The S_3 flavour symmetry: quarks, leptons and Higgs sector*

A. Mondragón
Instituto de Física, UNAM

DISCRETE 2012
Lisbon, Portugal

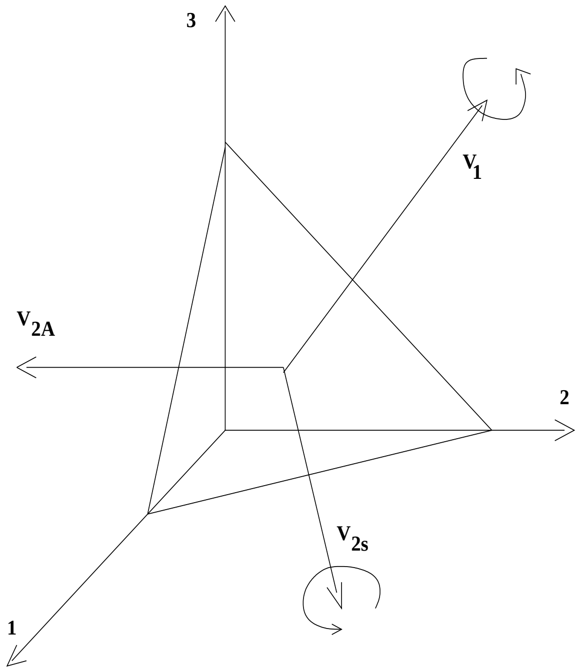
3-7 December 2012, Portugal.

*Work supported by DGAPA PAPIIT IN 113712 and CONACYT-MEXICO 132059

Contents

The Group S_3

The group S_3 of permutations of three objects



Permutations		Rotations
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$	$\Longleftrightarrow$	a $120^\circ-$ rotation around the invariant vector $\mathbf{V}_1$
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$	$\Longleftrightarrow$	a $180^\circ-$rotation around the invariant vector $\mathbf{V}_{2s}$

Symmetry adapted basis

$$|v_{2A}>= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \; |v_{2s}>= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \; |v_1>= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Irreducible representations of S_3

The group S_3 has two one-dimensional irreps (singlets) and one two-dimensional irrep (doublet)

- One dimensional: $\mathbf{1}_A$ antisymmetric singlet, $\mathbf{1}_s$ symmetric singlet
- Two - dimensional: 2 doublet

Direct product of irreps of S_3

$$\mathbf{1}_s \otimes \mathbf{1}_s = \mathbf{1}_s, \quad \mathbf{1}_s \otimes \mathbf{1}_A = \mathbf{1}_A, \quad \mathbf{1}_A \otimes \mathbf{1}_A = \mathbf{1}_s, \quad \mathbf{1}_s \otimes \mathbf{2} = \mathbf{2}, \quad \mathbf{1}_A \otimes \mathbf{2} = \mathbf{2}$$

$$\boxed{\mathbf{2} \otimes \mathbf{2} = \mathbf{1}_s \oplus \mathbf{1}_A \oplus \mathbf{2}}$$

the direct (tensor) product of two doublets

$$\mathbf{p_D} = \begin{pmatrix} p_{D1} \\ p_{D2} \end{pmatrix} \quad \text{and} \quad \mathbf{q}_D = \begin{pmatrix} q_{D1} \\ q_{D2} \end{pmatrix}$$

has two singlets, r_s and r_A, and one doublet r_D

$$r_s = p_{D1}q_{D1} + p_{D2}q_{D2} \quad \text{is invariant}, \quad r_A = p_{D1}q_{D2} - p_{D2}q_{D1} \quad \text{is not invariant}$$

$$r_D = \begin{pmatrix} p_{D1}q_{D2} + p_{D2}q_{D1} \\ p_{D1}q_{D1} - p_{D2}q_{D2} \end{pmatrix}$$

A Minimal S_3 invariant extension of the SM

The Higgs sector is modified,

$$\Phi \rightarrow H \quad = \quad (\Phi_1, \Phi_2, \Phi_3)^T$$

H is a reducible $\mathbf{1_s} \oplus 2$ rep. of S_3

$$H_s \quad = \quad \frac{1}{\sqrt{3}}\Big(\Phi_1 + \Phi_2 + \Phi_3\Big)$$

$$H_D \quad = \quad \begin{pmatrix} \frac{1}{\sqrt{2}}(\Phi_1 - \Phi_2) \\ \\ \frac{1}{\sqrt{6}}(\Phi_1 + \Phi_2 - 2\Phi_3) \end{pmatrix}$$

Quark, lepton and Higgs fields are

$$Q^T = (u_L, d_L), u_R, d_R, \qquad L^T = (\nu_L, e_L), e_R, \nu_R, \qquad H$$

All these fields have three species (flavours) and belong to a reducible $\mathbf{1}_s \oplus 2$ rep. of S_3

