C-parameter distribution at N^{3}LL and a determination of α_s

… a story of 3π's and 6's

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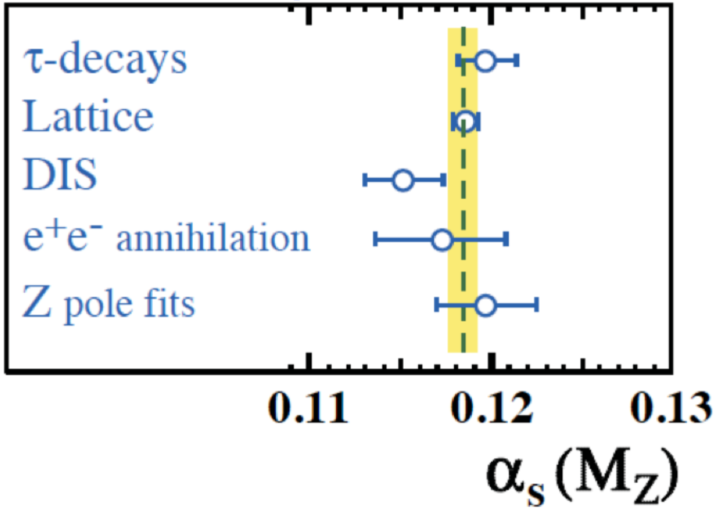
work done with Andre Hoang, Vicent Mateu and Iain Stewart

Outline

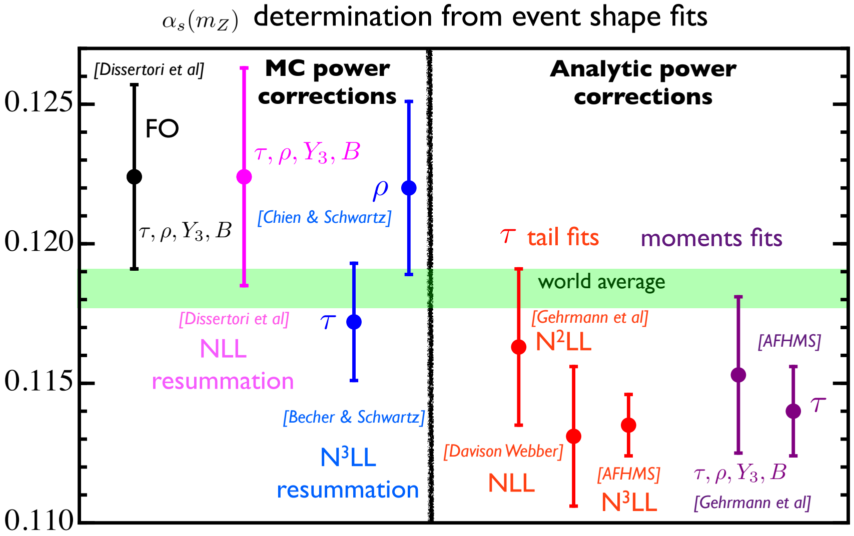
- Introduction
- C-parameter factorization
  - – Singular with resummation
  - – Nonsingular cross section
  - – Nonperturbative corrections
  - – Profile Functions
- Analysis and Results
- Conclusions

World average completely dominated by lattice result



α_s determination: compendium

Only consider analysis with 3-loop input



Definition of C-parameter

$$\theta_{\alpha\beta} = \frac{\sum_i \mathbf{p}_i^\alpha \mathbf{p}_i^\beta / |\mathbf{p}_i|}{\sum_j |\mathbf{p_j}|}$$

Linearized momentum tensor

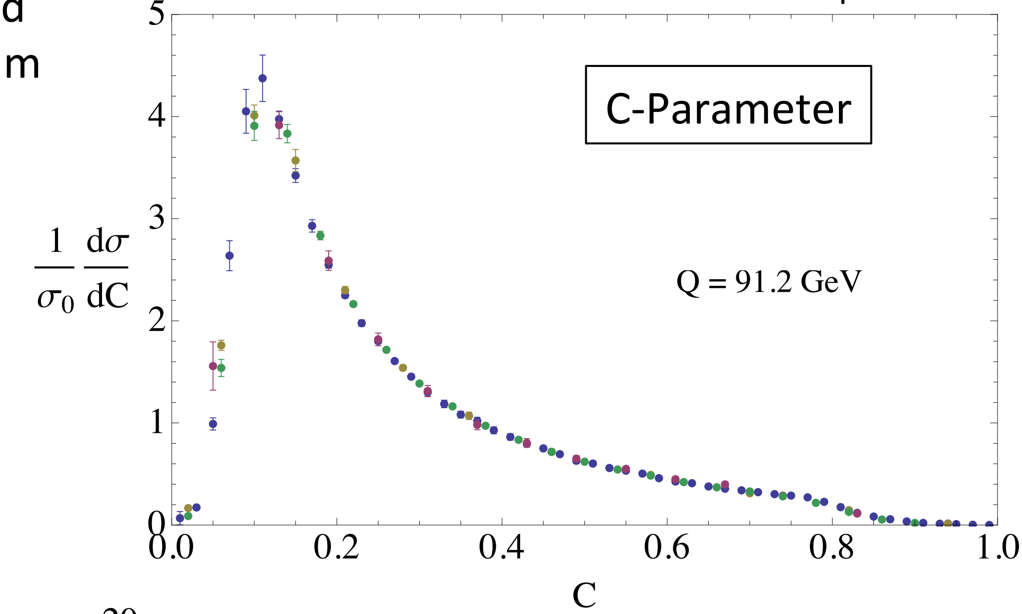
- Eigenvalues of θ: $\lambda_1, \ \lambda_2, \ \lambda_3$
- Definition of C-parameter:

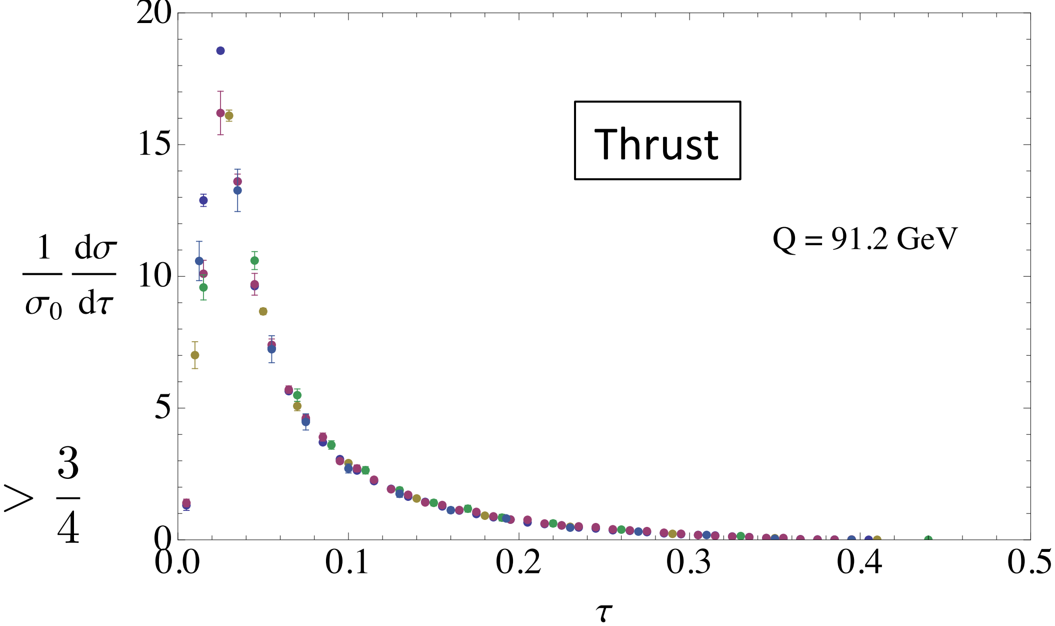
$$C = 3(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)$$

$$\boxed{C = \frac{3}{2}\frac{\sum_{ij} |\vec{p}_i||\vec{p}_j| \sin^2 \theta_{ij}}{\left(\sum_i |\vec{p}_i|\right)^2}}$$

- Dijet limit: $C \rightarrow 0$
- Multijet region: $C \rightarrow 1$
- $C > \dfrac{3}{4}$ only for $\mathcal{O}(\alpha_s^2)$
  - Only non planar events have $C > \dfrac{3}{4}$

Point color = Experiment





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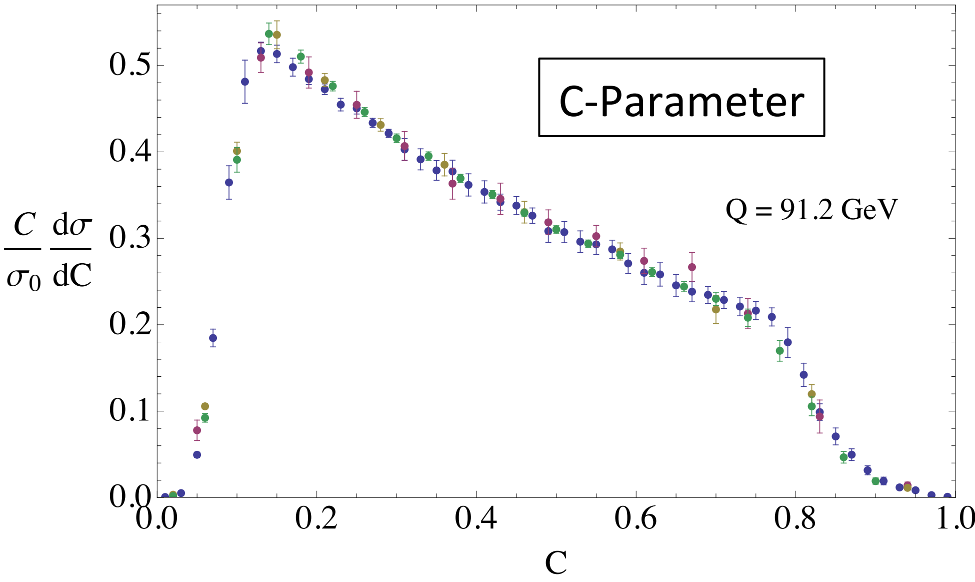
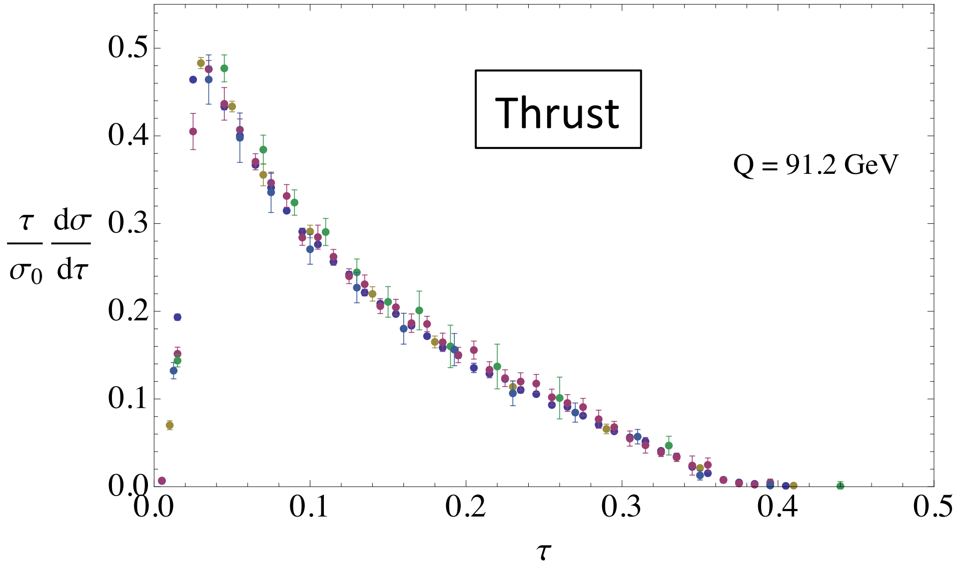
Contrast with thrust

- Thrust peaks at a lower value
- Thrust peak is much higher
- Thrust falls off faster in tail
- No minimization
- No thrust axis

