{\nu} \right) \eta^{\alpha\beta}$. Briefly: $\gamma' \sim \gamma + \partial$ (Ogievetskiĭ and Polubarinov, 1965).
- Affine transformation laws give tensorial Lie derivatives (Tashiro, 1950; Yano, 1957).
- Some (obscure) geometric objects are nonlinear. Some of them series transformation rules: $v' \sim \sum_i (v)^i$.
- Lie, covariant derivatives of nonlinear g.o. χ: usually only *pair* $\langle \chi, \mathcal{L}_{\xi} \chi \rangle$ is a g.o. Likewise $\langle \chi, \nabla \chi \rangle$ (Tashiro, 1950; Tashiro, 1952; Yano, 1957; Szybiak, 1966; Szybiak, 1963).
- Some nonlinear g.o. are equivalent to linear: "quasi-linear" (Aczél and Gołab, 1960).

- Nonlinear g.o. literature lacked good examples, but particle physicists invented nonlinear group representations!
- Symmetric square root of inverse metric $r^{\mu\nu}$, $g^{\mu\nu} = r^{\mu\alpha}\eta_{\alpha\beta}r^{\beta\nu}$ (Ogievetskiĭ and Polubarinov, 1965)?
- $r^{\mu\nu}$ is nonlinear g.o. (except coordinate restriction), equivalent (when defined) to $g^{\mu\nu}$.
- $r_{\mu\nu} = (r^{\mu\nu})^{-1}$.
- Transformation rule $r' \sim \ldots$ from binomial series for $(g_{\mu\nu}\eta^{\nu\alpha})^{\frac{1}{2}}$, even in r: $r'_{\mu\nu} = \sqrt{\frac{\partial x^{\alpha}}{\partial x^{\mu\prime}} r_{\alpha\beta}\eta^{\beta\gamma} r_{\gamma\rho}\frac{\partial x^{\rho}}{\partial x^{\sigma\prime}}\eta^{\sigma\delta}}\eta_{\delta\nu}$.
- $r'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x^{\mu\prime}} r_{\alpha\beta}\frac{\partial x^{\beta}}{\partial x^{\nu\prime}}\left|\frac{\partial x}{\partial x'}\right|^{-\frac{1}{4}}$: tensor density (linear) for 15-parameter conformal group (stability group), though most have only noticed Poincaré.
- Can $r^{\mu\nu}$ facilitate spinors in coordinates?

- Since Weyl, 'everyone knows' spinors not representation of general coordinate transformations; need for Lorentz group transformations, orthonormal tetrad (Belinfante et al., 1957; DeWitt, 1965; Weinberg, 1972; Deser and Isham, 1976; van Nieuwenhuizen, 1981; Faddeev, 1982; Lawson and Michelsohn, 1989; Fronsdal and Heidenreich, 1992; Kaku, 1993; Bardeen and Zumino, 1994; Weinberg, 2000; Fatibene and Francaviglia, 2003; Ferraris et al., 2003).
- But it *can be, has been done*; it's just hard and unfamiliar (DeWitt and DeWitt, 1952; Ogievetskiĭ and Polubarinov, 1965; Ogievetsky and Polubarinov, 1965; DeWitt, 1967; Isham et al., 1971; Borisov and Ogievetskii, 1974; Bilyalov, 1992; Bilyalov, 2002; Gates et al., 1983).

- Can symmetrize tetrad: gauge-fix local $O(1,3)$ (DeWitt and DeWitt, 1952; Isham et al., 1971; Isham et al., 1973; Deser and van Nieuwenhuizen, 1974; 't Hooft and Veltman, 1974; Borisov and Ogievetskii, 1974; Cho and Freund, 1975; Hamamoto, 1978; Boulware et al., 1979; Veltman, 1981; Ivanov and Niederle, 1982; Alvarez-Gaumé and Witten, 1984; Passarino, 1984; Fujikawa et al., 1985; Fujikawa et al., 1985; Choi et al., 1993; Aldrovandi et al., 1994; López-Pinto et al., 1995; Schücker, 2000; Tresguerres and Mielke, 2000; Bilyalov, 2002; Sardanashvily, 2002; Obukhov and Pereira, 2003; Vasiliev, 2003; Tiemblo and Tresguerres, 2004; Kirsch, 2005; Holstein, 2006; Leclerc, 2006a; Iochum and Schücker, 2006; Nibbelink et al., 2007). **Near-reinventors of OP spinors.**

- But few envision $r^{\mu\nu}$ as conceptually independent.
- Many even say one can't have spinors in coordinates, and then (or elsewhere) impose symmetric gauge (DeWitt, 1965; Deser and van Nieuwenhuizen, 1974; Boulware et al., 1979; Deser and Isham, 1976; Isham et al., 1971; Aldrovandi et al., 1994; Leclerc, 2006a; Leclerc, 2006b).
- Thus unwittingly having spinors in coordinates: curious long spectacle of mutual and self-contradiction.
- Worth an hour of philosophy to clear up.
- GR + spinors avoids Anderson-Friedman absolute object by OP spinors $r_{\mu\nu}$ (Pitts, 2006).
- Dodge 1 threat to analysis of substantive (not merely formal Kretschmann) general covariance.