S_3 - invariant Yukawa interactions

Quarks and leptons

$$\begin{aligned}\mathcal{L}_{Y_{D,E}} &= -Y_1^{(d,e)}\overline{\Psi}_I^{(Q,E)}H_s\psi_{IR}^{(d,e)} - Y_3^{(d,e)}\overline{\Psi}_3^{(Q,E)}H_s\psi_{3R}^{(d,e)}\\ &- Y_2^{(d,e)}\left[\overline{\Psi}_I^{(Q,E)}\kappa_{IJ}H_1\psi_{JR}^{(d,e)} + \overline{\Psi}_I^{(Q,E)}\eta_{IJ}H_2\psi_{JR}^{(d,e)}\right]\\ &- Y_4\overline{\Psi}_3^{(Q,E)}H_I\psi_{IR}^{(d,e)} - Y_5\overline{\Psi}_I^{(Q,E)}H_I\psi_{3R} + h.c.\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_{Y_{\nu,U}} &= -Y_1^{(u,\nu)}\overline{\Psi}_I^{(Q,E)}(i\sigma_2)H_s^*\psi_{IR}^{(u,\nu)} - Y_3^{(u,\nu)}\overline{\Psi}_3^{(Q,E)}(i\sigma_2)H_s^*\psi_{3R}^{(u,\nu)}\\ &- Y_2^{(u,\nu)}\left[\overline{\Psi}_I^{(Q,E)}\kappa_{IJ}(i\sigma_2)H_1^*\psi_{JR}^{(u,\nu)} + \overline{\Psi}_I^{(Q,E)}\eta_{IJ}(i\sigma_2)H_2^*\psi_{JR}^{(u,\nu)}\right]\\ &- Y_4^{(u,\nu)}\overline{\Psi}_3^{(Q,E)}(i\sigma_2)H_I^*\psi_{IR}^{(u,\nu)} - Y_5^{(u,\nu)}\overline{\Psi}_I^{(Q,E)}(i\sigma_2)H_I^*\psi_{3R} + h.c.\end{aligned}$$

where

$$\kappa = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad I, J = 1, 2$$

Furthermore, the Majorana mass terms for the right handed neutrinos are

$$\mathcal{L}_M = -\nu_{IR}^T C \mathbf{M}_I \nu_{IR} - M_3 \nu_{3R}^T C \nu_{3R},$$

C is the charge conjugation matrix.

Mass matrices

We will assume that

$$< H_{D1} > \;=\; < H_{D2} > \;\neq 0 \quad \text{and} \quad < H_3 > \neq\; 0$$

and

$$< H_3 >^2 + < H_{D1} >^2 + < H_{D2} >^2 \;\approx\; \left(\frac{246}{2} GeV\right)^2$$

Then, the Yukawa interactions yield mass matrices of the generic form

$$\mathbf{M} = \begin{pmatrix} \mu_1 + \mu_2 & \mu_2 & \mu_5 \\ \mu_2 & \mu_1 - \mu_2 & \mu_5 \\ \mu_4 & \mu_4 & \mu_3 \end{pmatrix}$$

The Majorana masses for ν_L are obtained from the see-saw mechanism

$$M_{\nu_L} = M_{\nu_D} \tilde{\mathsf{M}}^{-1} (M_{\nu_D})^T \qquad \text{with} \qquad \tilde{\mathsf{M}} = \mathrm{diag}(M_1, M_2, M_3)$$

Mixing matrices

The mass matrices are diagonalized by unitary matrices

$$U^\dagger_{d(u,e)L}\mathbf{M}_{d(u,e)}\mathbf{U}_{d(u,e)R} \quad = \quad \text{diag}\Big(m_{d(u,e)},\ m_{s(c,\mu)},\ m_{b(t,\tau)}\Big)$$

and

$$U^T_\nu M_\nu U_\nu \quad = \quad \text{diag}\Big(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}\Big)$$

The masses can be complex, and so, U_{eL} is such that

$$U^\dagger_{eL} M_e M^\dagger_e U_{eL} \quad = \quad \text{diag}\Big(|m_e|^2, |m_\mu|^2, |m_\tau|^2\Big), \qquad etc.$$

The quark mixing matrix is

$$V_{CKM} \quad = \quad U^\dagger_{uL} U_{dL}$$

and, the neutrino mixing matrix is

$$\mathbf{V}_{PMNS} \quad = \quad U^\dagger_{eL} U_\nu$$

Quark mass matrices

The quark mass matrices are obtained from $M^{generic}$ by means of a 45° rotation and a shift of the mass origin

$$\hat{M}_q = U^{\dagger}_{45^{\circ}} M^{generic} U_{45^{\circ}} - \mu_o \mathbf{1}$$