Point color = Experiment





Regions of Distribution

Peak

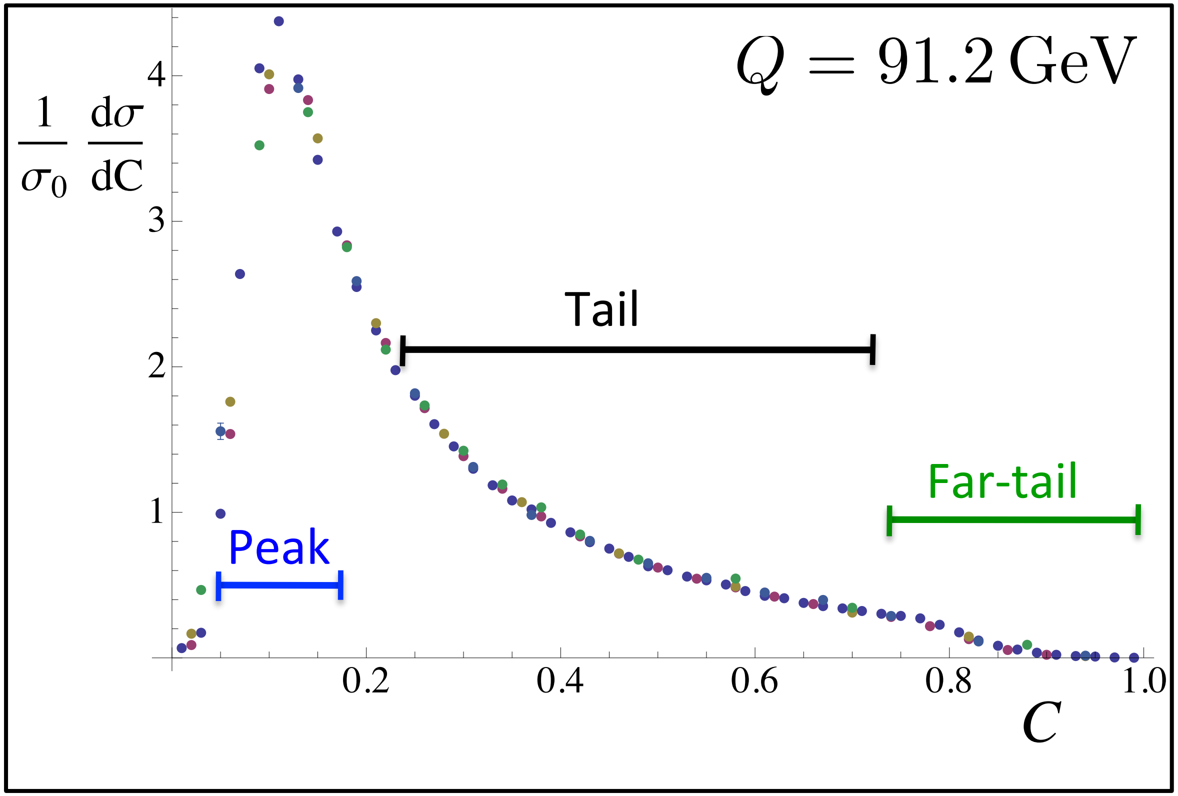
$C \sim 3\pi \frac{\Lambda_{\mathrm{QCD}}}{Q}$

Tail

$3\pi \frac{\Lambda_{\mathrm{QCD}}}{Q} \ll C \lesssim \frac{3}{4}$

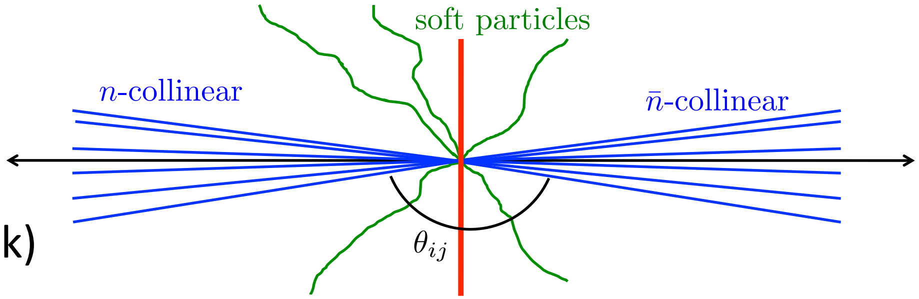
Far-tail

$\frac{3}{4} \lesssim C \leq 1$



C-parameter Kinematics

$$C = \frac{3}{2}\frac{\sum_{ij}|\vec{p}_i||\vec{p}_j|\sin^2\theta_{ij}}{\left(\sum_i|\vec{p}_i|\right)^2}$$



- In dijet limit C is small, expand in

  $\lambda \approx \sqrt{C}$ (tail) $\lambda \approx \sqrt{\frac{\Lambda_{\rm QCD}}{Q}}$ (peak)

- Split sum into pieces for various types of partons, keep only leading order terms

$$C_{\rm dijet} = C_{ss} + C_{ns} + C_{\bar{n}s} + C_{nn} + C_{\bar{n}\bar{n}} + C_{n\bar{n}}$$

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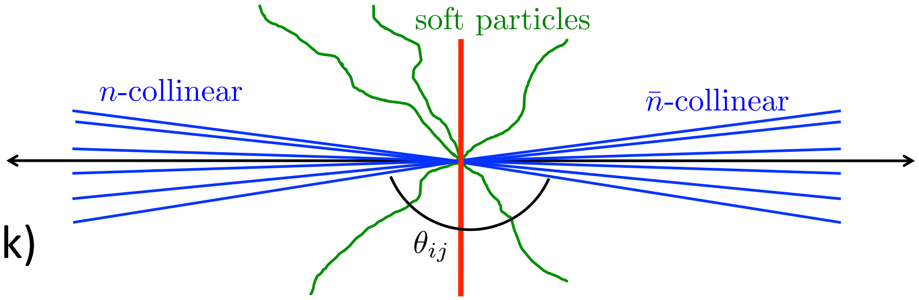

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$$C_{\text{dijet}} = \underset{\mathcal{O}(\lambda^4)}{\cancel{C_{ss}}} + C_{ns} + C_{\bar{n}s} + C_{nn} + C_{\bar{n}\bar{n}} + C_{n\bar{n}}$$

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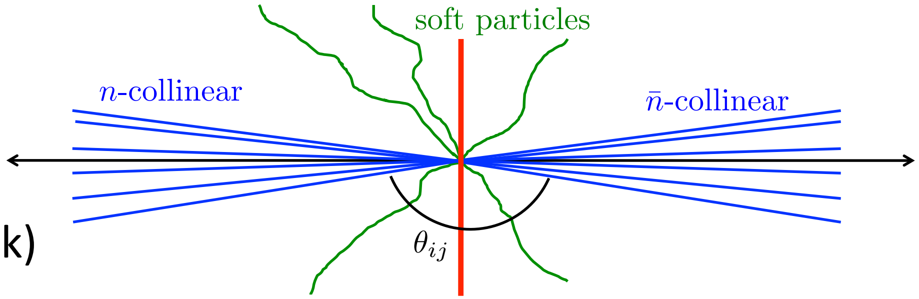

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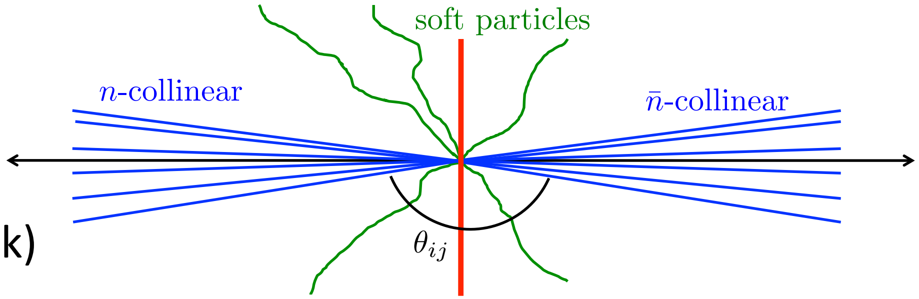
$$C_{\rm dijet} = \cancel{C_{ss}} + C_{ns} + C_{\bar{n}s} + C_{nn} + C_{\bar{n}\bar{n}} + C_{n\bar{n}}$$

Using $\sin\theta_{sn} \approx \sin\theta_s$:

$$C_{ns} = C_{\bar{n}s} = \frac{3}{Q}\sum_{i\in{\rm soft}}\frac{p_i^+p_i^-}{p_i^+ + p_i^-}$$

C-parameter Kinematics

$$C = \frac{3}{2}\frac{\sum_{ij} |\vec{p}_i||\vec{p}_j| \sin^2\theta_{ij}}{\left(\sum_i |\vec{p}_i|\right)^2}$$



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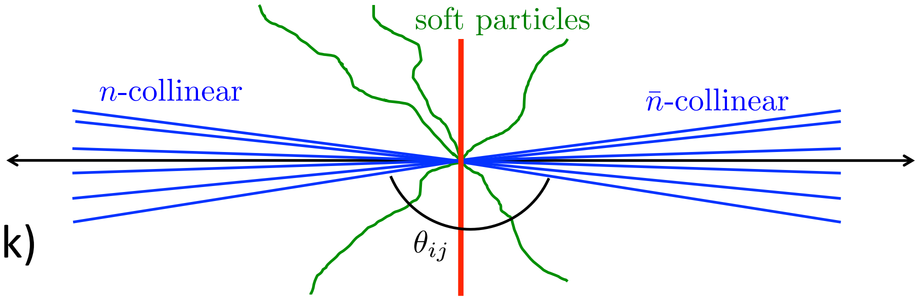
$$C_{\rm dijet} = \cancel{C_{ss}} + C_{ns} + C_{\bar{n}s} + C_{nn} + C_{\bar{n}\bar{n}} + C_{n\bar{n}}$$

Using momentum conservation and $\cos\theta_{nn} \approx 1$:

$$C_{nn} = \frac{3}{Q}\sum_{i\in n} p_i^+ \,, \quad C_{\bar{n}\bar{n}} = \frac{3}{Q}\sum_{i\in\bar{n}} p_i^-$$

C-parameter Kinematics

$$C = \frac{3}{2}\frac{\sum_{ij}|\vec{p}_i||\vec{p}_j|\sin^2\theta_{ij}}{\left(\sum_i|\vec{p}_i|\right)^2}$$



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$$C_{\mathrm{dijet}} = \cancel{C_{ss}} + C_{ns} + C_{\bar{n}s} + C_{nn} + C_{\bar{n}\bar{n}} + C_{n\bar{n}}$$

Using energy conservation, momentum conservation and $\cos\theta_{n\bar{n}} \approx -1$:

$$C_{n\bar{n}} = \frac{3}{Q}\sum_{i\in n} p_i^+ + \frac{3}{Q}\sum_{i\in\bar{n}} p_i^- = C_{nn} + C_{\bar{n}\bar{n}}$$

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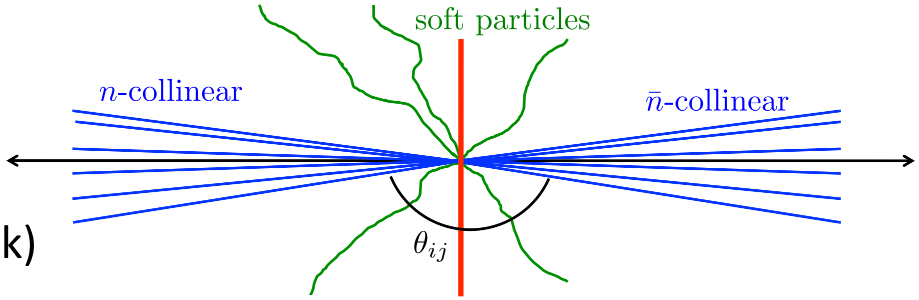
$$C = \frac{3}{2}\frac{\sum_{ij}|\vec{p}_i||\vec{p}_j|\sin^2\theta_{ij}}{\left(\sum_i|\vec{p}_i|\right)^2}$$



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$$C_{\mathrm{dijet}} = \cancel{C_{ss}} + C_{ns} + C_{\bar{n}s} + C_{nn} + C_{\bar{n}\bar{n}} + C_{n\bar{n}}$$

$$C_s = \frac{6}{Q}\sum_{i\in\mathrm{soft}}\frac{p_i^+p_i^-}{p_i^+ + p_i^-}\,,\quad C_n = \frac{6}{Q}\sum_{i\in n}p_i^+\,,\quad C_{\bar{n}} = \frac{6}{Q}\sum_{i\in\bar{n}}p_i^-$$

$$\boxed{\Longrightarrow\quad C_{\mathrm{dijet}} = C_s + C_n + C_{\bar{n}} = \frac{3}{Q}\sum_i\frac{p_i^\perp}{\cosh\eta_i}}$$