- OP: $r^{\mu\nu}$ *itself* represents arbitrary infinitesimal coordinate transformations, nonlinearly, with only coordinate indices.
- Tetrad-advocates Lawson and Michelsohn: "*the bundle of spinors itself depends in an essential way on the choice of riemannian* [*sic*] *structure on the manifold.*

  "These observations lead one to suspect that there must exist a local spinor calculus, like the tensor calculus, which should be an important component of local riemannian [*sic*] geometry. A satisfactory formalism of this type has not yet been developed." (Lawson and Michelsohn, 1989, p. 5)
- But it has (Ogievetskiĭ and Polubarinov, 1965).
- Eliminate $\frac{6}{16}$ of tetrad, local $O(1,3)$, to avoid surplus structure?

- $r^{\mu\nu}$ conceptually independent of orthonormal tetrad, not topologically obstructed.
- $r^{\mu\nu}$ fixes local Lorentz freedom of ψ in each coordinate system. Boost depends on coordinates and metric, hence nonlinear metric-dependent spinor transformation law.
- Spinors in coordinates not impossible, just hard and unfamiliar: spinors in nonlinear rep. (up to sign) of coord. trans., linear for global Lorentz. (Gates et al., 1983, p. 234)
- Can add local Lorentz group for *convenience*: "by enlarging the gauge group, we obtain linear spinor representations. The nonlinear spinor representations of the general coordinate group reappear only if we fix a gauge for the local Lorentz transformations." (Gates et al., 1983)

- OP mathematically refined, made less perturbative (Bilyalov, 2002; Bilyalov, 1992; Bourguignon and Gauduchon, 1992).
- As if choosing local Lorentz freedom to symmetrize tetrad $e^{\mu A} = e^{\alpha M}$, inferring spinor coordinate transformation from spinor Lorentz transformation.
- But $r^{\mu\nu}$ independent of tetrad, even exists when tetrad/n-ad is topologically obstructed.
- $r^{\mu\nu}$ is (almost) a g.o., analytic function of $g^{\mu\nu}$.
- Hairy ball theorem on sphere (Spivak, 1979), Stiefel-Whitney class restrictions (DeWitt et al., 1979) can exclude n-ad.
- OP: spinor ψ *with* $r^{\mu\nu}$ forms representation of general infinitesimal coordinate transformations.
- Numerical Dirac γ matrices: independent of $g^{\mu\nu}$, coordinates.

- Oddly, no-go theorem for spinors and OP refutation coexist over 45 years with negligible polemic (Pitts, 2012).
- OP: Cartan's no-go theorem lacks imagination. Not ψ, but $\langle r^{\mu\nu}, \psi \rangle$; ψ transformation depends on $r^{\mu\nu}$, hence nonlinear.
- So one needs only 10 components of $r^{\mu\nu}$, not 16 of e^{μ}_{A}.
- GR + spinor fits natural (coordinate) bundle, no need for gauge-natural bundle (*c.f.* (Godina and Matteucci, 2003; Matteucci, 2003; Godina and Matteucci, 2005)).
- $\langle r^{\mu\nu}, \psi \rangle$ equivalent as (almost) g.o. to $\langle g^{\mu\nu}, \psi \rangle$—if admit same coordinates. Must specify admissible ones (Kucharzewski and Kuczma, 1964).
- If OP spinors restrict coordinates, then *conflict between general covariance and Ockham's razor*.

- Are just any coordinates allowed? Binomial series:

$$r^{\mu\nu} = \sum_{k=0}^{\infty} \frac{\frac{1}{2}!}{(\frac{1}{2}-k)!k!} [(g^{\mu\bullet} - \eta^{\mu\bullet})\eta_{\bullet\bullet} \ldots (g^{\bullet\nu} - \eta^{\bullet\nu})]^{k \text{ factors}}$$

$$= \eta^{\mu\nu} + \frac{1}{2}(g^{\mu\nu} - \eta^{\mu\nu}) - \frac{1}{8}(g^{\mu\alpha} - \eta^{\mu\alpha})\eta_{\alpha\beta}(g^{\beta\nu} - \eta^{\beta\nu}) + \ldots .$$

- Convergence for coordinates not too far from Cartesian.
- Bilyalov's eigenvector formalism more general (Bilyalov, 2002), but not fully: put 'time' first with T-matrix.
- Need for arbitrary coordinates? Entrenched habit, not a fact.
- Equivalence principle, falling elevators, rotating disks don't need completely arbitrary coordinates.
- Is complete arbitrariness an artifact of using only linear geometric objects?

Nonlinear Geometric Objects and Modern-Style Geometry?