After reparametrizing $\hat{M}_q$ in terms of its eigenvalues, we get

$$\hat{M}_{u,d} \begin{pmatrix} 0 & \sqrt{\frac{\tilde{m}_1^{(u,d)} \tilde{m}_2^{(u,d)}}{1-\delta_{(u,d)}}} e^{i\phi} & 0 \\ \sqrt{\frac{\tilde{m}_1^{(u,d)} \tilde{m}_2^{(u,d)}}{1-\delta_{(u,d)}}} e^{-i\phi} & \tilde{m}_1^{(u,d)} - \tilde{m}_2^{(u,d)} + \delta_{(u,d)} & \sqrt{\frac{\delta_{(u,d)}}{1-\delta_{(u,d)}} \xi_1^{(u,d)} \xi_2^{(u,d)}} \\ 0 & \sqrt{\frac{\delta_{(u,d)}}{1-\delta_{(u,d)}} \xi_1^{(u,d)} \xi_2^{(u,d)}} & 1 - \delta_{(u,d)} \end{pmatrix}$$

$$\tilde{m}_i^{(u,d)} = \frac{m_i^{(u,d)}}{m_3^{(u,d)}}, \quad \xi_1^{(u,d)} = 1 - \tilde{m}_i^{(u,d)} - \delta_{(u,d)}, \quad \xi_2^{(u,d)} = 1 + \tilde{m}_2^{(u,d)} = 1 + \tilde{m}_2^{(u,d)} - \delta_{(u,d)}$$

The matrix U_q

The quark mass matrices may be brought to diagonal form by means of a unitary transformation $U_q = P_q Q_q$

$$M_q = \left(P_q O_q\right) M_{q,diag} \left(O_q^T P_q\right)$$

P_q is a diagonal matrix of phases and

$$O_q = \begin{pmatrix} \left(\tilde{m}_{2q}\xi_1/D_1\right)^{1/2} & -\left(\tilde{m}_{1q}\xi_2/D_2\right)^{1/2} & \left(\frac{\tilde{m}_{1q}\tilde{m}_{2q}\delta_q}{D_3}\right)^{1/2} \\ \left[\tilde{m}_{1q}(1-\delta_q)\xi_1/D_1\right]^{1/2} & \left[\tilde{m}_{2q}(1-\delta_q)\xi_2/D_2\right]^{1/2} & \left[(1-\delta_q)\delta_q/D_3\right]^{1/2} \\ -\left[\tilde{m}_{1q}\xi_2\xi_3/D_1\right]^{1/2} & -\left[\tilde{m}_{2q}\xi_1\xi_3/D_2\right]^{1/2} & \left[\xi_1\xi_2/D_3\right]^{1/2} \end{pmatrix}$$

$$\begin{aligned} \tilde{m}_{iq} &= \frac{m_{iq}}{m_{3q}}, \quad \xi_1 = 1 - \tilde{m}_{1q} - \delta_q, \quad \xi_2 = 1 + \tilde{m}_{2q} - \delta_q \\ D_1 &= \left(1-\delta_q\right)\left(\tilde{m}_{1q} + \tilde{m}_{2q}\right)\left(1 - \tilde{m}_{1q}\right) \\ D_2 &= \left(1-\delta_q\right)\left(\tilde{m}_{1q} + \tilde{m}_{2q}\right)\left(1 + \tilde{m}_{2q}\right), \\ D_3 &= \left(1-\delta_q\right)\left(1 - \tilde{m}_{1q}\right)\left(1 + \tilde{m}_{2q}\right) \\ q &= u, d \end{aligned}$$

The quark mixing matrix V_{CKM}

After the diagonalization of the mass matrices M_q, one obtains the mixing matrix as

$$\mathbf{V}_{CKM}^{th} = O_u^T \mathbf{P}^{u-d} O_d$$

where

$$\mathbf{P}^{u-d} = diag\left[1, e^{i\phi}, e^{i\phi}\right], \quad \phi = \phi_u - \phi_d$$

In this way, the elements of the mixing matrix $V_{CKM}$are written as functions of the quark mass ratios and two free parameters. For example:

$$\begin{aligned}
\mathbf{V}_{us}^{th} &= -\Big(\frac{\tilde{m}_c\big(1-\tilde{m}_u-\delta_u\big)\tilde{m}_d\big(1+\tilde{m}_s-\delta_q\big)}{\big(1-\delta_u\big)\big(1-\tilde{m}_u\big)\big(\tilde{m}_c+\tilde{m}_u\big)\big(1-\delta_q\big)\big(1+\tilde{m}_s\big)\big(m_s+m_d\big)}\Big)^{1/2} \\
&+ \Big(\frac{\tilde{m}_u\tilde{m}_s}{\big(1-\tilde{m}_u\big)\big(\tilde{m}_c+\tilde{m}_u\big)\big(\tilde{m}_d+\tilde{m}_s\big)}\Big)^{1/2}\Big\{\Big(\frac{\big(1-\tilde{m}_u-\delta_u\big)\big(1+\tilde{m}_s-\delta_d\big)}{\big(1+\tilde{m}_s\big)}\Big)^{1/2} \\
&+ \Big(\frac{\big(1+\tilde{m}_c-\delta_u\big)\delta_u\big(1-\tilde{m}_d-\delta_d\big)\delta_d}{\big(1-\delta_u\big)\big(1-\delta_d\big)\big(1+\tilde{m}_s\big)}\Big)^{1/2}\Big\}e^{i\phi}
\end{aligned}$$