C-parameter Kinematics

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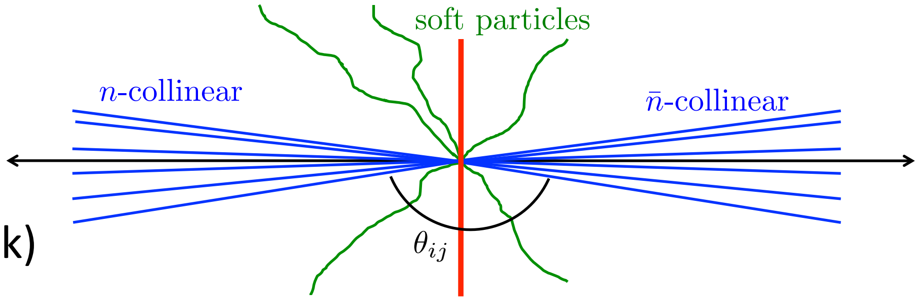

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For the purposes of logs in factorization theorem (dijet)

(C-parameter) $\widetilde{C} \equiv \frac{C}{6} \approx \tau$ (Thrust)

Factorization

Catani, Weber: NLL

Korchemsky, Sterman, Tafat: NLL+Shape Function

Bauer, Fleming, Lee, Sterman: Factorization

$$\frac{1}{\sigma_0}\frac{\mathrm{d}\sigma}{\mathrm{d}C} = \frac{Q}{6} H(Q,\mu) \int \mathrm{d}s\, J_\tau(s,\mu) S_{\tilde{C}}\left(\frac{QC}{6} - \frac{s}{Q}, \mu\right)$$

- Both Jet and Hard functions are the same as in thrust
- Soft function differs, given by:

$$S_{\tilde{C}}(\ell,\mu) = \langle 0|\, \underbrace{\overline{Y}^{\dagger}_{\bar{n}} Y^{\dagger}_{n}}_{\text{Soft Wilson Lines}} \delta\left(\ell - \underbrace{\frac{Q\hat{C}}{6}}_{\text{Measurement}}\right) Y_n \overline{Y}_{\bar{n}}\, |0\rangle$$

- Anomalous dimension for soft function same as thrust!

$$\frac{\mathrm{d}\sigma^{\mathrm{NLL}}}{\mathrm{d}C} = \frac{1}{6}\frac{\mathrm{d}\sigma^{\mathrm{NLL}}}{\mathrm{d}\tau}\bigg|_{\tau=\frac{C}{6}}$$

  - Relations between large logs to all orders in α_s (which are nontrivial for NNLL and beyond)

Soft Function

$$S_{\widetilde{C}}(\ell,\mu) = \langle 0|\,\overline{Y}_{\bar{n}}^{\dagger} Y_{n}^{\dagger}\,\delta\left(\ell - \frac{Q\hat{C}}{6}\right) Y_{n}\overline{Y}_{\bar{n}}\,|0\rangle$$

- In Fourier space

$$S_{\widetilde{C}}(y,\mu) = e^{K(y,\mu)} e^{S_0(\alpha_s(\mu_y))}$$

$$\boxed{\mu_y^{-1} = i e^{\gamma_E} y}$$

- $K(y,\mu)$ contains logs and is the same as thrust
- S_0 is C-parameter dependent piece

$$S_0(\alpha_s) = \frac{\alpha_s}{4\pi} s_1 + 2\left(\frac{\alpha_s}{4\pi}\right)^2 s_2 + \ldots$$

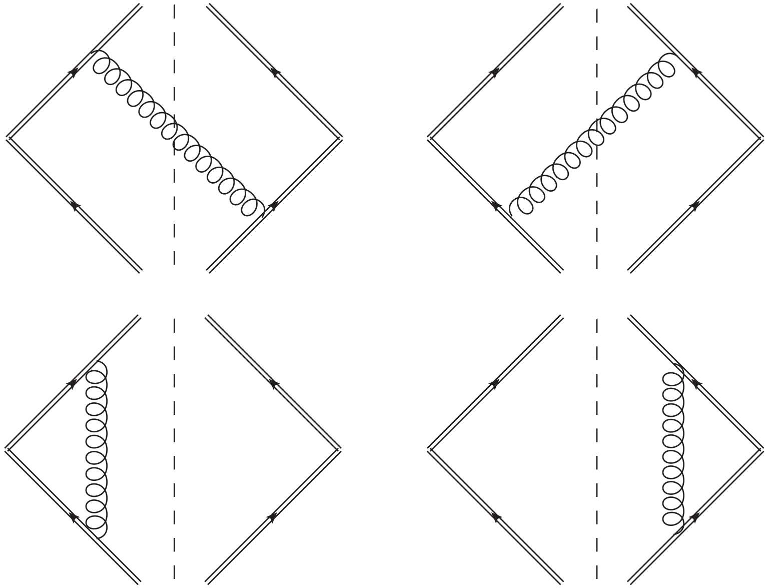
$$s_1 = -\frac{C_F\pi^2}{3}$$

Thrust Result

$$s_1^{\tau} = -C_F\pi^2$$

$$s_2 = -43 \pm 1$$

(From later fit to EVENT2)



Diagrams to calculate one loop soft function

Full Cross Section

From renormalon free definition of first moment

$$\frac{\mathrm{d}\sigma}{\mathrm{d}C} = \int \mathrm{d}k \left(\frac{\mathrm{d}\hat{\sigma}_{\mathrm{s}}}{\mathrm{d}C} + \frac{\mathrm{d}\hat{\sigma}_{\mathrm{ns}}}{\mathrm{d}C} \right) \left[C - \frac{6k}{Q} \right] F_{\widetilde{C}} \left(k - \frac{3\pi}{6} \underset{\downarrow}{\bar{\Delta}}(R, \mu_S) \right)$$

- Singular piece calculated from factorization theorem

$$\frac{\mathrm{d}\hat{\sigma}_{\mathrm{s}}}{\mathrm{d}C} \sim \frac{\alpha_s^k \ln^\ell \left(\frac{C}{6}\right)}{C}$$

At $\mathrm{N}^3\mathrm{LL}$

- Nonsingular piece calculated numerically to $\mathcal{O}(\alpha_s^3)$

$$\frac{\mathrm{d}\hat{\sigma}_{ns}}{\mathrm{d}C} = \frac{\alpha_s}{2\pi} f_1 + \left(\frac{\alpha_s}{2\pi}\right)^2 f_2 + \left(\frac{\alpha_s}{2\pi}\right)^3 f_3 + \ldots$$

- Shape function gives nonperturbative corrections

In tail region

$$\frac{1}{6} F\left(\frac{\ell}{6}\right) = \delta(\ell) - 3\pi\, \Omega_1^{gC}\, \delta'(\ell) + \ldots$$

[Adopting a notation that makes is easier to compare with previous talk by Vicent]

Resummation

$$\frac{1}{\sigma_0}\frac{\mathrm{d}\sigma_s}{\mathrm{d}C} = \frac{Q}{6} H(Q,\mu_H) U_H(\mu_H,\mu_J)\int \mathrm{d}s'\mathrm{d}s\, J_\tau(s',\mu_J) U_S(s-s',\mu_J,\mu_S) S_{\widetilde{C}}\left(\frac{QC}{6}-\frac{s}{Q},\mu_S\right)$$

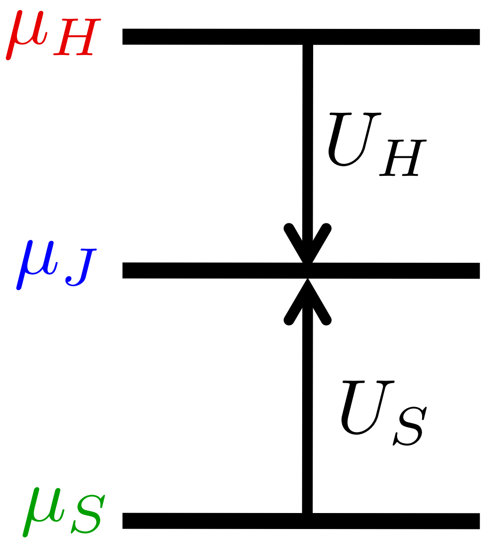
- $H(Q,\mu_H)$, $J_\tau(s,\mu_J)$, $S_{\widetilde{C}}(\ell,\mu_S)$ will have no more large logs
- Running U_H and U_S resum the large logs
- To get N^{3}LL need 3-loop anomalous dimension and 4-loop Γ_{cusp}

Padé Approximation ± 200%

- N^{3}LL + $\mathcal{O}(\alpha_s^3)$ needs 3-loop matching for

$H(Q,\mu_H)$, $J_\tau(s,\mu_J)$, $S_{\widetilde{C}}(\ell,\mu_S)$

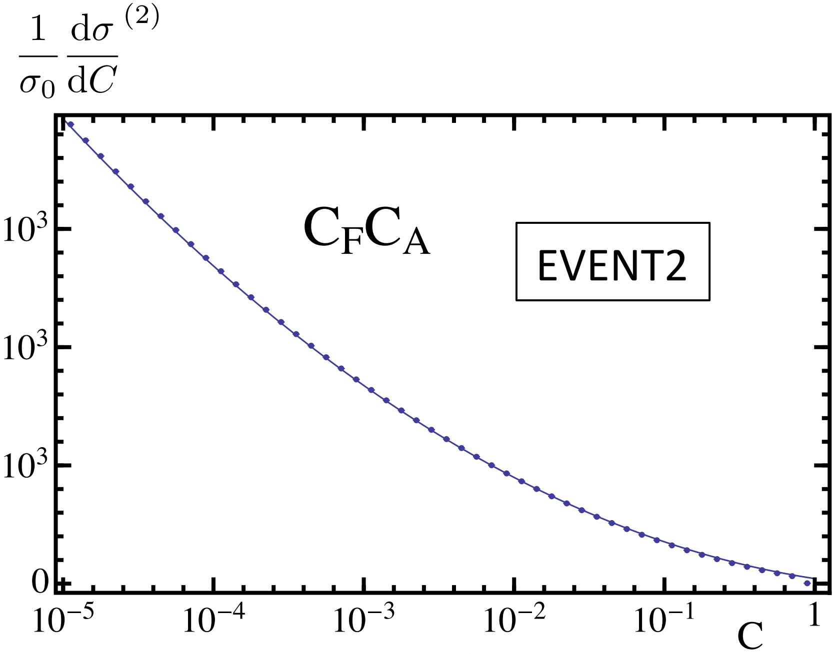
  - j_3, s_3 varied as part of uncertainty



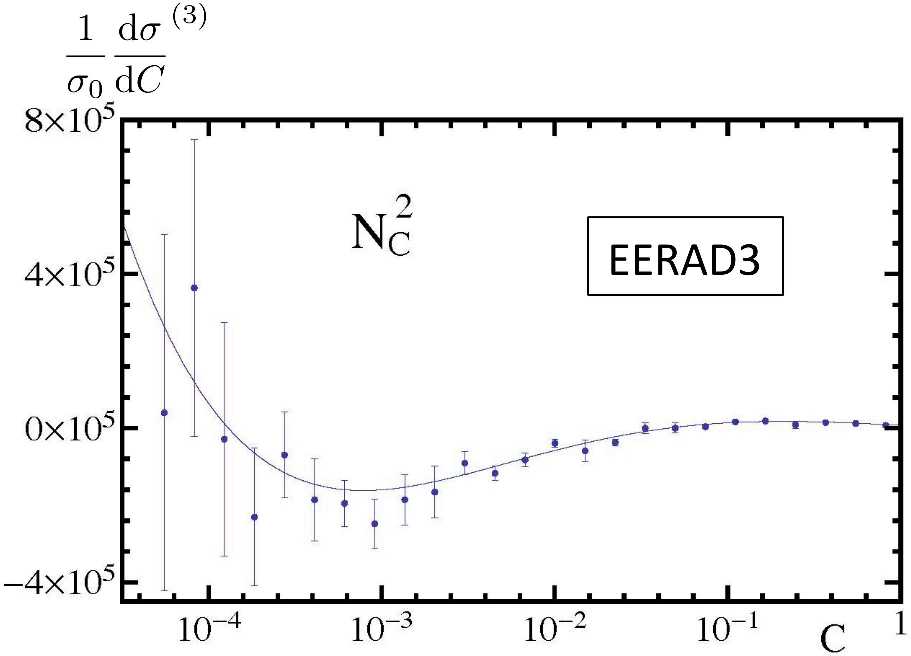
Nonsingular Distributions

$$\sigma_{\text{nonsingular}} = \sigma_{\text{full}}^{\text{Fixed Order}} - \sigma_{\text{singular}}^{\text{Fixed Order}}$$