- Nonlinear g.o. rarely discussed in 'coordinate-free' geometry.
- Suppose one tries: viewed as machines with slots, what sorts of machines are nonlinear geometric objects?
- Metric $g_{\mu\nu}$: $\mathbf{g}$ eats 2 vectors, gives a real number.
- Construction of $\mathbf{g} = \underset{\sim}{\underset{\sim}{g}}$ from components $g_{\mu\nu}$ or *vice versa*:

  $g_{\mu\nu} = \underset{\sim}{\underset{\sim}{g}}(\frac{\vec{\partial}}{\partial x^\mu}, \frac{\vec{\partial}}{\partial x^\nu}) = (\frac{\vec{\partial}}{\partial x^\mu}) \cdot \underset{\sim}{\underset{\sim}{g}} \cdot \frac{\vec{\partial}}{\partial x^\nu}.$
- Feeding in basis means placing outside, taking inner product.
- Reverse: $\underset{\sim}{d}x^\mu g_{\mu\nu} \underset{\sim}{d}x^\nu = \underset{\sim}{d}x^\mu \frac{\vec{\partial}}{\partial x^\mu} \cdot \underset{\sim}{\underset{\sim}{g}} \cdot \frac{\vec{\partial}}{\partial x^\nu} \underset{\sim}{d}x^\nu = \underset{\sim}{\vec{I}} \cdot \underset{\sim}{\underset{\sim}{g}} \cdot \underset{\sim}{\vec{I}} = \underset{\sim}{\underset{\sim}{g}}.$
- But $\mathbf{r}$ gives no number(s) $r_{\mu\nu}$ without **4** vectors, a basis.
- $\mathbf{r}$ cares if vector(s) intended for 0th position, like $\gamma^{\mu\nu}$ *does*.

Nonlinear Geometric Objects and Modern-Style Geometry?

- $r_{\mu\nu}$: Two indices, two external slots. But 'internal' slots also!
- Try coordinate-free square root of $\mathbf{g}$:
  $\mathbf{r} \stackrel{?}{=} c_0\eta + c_1\underset{\sim}{g} + c_2\underset{\sim}{g}\eta\underset{\sim}{g} + c_3\underset{\sim}{g}\eta\underset{\sim}{g}\eta\underset{\sim}{g} + \ldots$
- 0th term $c_0\eta$ has no slots, first has 2 external slots, second has 2 external and 2 internal (summed over), third has 2 external and 4 four internal (summed over), *etc.*, *ad infinitum*.
- Viewed as machine, $\mathbf{r}$ has 2 external, ∞ internal slots.
- Must photocopy basis and feed in over and over.
- Unfortunately this expression diverges. Rewrite binomial series more abstractly: $\mathbf{r} = \sum_{k=0}^{\infty} \frac{\frac{1}{2}!}{(\frac{1}{2}-k)!k!}(\underset{\sim}{g}\eta - I)^k\eta$.

- Still **r** has 2 external slots, ∞ internal.
- Is modern-style **r** better, cleaner, more illuminating than $r_{\mu\nu}$?
- Note: this **r** is 'coordinate-free' but still perturbative.
- What is non-perturbative **r**? Admitting polar coordinates?
- Can one do more than write $\sqrt{\mathbf{g}\eta}\eta$? That is, can one make sense of $\sqrt{\ }$ without first feeding in basis?
- Or is non-perturbative coordinate-free **r** ineffable?
- Coordinate-free $\mathbf{r} = \sqrt{\mathbf{g}\eta}\eta$ compromised because $r_{\mu\nu}$ doesn't exist for all coordinates!
- Consequence of indefinite $\eta - + ++$, not $I + + + +$.
- *C.f.* components: nonlinear g.o. technically intricate, but not essentially different or ineffable.

Conformal *In*variance of Massless Dirac Equation

- Conformal *co*variance of massless Dirac equation fairly well known: transformation under local rescaling of $g_{\mu\nu}$.
- SR *co*variance of massless Dirac under 15-parameter conformal group: also widely known.
- Conformal *in*variance apparently a secret (Schouten and Haantjes, 1936; Haantjes, 1941; Pitts, 2008): volume element $\sqrt{-g}$ *simply absent* from theory, using weighted spinor ψ_w.
- Only conformal metric density needed: $\langle \hat{g}_{\mu\nu}, \psi_w \rangle$ with $|\hat{g}_{\mu\nu}| = -1$, $g_{\mu\nu} = \hat{g}_{\mu\nu}\sqrt{-g}^{\frac{1}{2}}$ in 4-d, $g_{\mu\nu} = \hat{g}_{\mu\nu}\sqrt{-g}^{\frac{2}{n}}$ in n-d.
- ψ_w weight $\frac{3}{8}$ in 4-d (Schouten and Haantjes, 1936; Haantjes, 1941), weight $w = \frac{n-1}{2n}$ in n dimensions (Pitts, 2008).

- Novelty? Combine OP and Haantjes economies (Pitts, 2012): $\mathcal{L} = \sqrt{-g}\bar{\psi}\gamma_\nu r^{\nu\mu}\nabla_\mu\psi = \bar{\psi}_w\gamma_\nu\hat{r}^{\nu\mu}\nabla_\mu\psi_w$.
- Weighted Dirac operator $\gamma_\nu\hat{r}^{\nu\mu}\nabla_\mu\psi_w$ does not contain $\sqrt{-g}$ (though $\nabla_\mu\psi_w$ does). γ_ν are *numerical* Dirac matrices.
- $\hat{r}^{\mu\nu}$ is symmetric square root of $\hat{g}^{\mu\nu}$, has weight $\frac{1}{n}$.
- For general coordinate transformations $\hat{r}'_{\mu\nu} = \sqrt{\left|\frac{\partial x}{\partial x'}\right|^{-\frac{2}{n}} \frac{\partial x^\alpha}{\partial x^{\mu'}}\hat{r}_{\alpha\beta}\eta^{\beta\gamma}\hat{r}_{\gamma\rho}\frac{\partial x^\rho}{\partial x^{\sigma'}}\eta^{\sigma\delta}\eta_{\delta\nu}}$.
- $\mathcal{L}$ has weight $\frac{n-1}{2n} + \frac{1}{n} + \frac{n-1}{2n} = 1$, as required.
- *Manifest* covariance under 15-parameter conformal group when $\hat{g}_{\mu\nu}$ Weyl-flat: no long proof (*c.f.* (Vachaspati, 1960)).

- Conformal Killing vectors $\mathcal{L}_\xi \hat{g}_{\mu\nu} = 0$ (not just Killing) are fully special, a fact often overlooked (Kosmann, 1966a; Kosmann, 1966b; Fatibene et al., 1998; Cotăescu, 2000) (*c.f* (Kosmann, 1967)) or noticed with effort.
- Cooperation of $\sqrt{-g}$ is unnecessary because $\sqrt{-g}$ is absent, like the aether frame in SR.
- *Manifestly* massless spinor stress-energy is traceless *without* field equations (Pitts, 2011): $\frac{1}{2\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} g_{\mu\nu} = \frac{\delta \mathcal{L}}{\delta \sqrt{-g}} \equiv 0$.
- *C.f.* roundabout results using field equations (Callan et al., 1970; Sorkin, 1977; Dehnen et al., 1990).
- $\frac{\mathbf{7}}{16}$ eliminated: not general matrix, but symmetric unimodular.

- Irreducible geometric objects, elimination of surplus structure $\sqrt{-g}$ yield important results effortlessly.
- $(\mathcal{L}_\xi \psi_w)' \sim \left(\frac{\partial \psi'_w}{\partial \psi_w} \mathcal{L}_\xi \psi_w + \frac{\partial \psi'_w}{\partial \hat{g}_{\mu\nu}} \mathcal{L}_\xi \hat{g}_{\mu\nu} \right)$: Lie derivative with respect to *conformal* Killing vector is nicer, but exists generally (*c.f.* (Benn and Tucker, 1987; Penrose and Rindler, 1986)).
- $(\nabla \psi_w)' \sim \left(\frac{\partial \psi'_w}{\partial \psi_w} \nabla \psi_w + \frac{\partial \psi'_w}{\partial \hat{g}_{\mu\nu}} \nabla \hat{g}_{\mu\nu} \right) \sim \left(\nabla \psi_w + \frac{\partial \psi'_w}{\partial \hat{g}_{\mu\nu}} \nabla \hat{g}_{\mu\nu} \right)$.
- $\nabla \hat{g}_{\mu\nu} = 0$, so $\nabla \psi_w$ is spinor-covector density.
- With symmetrization and conformal invariance, general matrix is replaced by symmetric unimodular matrix.
- Savings of $6 + 1 = 7$ components out of 16, or $\frac{n^2-n+2}{2}$ of n^2.
- Better to avoid surplus structure than to introduce it and gauge it away. Why build something to knock it down?

Differentiating OP Spinors as Nonlinear Geometric Objects

- Unlike other spinors, OP spinors have classical Lie derivative: $\langle r_{\mu\nu}, \psi, \mathcal{L}_\xi r_{\mu\nu}, \mathcal{L}_\xi \psi \rangle$ is a g.o.; $[\mathcal{L}_\xi, \mathcal{L}_\phi]\psi = \mathcal{L}_{[\xi,\phi]}\psi$.
- Facilitates conservation laws (Pitts, 2010).
- $\nabla g_{\mu\nu} = 0 \leftrightarrow \nabla r_{\mu\nu} = 0$.
- Only conformal metric density needed: $\langle \hat{g}_{\mu\nu}, \psi \rangle$ with $|\hat{g}_{\mu\nu}| = -1$, $g_{\mu\nu} = \hat{g}_{\mu\nu}\sqrt{-g}^{\,\frac{1}{2}}$. $\hat{g}_{\mu\nu}$ gives null cones.
- Transformation rules for $\mathcal{L}_\xi \chi$, $\nabla \chi$, for χ a nonlinear g.o. (Szybiak, 1963; Szybiak, 1966):
- $(\mathcal{L}_\xi \psi)' \sim \left(\frac{\partial \psi'}{\partial \psi} \mathcal{L}_\xi \psi + \frac{\partial \psi'}{\partial \hat{g}_{\mu\nu}} \mathcal{L}_\xi \hat{g}_{\mu\nu} \right)$: Lie derivative with respect to conformal Killing vector is *nicer*, but exists in general.