First compare the EVENT2 and EERAD3 results with singular cross section very close to zero



$\mathcal{O}(\alpha_s^2)$

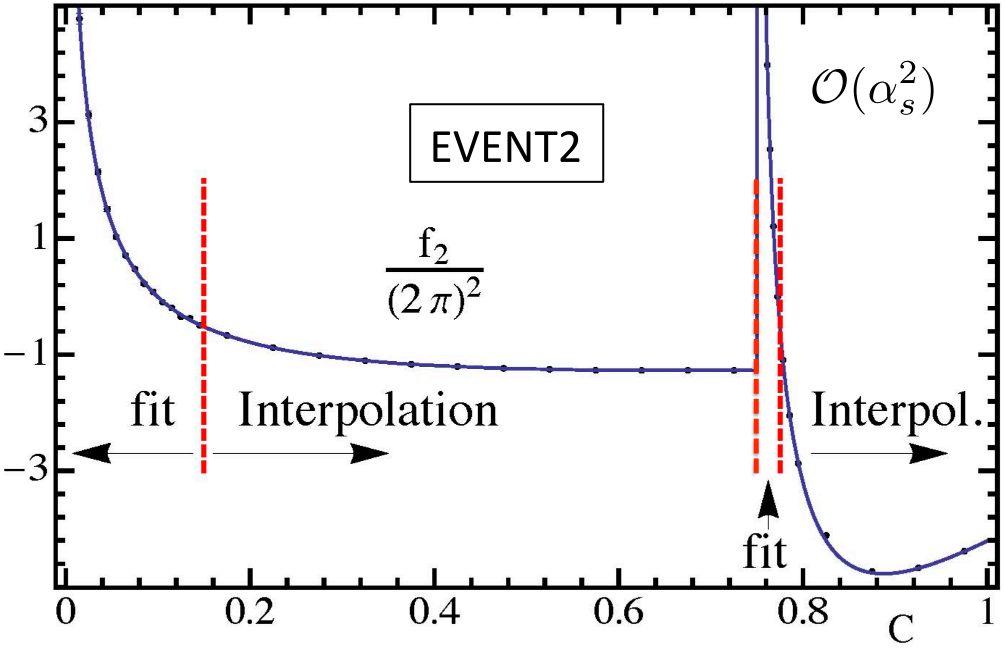


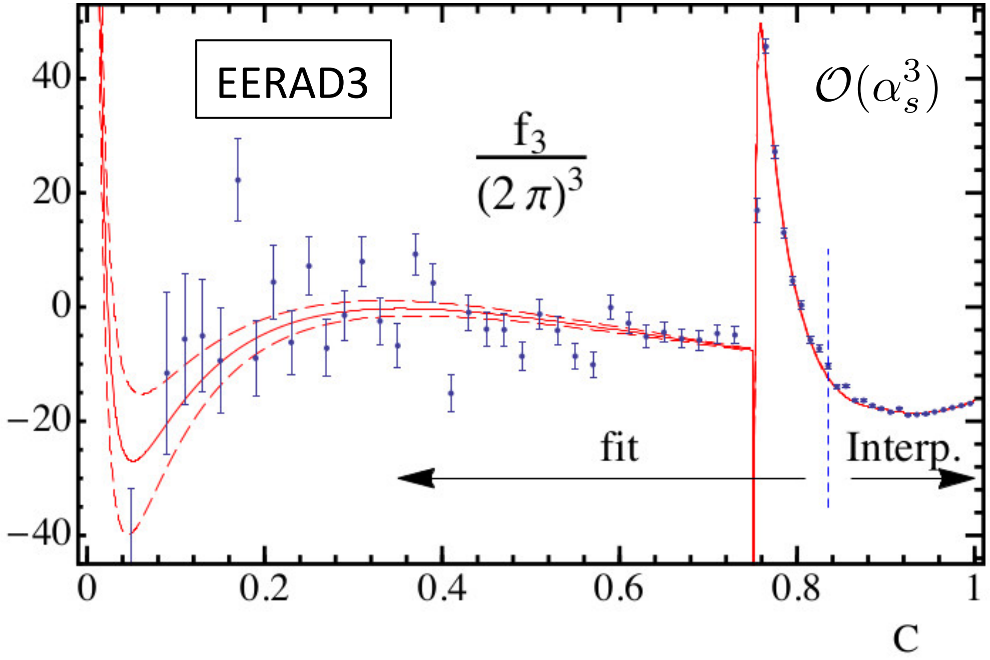
$\mathcal{O}(\alpha_s^3)$

Nonsingular Distributions

$$\sigma_{\rm ns} = \sigma^{\rm FO}_{\rm full} - \sigma^{\rm FO}_{\rm s}$$

- Exact at one loop
- Subtract singular from EVENT2 (3×10^{11} events) and EERAD3 data (Gehrmann)
- Use norm to fit for
  $$s_2 = -43 \pm 1$$
- Fit in low region
- Interpolate in tail
- Fit around shoulder at $C = 0.75$
  - More on this later



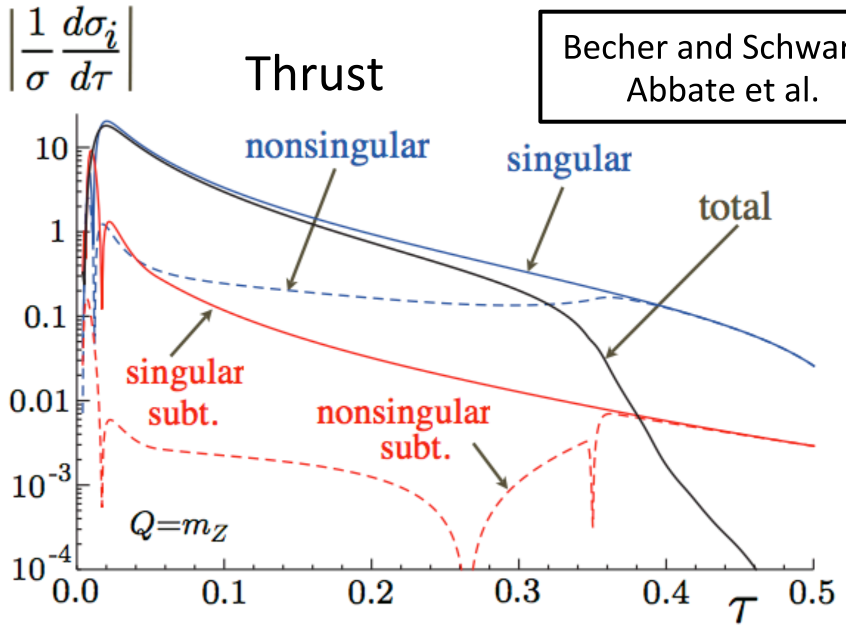


Nonsingular Distributions

$$\sigma_{\rm ns} = \sigma^{\rm FO}_{\rm full} - \sigma^{\rm FO}_{\rm s}$$

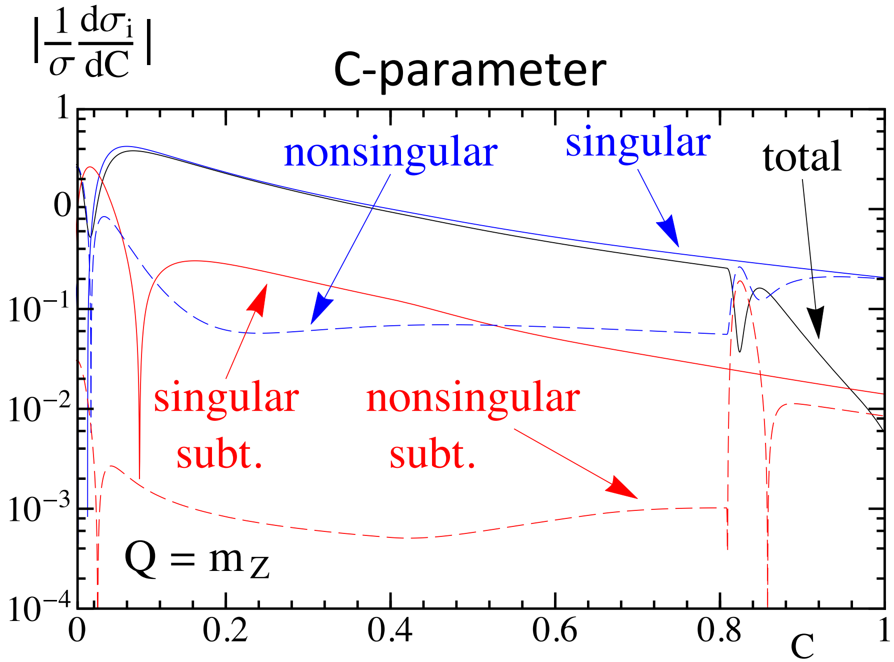
Contrast with thrust

- Nonsingular is smaller up to $C = 0.8$
- Singular dominant region is larger in C-parameter than thrust



Thrust

Becher and Schwartz, Abbate et al.