- Explains why one reads that spinors don't have Lie derivative except using conformal Killing vectors (Benn and Tucker, 1987) (Penrose and Rindler, 1986, p. 101).
- OP spinors do have $\mathcal{L}_\xi \psi$ for $\mathcal{L}_\xi \hat{g}_{\mu\nu} \neq 0$, but messier.
- $(\nabla\psi)' \sim \left(\frac{\partial \psi'}{\partial \psi} \nabla\psi + \frac{\partial \psi'}{\partial \hat{g}_{\mu\nu}} \nabla \hat{g}_{\mu\nu} \right) \sim \left(\nabla\psi + \frac{\partial \psi'}{\partial \hat{g}_{\mu\nu}} \nabla \hat{g}_{\mu\nu} \right) \sim \nabla\psi$.
- Covariant derivative also exists.
- *C.f.* "*insurmountable*" difficulties of $\nabla\psi$ (Cartan, 1966, French 1937).
- $\mathcal{L}_\xi$, ∇ need only infinitesimal transformations.

Unified Noether Treatment of Conservation Laws

- (Supposed) lack of spinor Lie derivative makes it tricky getting conserved quantities given Killing vectors ξ^μ ($\mathcal{L}_\xi g_{\mu\nu} = 0$) (Møller, 1961; Fatibene et al., 1998; Cotăescu, 2000; Fatibene and Francaviglia, 2003; Ferraris et al., 2003).
- Have to make everything locally $O(1,3)$ covariant also.
- But since OP spinors have Lie derivative, unified conservation laws *via* $\mathcal{L}_\xi \psi$ (Ogievetskiĭ and Polubarinov, 1965; Ogievetsky and Polubarinov, 1965; Bilyalov, 1992; Pitts, 2010).
- Less confusion regarding gravitational energy in GR.
- x^0 translation invariance gives conserved 'energy,' without local $O(1,3)$ detour.

- Do, should we have truly arbitrary coordinates?
- Amount of coordinate conventionality is conventional.
- Einstein's 1916: coordinates directly yielding lengths impossible in GR, "and there seems to be no other way which would allow us to adapt systems of co-ordinates to the four-dimensional universe so that we might expect from their application a particularly **simple** formulation of the laws of nature. So there is nothing for it but to regard all imaginable systems of co-ordinates, on principle, as equally suitable for the description of nature." (emphasis added) (Einstein, 1923, p. 117)
- Convention: not arbitrary, not dictated by facts, but guided by facts (Poincaré, 1902).

- 1960s OP spinors: either $r^{\mu\nu}$ exists nonperturbatively, or coordinates for which series converges *deflate* GR + spinor by 6 fields *vs.* tetrad.
- How much tension between Ockham's razor and general covariance?
- Bilyalov (Bilyalov, 1992; Bilyalov, 2002): $r^{\mu\nu}$ exists nonperturbatively, but not quite always.
- Arbitrary set of coordinates, but not in *arbitrary* order.
- Can't always do $\langle t, x, y, z \rangle \rightarrow \langle x, t, y, z \rangle$; avoid wrong x^0.
- But who has ever cared about $\langle t, x, y, z \rangle \rightarrow \langle x, t, y, z \rangle$?
- Roots in a customary asymmetry: one feels free to put coordinates in any order, but keeps time-leg of tetrad in 0th slot: signature not $diag(1, -1, 1, 1)$. Minus sign not mobile.

- Generalized eigenvalue problem: metric $g_{\mu\nu}$, $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ (Bilyalov, 1992).
- Both indefinite, so usual theorems about complete set of real eigenvalues with orthogonal complete eigenvectors fail.
- What can go 'wrong' in generalized eigenvalue formalism?
- Eigenvalues need not be real.
- Can lack complete set of eigenvectors due to Jordan blocks.
- 4-d Segré: 4 real eigenvalues with 4 independent eigenvectors, or 4 real eigenvectors with 3 independent eigenvectors, or 4 real eigenvalues with two 2 independent eigenvectors, or 2 real and 2 complex conjugate eigenvalues with two real and two complex conjugate eigenvectors (Hall, 1984; Bilyalov, 1992; Bilyalov, 2002).

- Neither complex eigenvalues nor Jordan blocks cause trouble.
- Problem iff some eigenvalue(s) real but *negative*.
- Negative real parts for complex eigenvalues not a problem: take roots in right half-plane.
- "Any matrix with no nonpositive real eigenvalues has a unique square root for which every eigenvalue lies in the open right half-plane." (Higham, 1997)
- Matrix for square root: $g^{\mu\nu}\eta_{\nu\alpha}$ in *each* coordinate system.
- Having no negative eigenvalues is not merely sufficient, but also necessary for real square root determined by metric:

"Let $A \in \mathbb{R}^{n \times n}$ be nonsingular. If A has a real negative eigenvalue, then A has no real square roots which are functions of A." (Higham, 1987)

- Square root that is not function of original matrix has surplus structure (*e.g.*, tetrad). Seeking *principal* square root.
- Generalized polar decomposition with $\eta = diag(-1, 1, 1, 1)$ (Bolshakov et al., 1997; Higham, 2003).
- M is η-orthogonal iff $M^T \eta M = \eta$. η-symmetric iff symmetric with index moved by η.
- Tweaking notation, decompose tetrad into symmetric square root and boost-rotation:

"Theorem 5.1. If [tetrad component matrix at a point] $E \in R^{n \times n}$ and $\eta E^T \eta E$ has no eigenvalues on the nonpositive real axis, then E has a unique indefinite polar decomposition $E = QS$, where Q is η-orthogonal and S is η-symmetric with eigenvalues in the open right half-plane." (Higham, 2003, p. 513, modified)

- If $-$ eigenvalues of $g^{\mu\nu}\eta_{\nu\alpha}$ in some coordinates, no principal $\sqrt{g^{\mu\nu}\eta_{\nu\alpha}}$ in those coordinates.
- Such coordinates are exotic but do exist.
- Null/light-cone coordinates cause no trouble.
- Decomposing tetrad E, OP works with $R = \eta S$, discarding local $O(1,3)$ gauge group.
- Boost-rotation $Q = E(\eta E^T \eta E)^{-\frac{1}{2}}$ fixes local $O(1,3)$.
- Bilyalov (Bilyalov, 2002) accommodates arbitrary set of coordinates, but not in *arbitrary* order.
- Einstein's simplicity criterion: simplify laws of nature by restricting coordinates?
- Parity inversion P, T time reversal metric-dependent.

Necessity and Sufficiency of Coordinate Reordering: 2-d $-+$

- 2-d readily find eigenvalues by quadratic formula.

$$|\hat{g}^{\mu\nu} - \lambda\eta^{\mu\nu}| = \left| \begin{bmatrix} \hat{g}^{00} - \lambda\eta^{00} & \hat{g}^{01} \\ \hat{g}^{01} & \hat{g}^{11} - \lambda\eta^{11} \end{bmatrix} \right| = -\lambda^2 + \eta_{\mu\nu}\hat{g}^{\mu\nu}\lambda - 1 = 0.$$

- Find $\lambda = \frac{\eta_{\mu\nu}\hat{g}^{\mu\nu} \pm \sqrt{(\eta_{\mu\nu}\hat{g}^{\mu\nu})^2 - 4}}{2}$.
- If $|\eta_{\mu\nu}\hat{g}^{\mu\nu}| < 2$, eigenvalues and eigenvectors complex, so square root real.
- If $\eta_{\mu\nu}\hat{g}^{\mu\nu} \geq 2$, eigenvalues positive, square root real.
- But if $\eta_{\mu\nu}\hat{g}^{\mu\nu} \leq -2$, eigenvalues negative, square root bad.

- 2-d necessary and sufficient condition for local admissibility of coordinates: $\eta_{\mu\nu}\hat{g}^{\mu\nu} > -2$. Equivalent to $\eta^{\mu\nu}\hat{g}_{\mu\nu} > -2$.
- Not all coordinate systems are admissible, unlike tensors.
- Goodbye, $GL(2,\mathbb{R})$: false that $\frac{\partial x^{\mu\prime}}{\partial x^{\nu}}$ is arbitrary at a point.
- $\eta_{\mu\nu}$ indefinite, so nonlinear reps care which is x^0.
- Spinors say what a time coordinate is; lax but not empty.
- Swapping x^0 and x^1, and thus g^{00} and g^{11}, flips sign of $\eta_{\mu\nu}\hat{g}^{\mu\nu}$, makes inadmissible coordinates admissible.
- Not any (ordered) coordinate system admissible, but any *rightly ordered* coordinate system is, at a point.

- Swapping is part of what Bilyalov's T matrix does (Bilyalov, 1992).
- Higher dimensions less trivial; Bilyalov's proof is different (Bilyalov, 1992).
- Why not noticed earlier?
- Because most work on nonlinear reps. of (a big enough piece of) $GL(4, \mathbb{R})$, linear for Lorentz subgroup, have been *near* 1 (Ogievetskiĭ and Polubarinov, 1965; Isham et al., 1971; Isham et al., 1973; Borisov and Ogievetskii, 1974; Cho and Freund, 1975; Borisov, 1978; Giachetta, 1999).