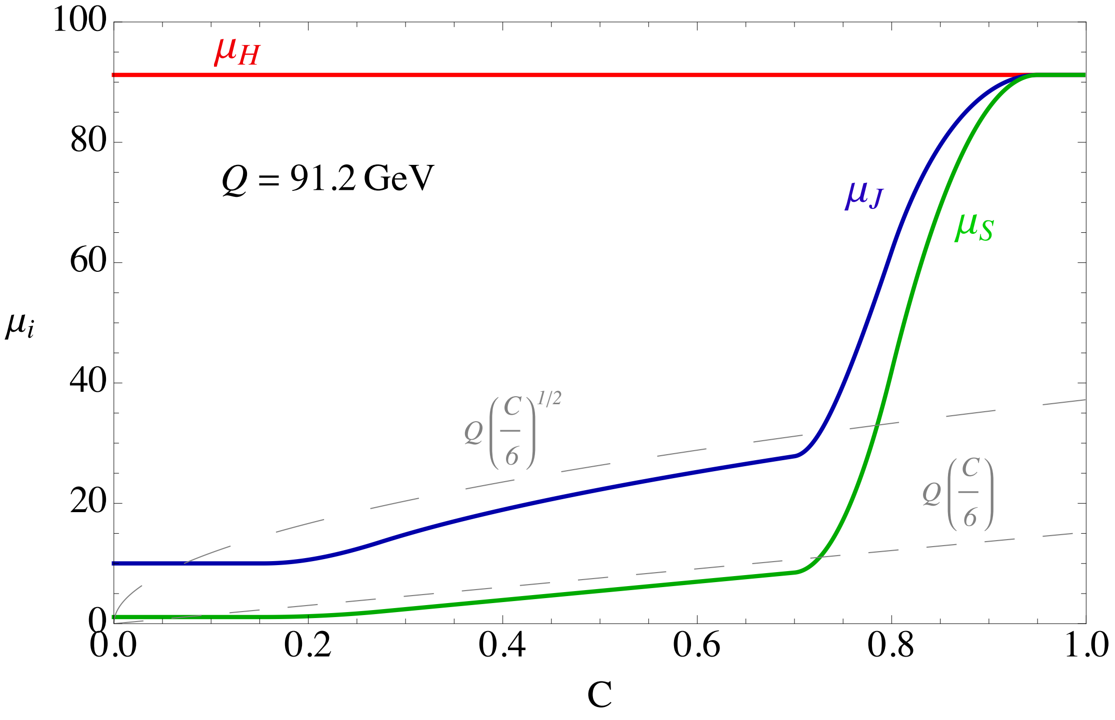


C-parameter

Profile Functions

Used by many authors

Pick scales to avoid large logs



$$\mu_H \sim Q$$

$$\mu_J \sim \sqrt{\mu_S \mu_H}$$

- Peak: $\mu_S \sim \Lambda_{\mathrm{QCD}}$
- Tail: $\mu_S \sim \frac{QC}{6}$

Much gentler slope than for thrust

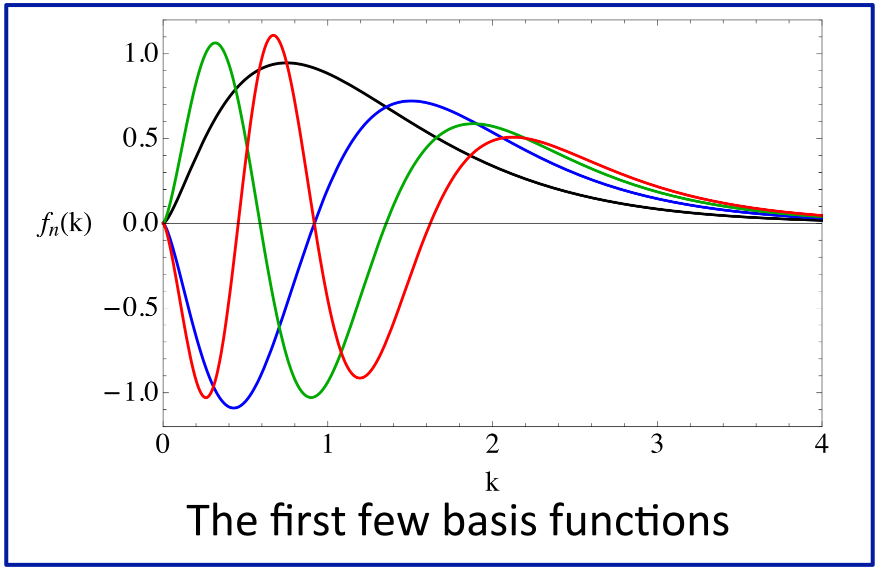
- Far-tail: $\mu_S \sim Q$

Shape Function

- Complete basis

Ligeti, Stewart, Tackmann

$$F(k,\lambda,\{c_i\}) = \frac{1}{\lambda}\left[\sum_{n=0}^{N} c_n f_n\left(\frac{k}{\lambda}\right)\right]^2$$



The first few basis functions

- First moment is $\Omega_1^C = 3\pi\,\Omega_1^{gC}$
- Cannot expand in peak region
- Can expand in tail

$$F_C(k,\lambda,\{c_i\},\mu) = \delta(k) - 3\pi\,\bar{\Omega}_1^{gC}(\lambda,\{c_i\},\mu)\delta'(k) + \ldots$$

- Consider two renormalization schemes:

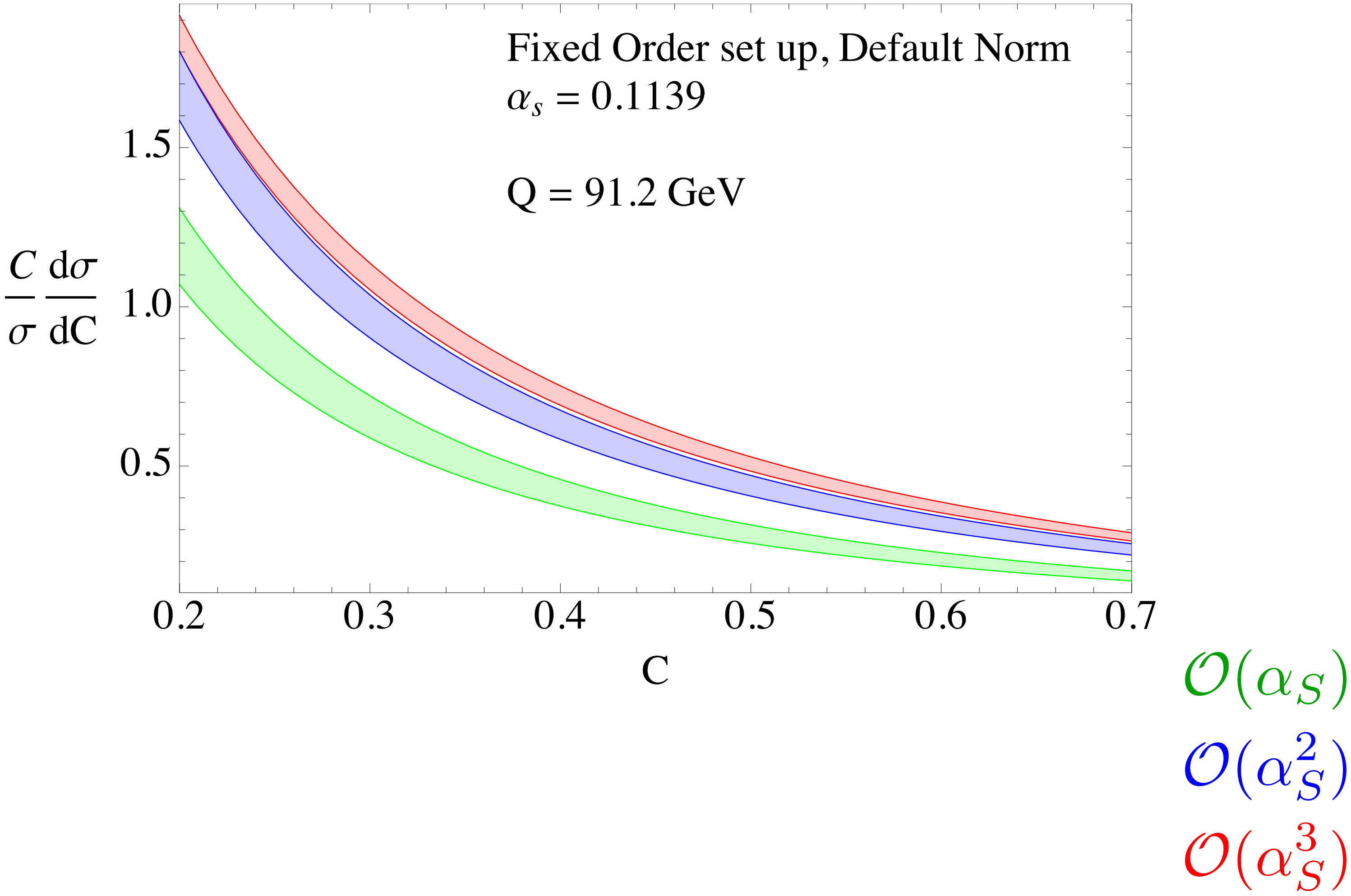
$\overline{\mathrm{MS}}$ scheme: $\bar{\Omega}_1^{gC}$

Renormalon free gap scheme: $\Omega_1^{gC}(R,\mu_S) = \bar{\Omega}_1^{gC} - \delta(R,\mu_S)$

$$\delta(R,\mu_S) = e^{\gamma_E} R \sum_k \alpha_s^k(\mu_S)\delta_k(R,\mu_S)\,, \quad \frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}C} = e^{-\left(\frac{3\pi\,\delta(R,\mu_S)}{Q}\frac{\partial}{\partial C}\right)}\frac{\mathrm{d}\sigma}{\mathrm{d}C}$$

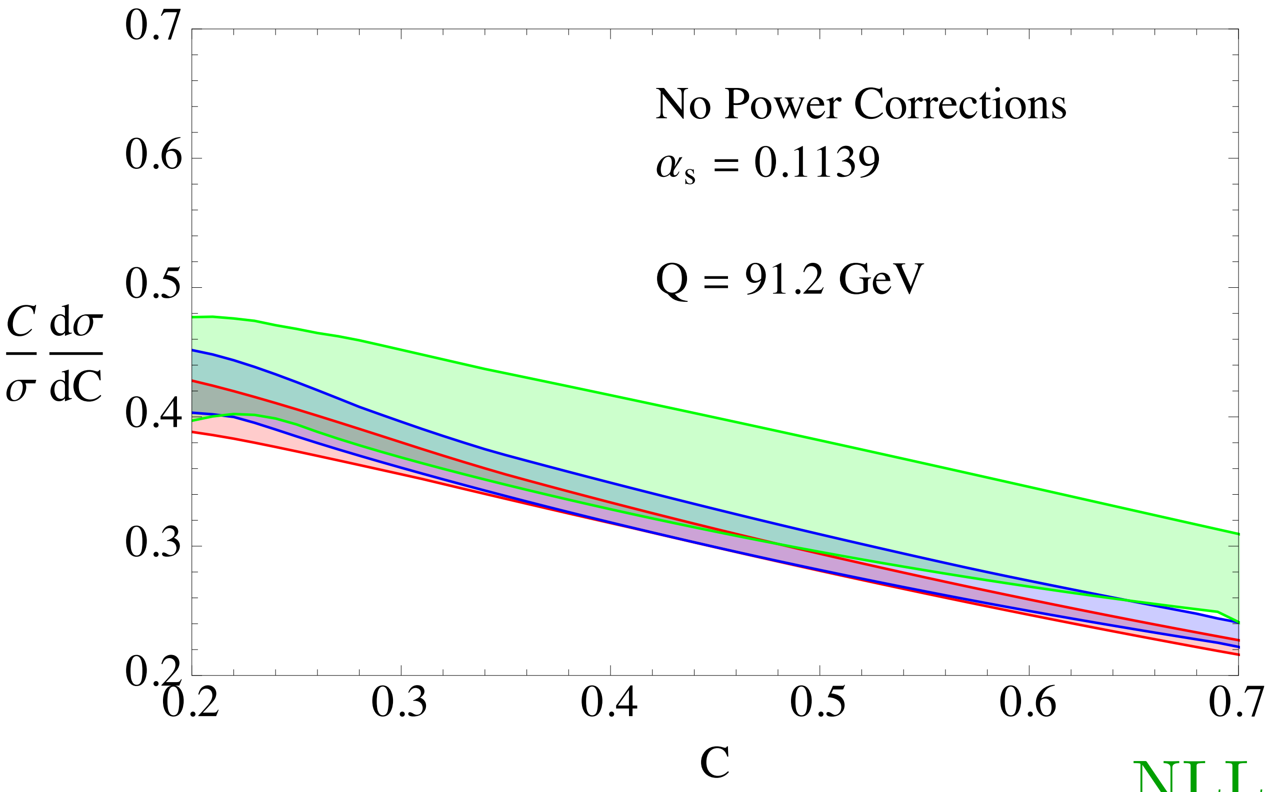
Full Distributions

Fixed Order



Full Distributions

Pure Resummation
(no nonperturbative shape function)