Spinors and the Partial Conventionality of Simultaneity? A Worked Example

- Addressing literature on time coordinates (*c.f.* (Havas, 1987)), not definability and causal relations (*c.f.* (Malament, 1977)).
- Claim: spinors refute conventionality by not admitting non-standard simultaneity (Zangari, 1994).
- Claim: spinors as coordinate scalars don't notice time, so no difficulty at all (Gunn and Vetharianam, 1995).
- Claim: spinors refute conventionality (Karakostas, 1997).
- Spinors don't refute conventionality (Bain, 2000).
- It's weird to sever spinors from coordinate rotations (Cartan, 1966, pp. 150, 151) (Bain, 2000)—but correct, they say.

- I mostly want a tractable interesting example (Pitts, 2012).
- OP spinors: spinorial coordinate transformation, so spinor notices time coordinate. How does it behave?
- Generalizes SR particle physics textbook spinors.
- I find *many* non-standard simultaneities permitted, *but not all*.
- Standard simultaneity makes transformation linear.
- $x^{\mu'} = (x^0 + (2\epsilon_1 - 1)x^1 + (2\epsilon_2 - 1)x^2 + (2\epsilon_3 - 1)x^3, x^1, x^2, x^3)$.
- Standard simultaneity $\epsilon_1 = \epsilon_2 = \epsilon_3 = \frac{1}{2}$. Let $\vec{n} = (2\epsilon_1 - 1, 2\epsilon_2 - 1, 2\epsilon_3 - 1)$, so $x^{\mu'} = x^\mu + \delta^\mu_0 n_i x^i$.
- $g^{\mu\nu} = \begin{bmatrix} -1 + \vec{n}^2 & n_1 & n_2 & n_3 \\ n_1 & 1 & 0 & 0 \\ n_2 & 0 & 1 & 0 \\ n_3 & 0 & 0 & 1 \end{bmatrix}$. Let $n_1 = n$, $n_2 = n_3 = 0$.

- Generalized eigenvalue problem (2-d part) $|g'^{\mu\nu} - \lambda\eta^{\mu\nu}| =$

$$\left| \begin{bmatrix} -1 + n^2 - \lambda & n \\ n & 1 - \lambda \end{bmatrix} \right| = -\lambda^2 + (2 - n^2)\lambda - 1 = 0.$$

- $\lambda = \frac{2 - n^2 \pm \sqrt{n^4 - 4n^2}}{2}$.
- Coordinates with $|n| \geq 2$ inadmissible (in this ordering).
- Coordinates with $|n| < 2$ permitted ($-\frac{1}{2} < \epsilon < \frac{3}{2}$).
- Usual range of n is between -1 and 1 ($0 < \epsilon < 1$) (Grünbaum, 1973).
- OP's $-2 < n < 2$ is more than standard range.
- But not anything goes, *pace* 'Kretschmann' platitude of trivial admissibility of any coordinates.
- Platitude an artifact of using only *linear* geometric objects.

For non-standard simultaneity,

$$r'^{\mu\nu} = \begin{bmatrix} -\left(1-\frac{n^2}{2}\right)\left(1-\frac{n^2}{4}\right)^{-\frac{1}{2}} & \frac{n}{2}\left(1-\frac{n^2}{4}\right)^{-\frac{1}{2}} \\ \frac{n}{2}\left(1-\frac{n^2}{4}\right)^{-\frac{1}{2}} & \left(1-\frac{n^2}{4}\right)^{-\frac{1}{2}} \end{bmatrix} \oplus I_{2\times 2}$$

$$= \begin{bmatrix} -1+\frac{3n^2}{8}+\ldots & \frac{n}{2}+\ldots \\ \frac{n}{2}+\ldots & 1+\frac{n^2}{8}+\ldots \end{bmatrix} \oplus I_{2\times 2}.$$

Spinor transformation $\psi' = S\psi$ is
$S = I\cosh\left[\frac{1}{2}\ln\sqrt{\frac{1-n/2}{1+n/2}}\right] + \gamma_0\gamma^1 \sinh\left[\frac{1}{2}\ln\sqrt{\frac{1-n/2}{1+n/2}}\right]$
$= \frac{1}{2}I\left[\sqrt[4]{\frac{1-n/2}{1+n/2}} + \sqrt[4]{\frac{1+n/2}{1-n/2}}\right] + \frac{1}{2}\gamma_0\gamma^1\left[\sqrt[4]{\frac{1-n/2}{1+n/2}} - \sqrt[4]{\frac{1+n/2}{1-n/2}}\right] =$
$I - \frac{n}{4}\gamma_0\gamma^1 + \ldots.$

- Exact expression confirms $-2 < n < 2$.
- Change to non-standard simultaneity induces n-dependent boost of the spinor *via* dependence on metric.
- Not any coordinate order: electrons know what a time is.
- Dependence on n inherited from symmetric square root of conformal part of metric.
- Spinor trans. rule linear in spinor but nonlinear in metric.
- But otherwise, any coordinates will do: tractable example of how spinors (almost) fit into geometric objects.
- Regarding transformation laws, conventional choice has wide scope, but not universal—a novel intermediate position dictated by Ockham's razor.