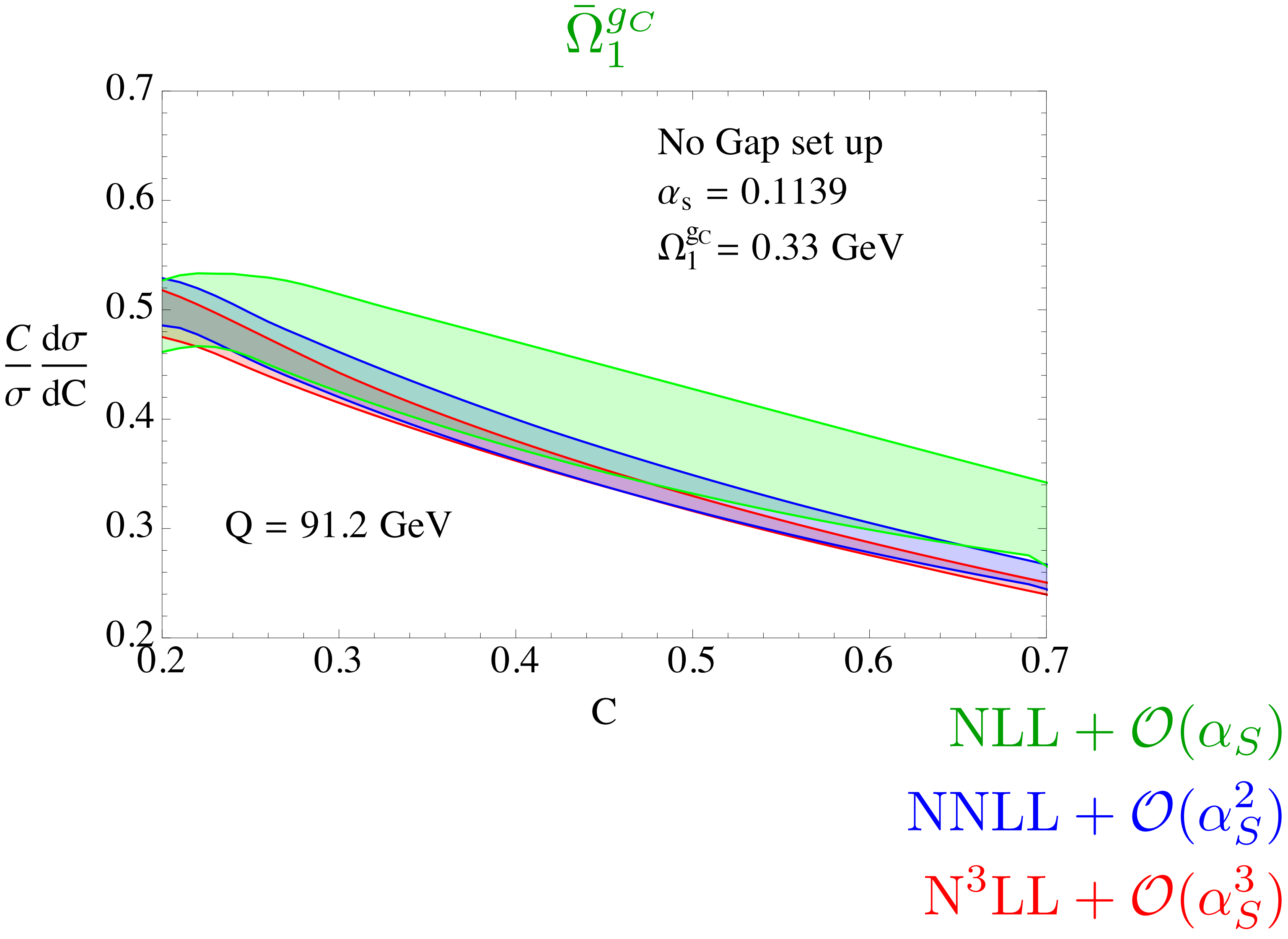
$\mathrm{NLL} + \mathcal{O}(\alpha_S)$

$\mathrm{NNLL} + \mathcal{O}(\alpha_S^2)$

$\mathrm{N^3LL} + \mathcal{O}(\alpha_S^3)$

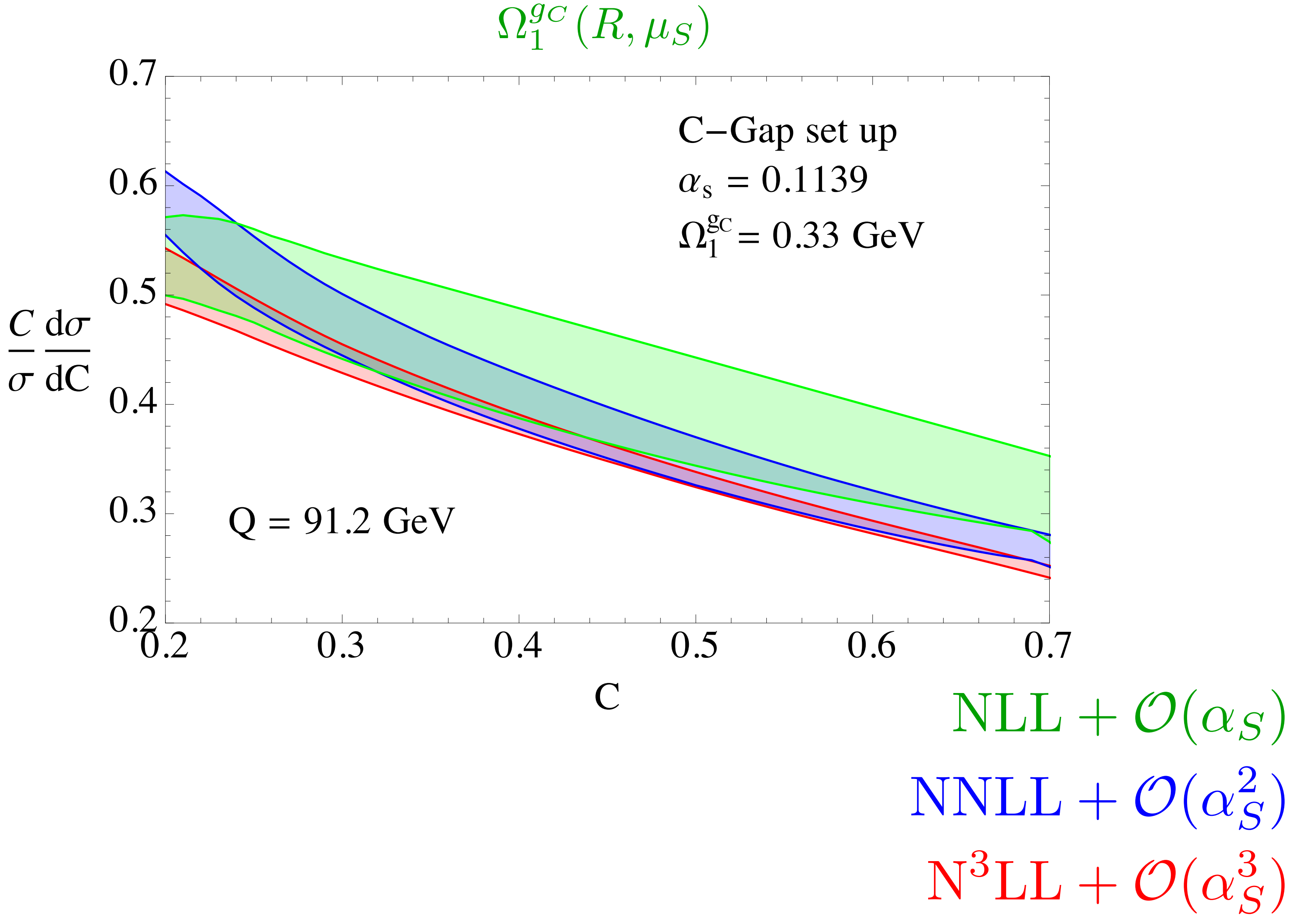
Full Distributions

Resummation and shape function



Full Distributions

Full result: Resummation, Shape function and renormalon free scheme

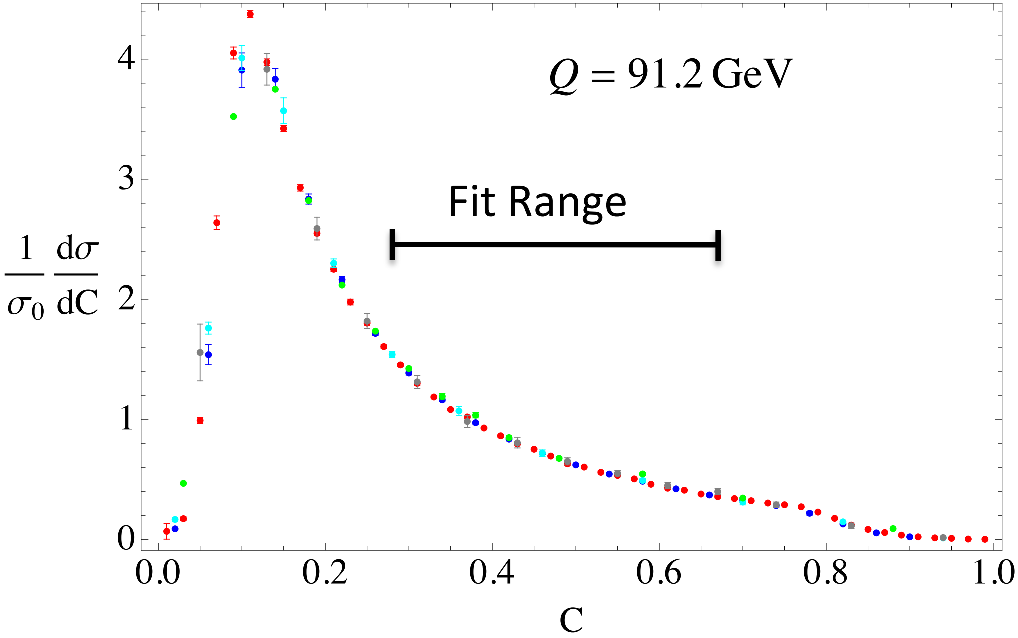


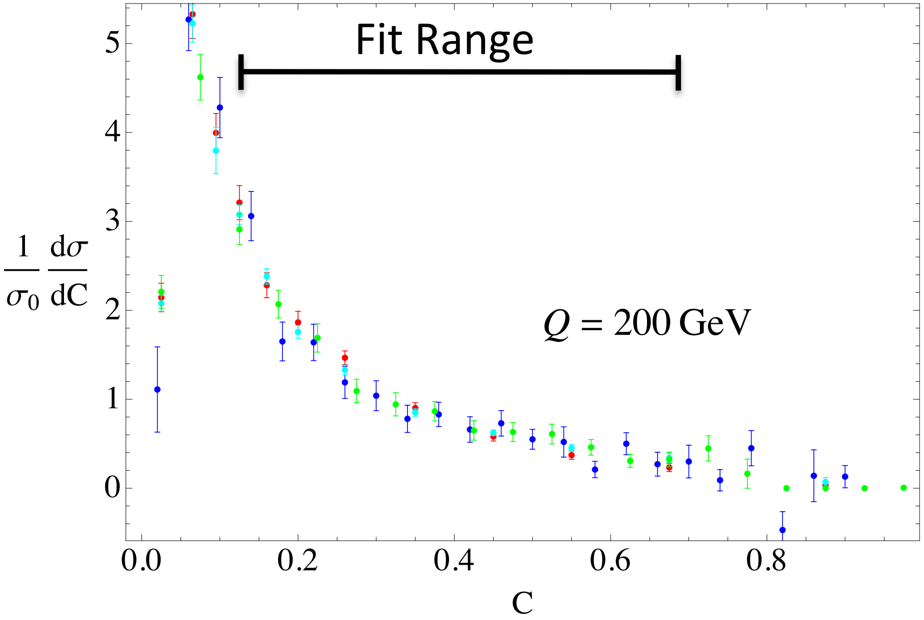
Data

- Six experiments:
  ALEPH, DELPHI, OPAL ,
  L3, SLD and JADE
- Center of mass energies:
  Q = 35 GeV – 207 GeV
- Default fit range:

$$\frac{25\,\mathrm{GeV}}{Q} \leq C \leq 0.7$$

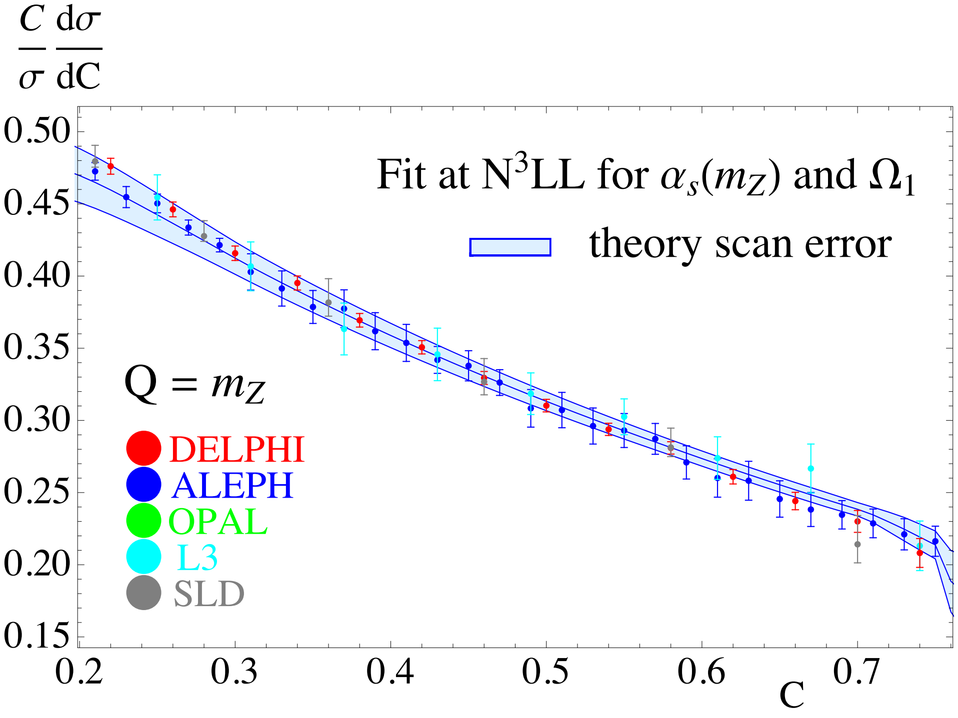
- 404 bins in default fit region

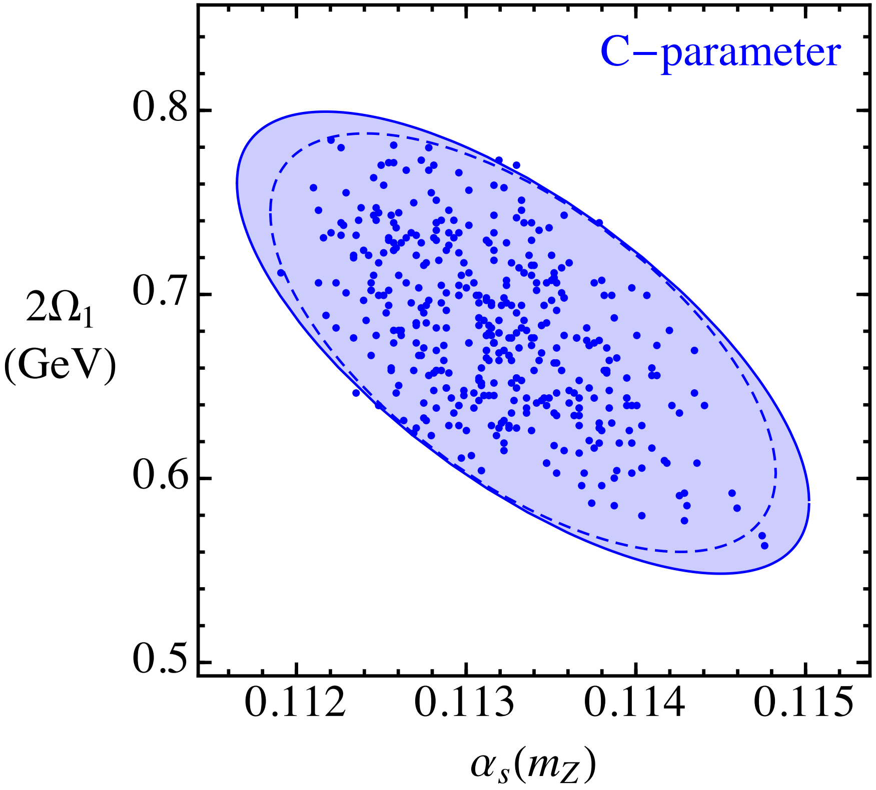




Fit in the tail

Fit for Ω_1^{gC} and $\alpha_s(m_Z)$ simultaneously





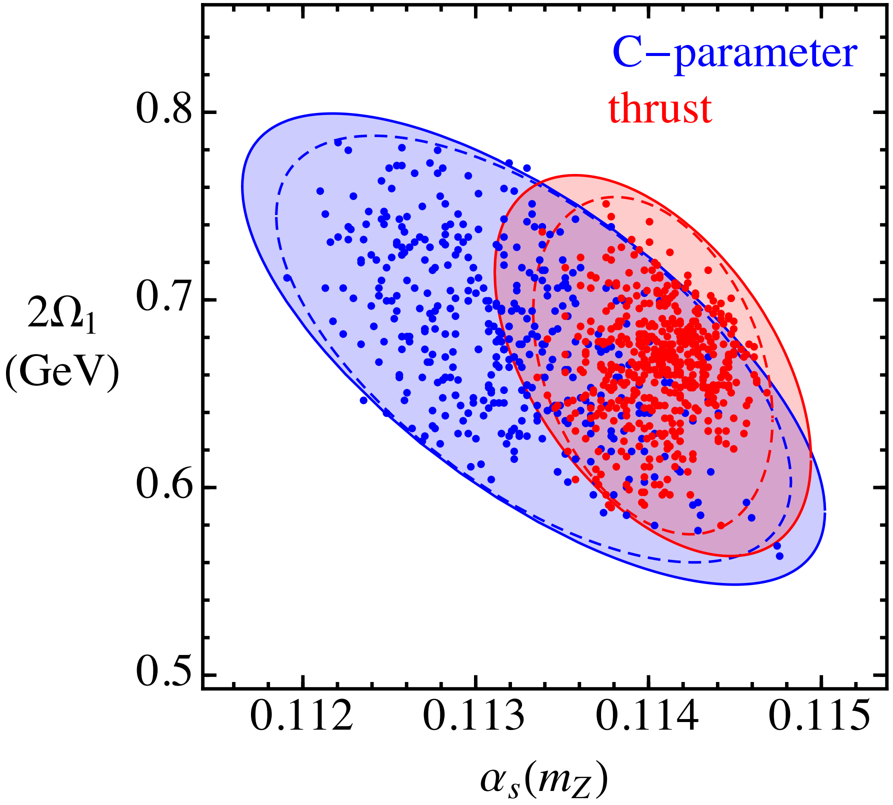
$$\boxed{\alpha_s(m_z) = 0.1133 \pm 0.0016}$$

(Preliminary)

$$\boxed{\Omega_1^{gC} = 0.34 \pm 0.06\,\text{GeV}}$$

$$\boxed{\frac{\chi^2}{\text{d.o.f.}} = 0.97}$$

Comparison with Thrust Results



C-Parameter (pure QCD)

$$\alpha_s(m_z) = 0.1133 \pm 0.0016$$

$$\Omega_1^{gC} = 0.34 \pm 0.06\,\mathrm{GeV}$$

Thrust (pure QCD, Abbate et al.)

$$\alpha_s(m_z) = 0.1140 \pm 0.0011$$

$$\Omega_1^{g_\tau} = 0.33 \pm 0.05\,\mathrm{GeV}$$

Even including hadron mass effects,

$$\Omega_1^{gC} = \Omega_1^{g_\tau} + \mathcal{O}(10\%)$$

(Salam and Wicke;
Mateu, Stewart, Thaler)

Thrust (with QED and m_b)

$$\alpha_s(m_z) = 0.1135 \pm 0.0011$$

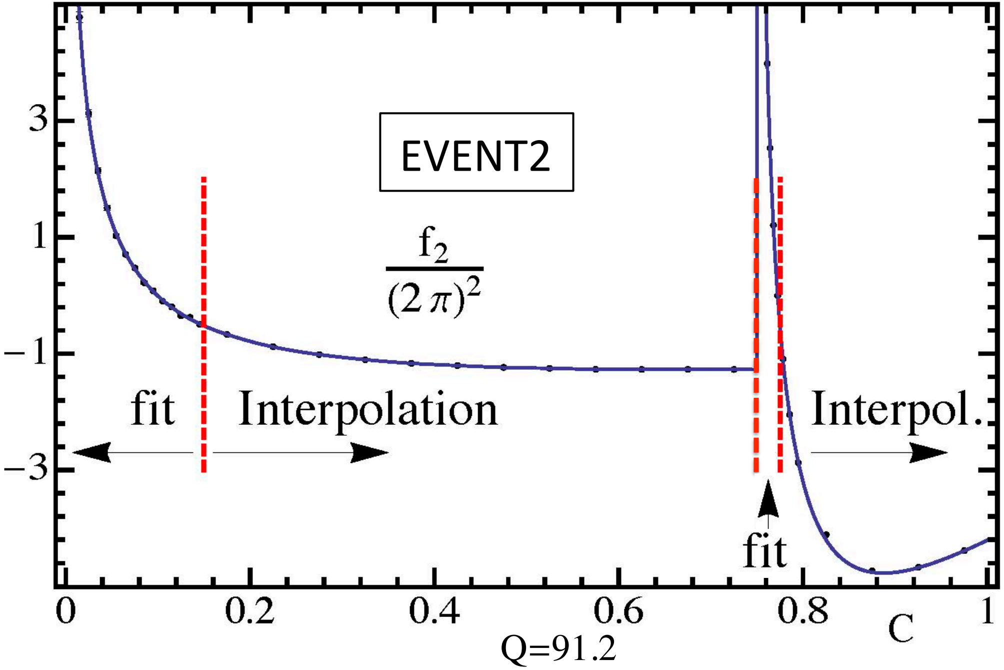
$$\Omega_1^{g_\tau} = 0.32 \pm 0.05\,\mathrm{GeV}$$

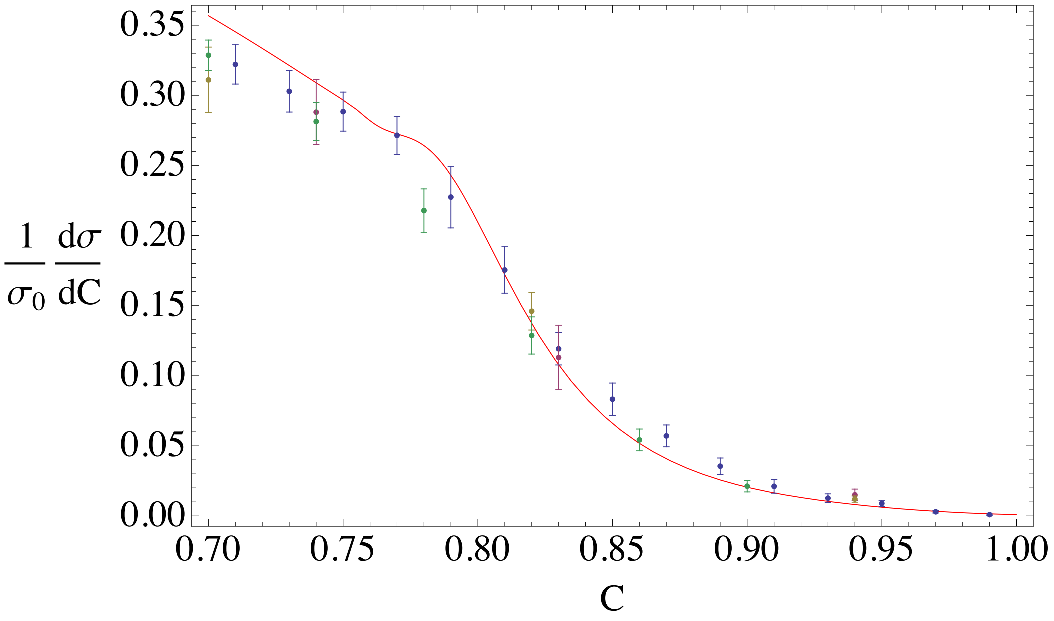
Shoulder Region

- Treat to order α_s^3
- Smear with F
- No subtractions
- This region is not in fit
- Proper treatment needs another resummation

$$\sim \alpha_s \left(\alpha_s \ln^2 \left(C - \frac{3}{4} \right) + \ldots \right)$$

(NLL exists, Catani and Webber)