Infalling Coordinates, Black Holes and 'Time'

- Another interesting example (Pitts, 2012), perhaps important.
- Thanks to Charles Misner for question.
- 'Eddington-Finkelstein' coordinates help to overcome 'Schwarzschild singularity' at $r = 2M$ (now "horizon").
- On to curvature singularity at $r = 0$ (Finkelstein, 1958; Misner et al., 1973).
- Context of discovery: don't regiment as time, space.
- Does OP formalism un-learn lesson of Schwarzschild radius?
- Infalling 'Eddington-Finkelstein': r radial, $\tilde{V}$ null, angles θ, ϕ.
- $ds^2 = -\left(1 - \frac{2M}{r}\right) d\tilde{V}^2 + 2d\tilde{V}dr + r^2(d\theta^2 + sin^2\theta d\phi^2)$ (Misner et al., 1973, p. 828).

- For metric, null $\tilde{V}$ is half space, half time; r is space.
- OP admissibility of $\langle \tilde{V}, r, \theta, \phi \rangle$ (note *order*)? For $r > \frac{2M}{3}$.
- OP admissibility of $\langle r, \tilde{V}, \theta, \phi \rangle$ (note *order*)? For $r > 0$!
- More than one right answer for $r > \frac{2M}{3}$; complex eigenvalues.
- Ironically, null $\tilde{V}$ as OP 'time' x^0, r as space gives *smaller* admissible region than r as OP 'time,' $\tilde{V}$ as space!
- 'Wrong' order works down to curvature singularity.
- Lesson of Schwarzschild radius not un-learned by OP.
- What fermions call 'time' x^0 *need not be unique or intuitive*.
- Very different from Hilbert-Møller-type conditions to enforce 1 time, 3 space (Hilbert, 2007; Møller, 1952).
- Ockham, calculation-driven; not intuitions imposed by hand.

De-Ockhamization or OP Spinors

- Other things being equal, Ockham's razor is a good idea, de-Ockhamization (adding stuff not needed) is a bad idea.
- Remove *irrelevant* fields in testing for general covariance (Anderson, 1967; Thorne et al., 1973; Lee et al., 1974).
- Force = gorce + morce: why bother? (Glymour, 1977)
- 'Kretschmann' triviality of admissibility of arbitrary coordinates in any order for spinors?
- *Only* by inflating with 6 gauge compensation fields in tetrad.
- Due to historical contingencies, before ((Ogievetskiĭ and Polubarinov, 1965)) one didn't know tetrad was de-Ockhamized.

- Now one knows, so question arises: why keep the more common but de-Ockhamized tetrad formalism?
- Tetrad de-Ockhamization hardly more compelling than Stueckelberg trick in massive electromagnetism: $-\frac{m^2}{2}(A^\mu - \partial^\mu \chi)(A_\mu - \partial_\mu \chi)$ instead of Proca $-\frac{m^2}{2}A^\mu A_\mu$.
- Gauge compensation field $\chi \rightarrow \chi + \phi$ invented to permit *artificial* gauge freedom $A_\mu \rightarrow A_\mu + \partial_\mu \phi$.
- χ useful in some applications, but conceptually regressive.
- Massive electromagnetism *naturally* lacks gauge freedom.
- Conceptual issues best addressed in economical formalism, not de-Ockhamized one.
- Hence OP spinors right for foundations of physics.

Conclusions and Questions

- Spinors do exist in (most, not all) coordinates in $-+++$.
- GR + spinors avoids Anderson-Friedman absolute object by OP spinors.
- $r_{\mu\nu}$ restricts which coordinate is x^0.
- Arbitrary coordinates permitted? Why? Conventional choice.
- Simplicity of Ockham's razor (fewer fields) $\rightarrow$ OP with coordinate restrictions.
- OP spinors in modern natural bundle context? (Giachetta, 1999); not obvious where x^0 order issue arises; near 1.
- OP spinors and topology—more general than *n*-ad formalism (not 4-d or $-+++$ (Choquet-Bruhat et al., 1996))?

- Coordinate restriction an echo of conformal group, the stability subgroup of nonlinear 'group' realization.
- Def. of manifold: atlas of admissible charts should be sensitive to types of fields, values of fields.
- Unification: bosons, fermions more alike in differential geometry.
- Unified treatment of symmetries and conservation laws from Lie derivative of OP spinors.
- OP spinors in coordinates avoid much surplus structure: $\frac{6}{16}$.
- Densities give conformal *invariance*, avoid more surplus structure: $\frac{7}{16}$ eliminated.
- Particle physics relevant to space-time theory, contrary to widespread literature-reading habits.

- Treatment above purely local and classical: walking before running.
- For historians, sociologists: why do Weyl-Cartan claims persist 40+ years after constructive refutation?
- Need spinor differential geometry of 1960s (OP), not 1929 (Weyl).
- Should make space-time theory quantization-ready, hence include spinors.

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