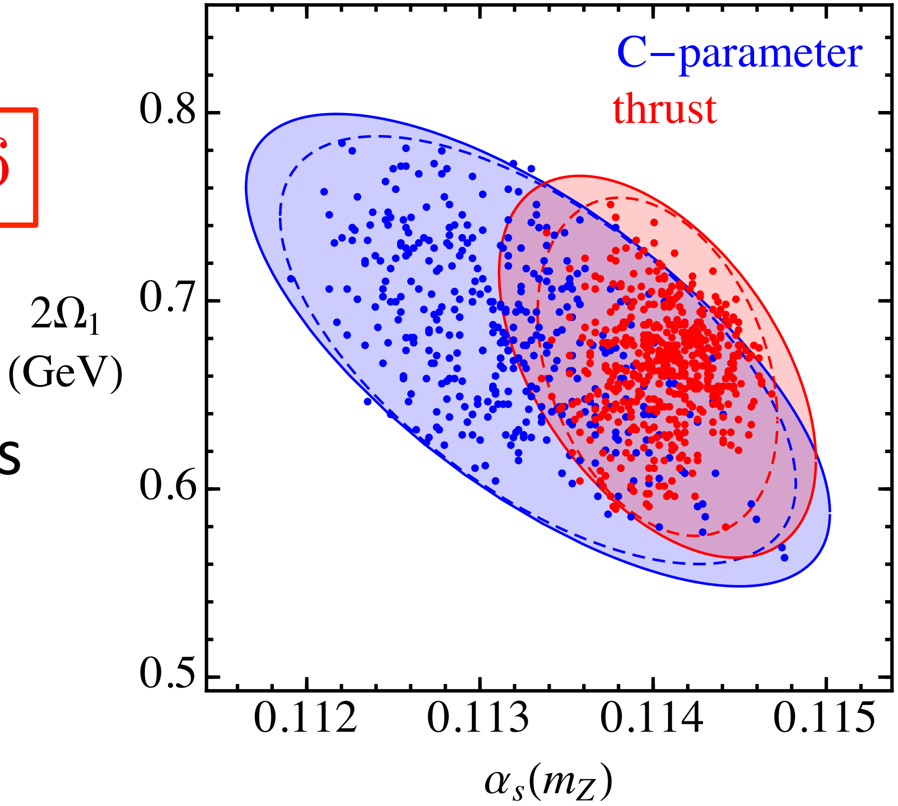
Conclusions

- Have used factorization for C-parameter to obtain
  - N^{3}LL + $\mathcal{O}(\alpha_s^3)$result
  - Field theory treatment of power corrections with renormalon subtractions
  - Valid treatment of all kinematic regions
- Global fit gives values

$$\boxed{\alpha_s(m_z) = 0.1133 \pm 0.0016}$$

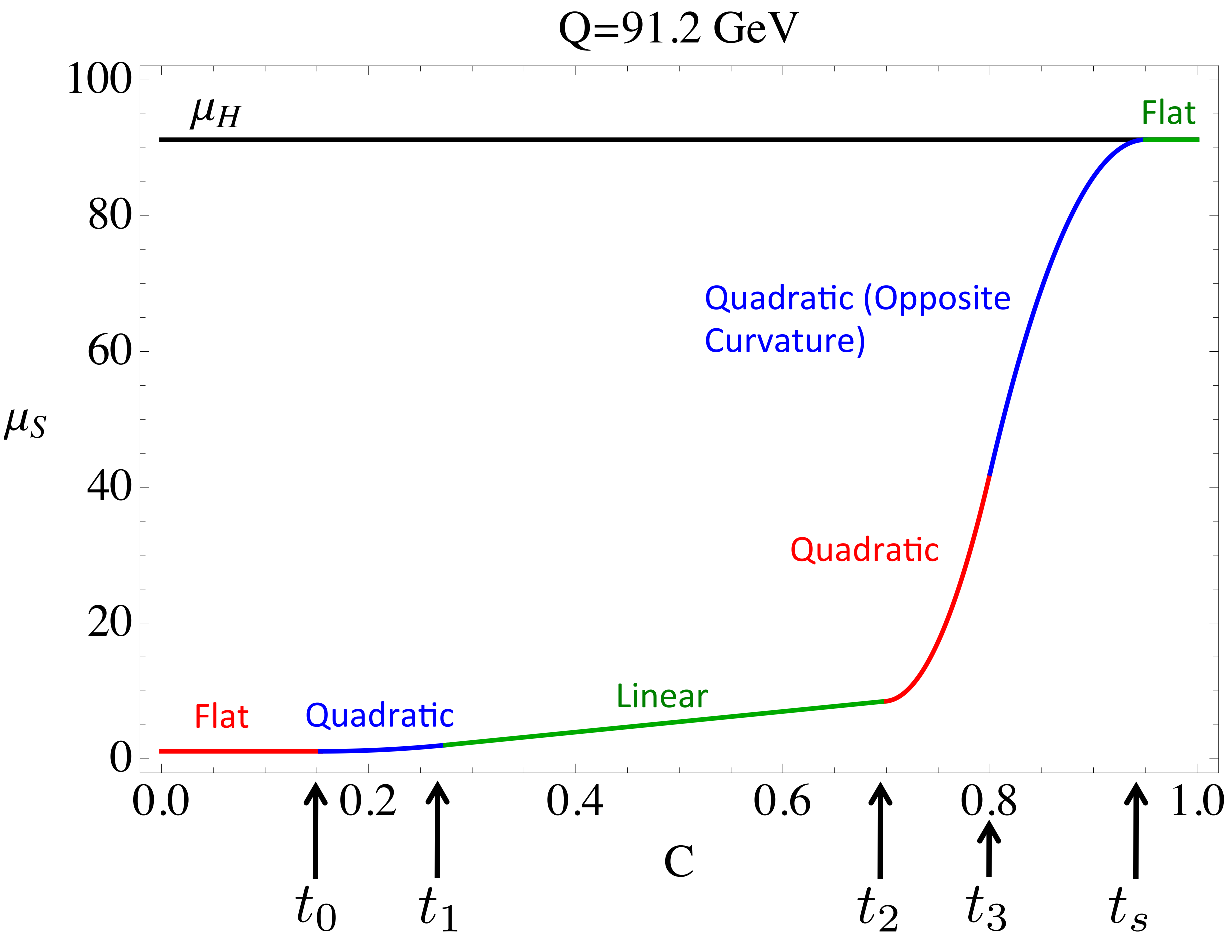
$$\boxed{\Omega_1^{gC} = 0.34 \pm 0.06\,\mathrm{GeV}}$$

- Agrees with earlier thrust results



Backup Slides

Profile Functions

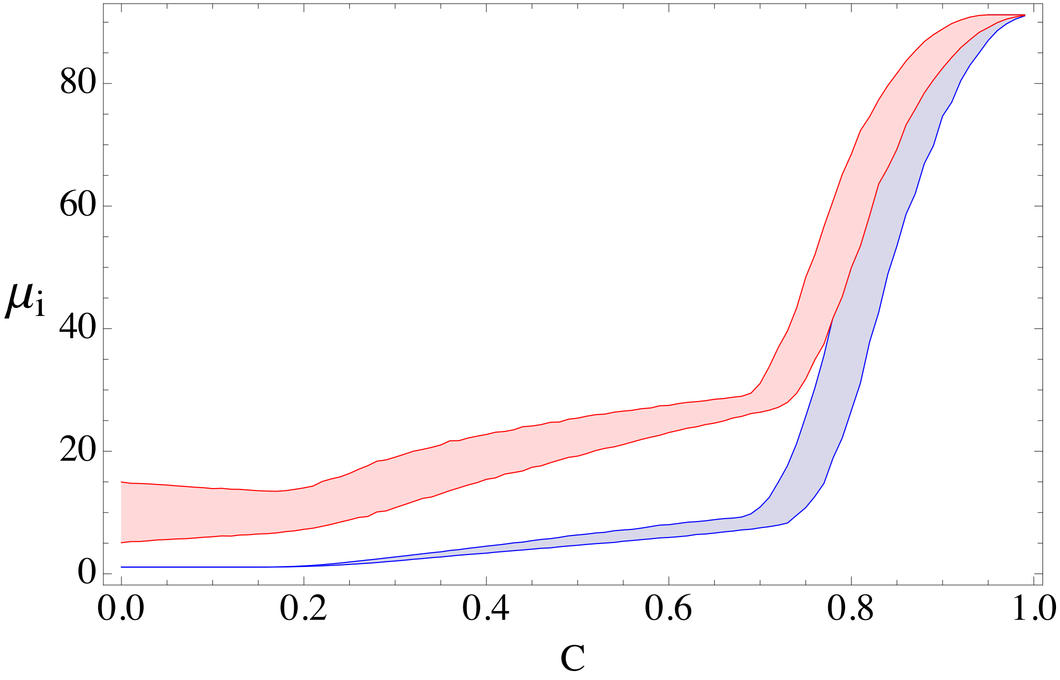


Parameter	Default Value
μ_0	$1.1\,\mathrm{GeV}$
r	$\frac{1}{6}$
t_0	$\frac{14\,\mathrm{GeV}}{Q}$
t_1	$\frac{25\,\mathrm{GeV}}{Q}$
t_2	0.7
t_3	0.8
t_s	0.95
$e_H = \frac{\mu_H}{Q}$	1

Parameters varied to get the perturbative uncertainty

Profile Variation for Fits

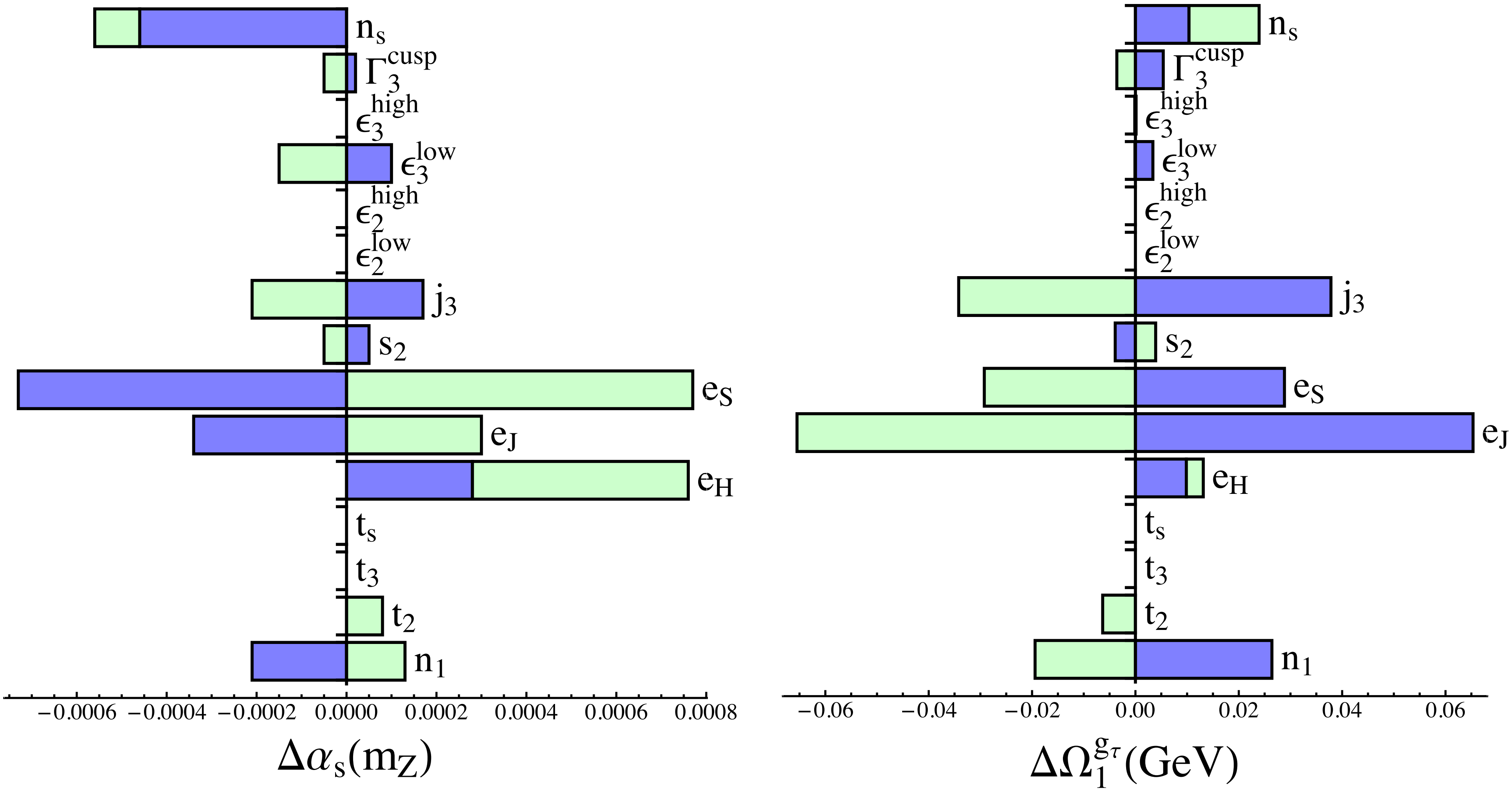
Variation of the soft (blue) and jet (red) scales at Q=91.2 GeV



Plus there is an additional up & down variation of all scales by factor of 2 & ½

Theory Error

Vary parameters in profile and model to estimate theory error



Preliminary