V_{CKM}^{th} from χ^2 minimization

We construct the χ^2 function as (in collaboration with F. González Canales, U.J. Saldaña Salazar and L. Velasco-Sevilla)

$$\chi^2 = \frac{\left(|V_{ud}^{\text{th}}| - |V_{ud}|\right)^2}{\sigma_{V_{ud}}^2} + \frac{\left(|V_{us}^{\text{th}}| - |V_{us}|\right)^2}{\sigma_{V_{us}}^2} + \frac{\left(|V_{ub}^{\text{th}}| - |V_{ub}|\right)^2}{\sigma_{V_{ub}}^2} + \frac{\left(\mathcal{J}_q^{\text{th}} - \mathcal{J}_q\right)^2}{\sigma_{\mathcal{J}_q}^2},$$

In the fitting procedure, the mass ratios in V_{CKM}^{th} are not free parameters !!!

We took values for $\tilde{m}_i = m_i/m_{3q}$ in their allowed 3σ regions

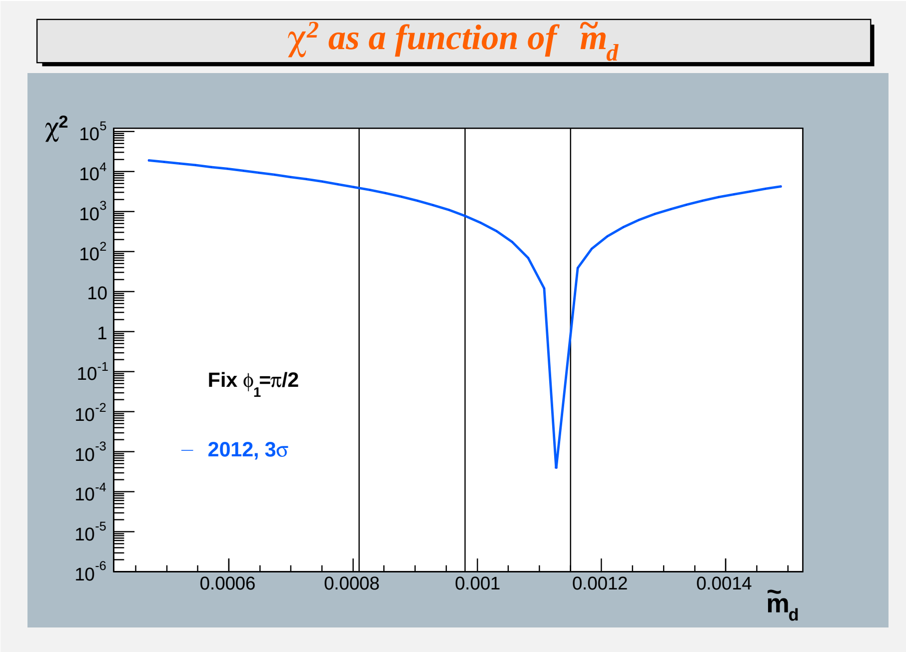
	2012
$\widetilde{m}_u\,(M_Z)$	0.0000083 ± 0.0000030
$\widetilde{m}_c\,(M_Z)$	0.0037 ± 0.00008
$\widetilde{m}_d\,(M_Z)$	0.00098 ± 0.00017
$\widetilde{m}_s\,(M_Z)$	0.0190 ± 0.0014

Table 1: Values of the mass ratios at M_Z.

We fitted our theoretical expressions to the following experimental values

$$|V_{ud}^{PDG}| = 0.97427 \pm 0.00015, \quad |V_{us}^{PDG}| = 0.2253 \pm 0.007,$$
$$|V_{ub}^{PDG}| = 0.00351 \pm 0.00015, \quad J^{PDG} = (2.96 \pm 0.18) \times 10^{-5},$$

Results from a χ^2 fit of V_{CKM}^{th} to V_{CKM}^{exp}



The best values of $|V_{ud}^{th}|$, $|V_{us}^{th}|$, $|V_{ub}^{th}|$ and J^{th}, obtained from a χ^2 fit to the experimental values, as given in PDG (2012), are shown below

$$|V_{ud}^{th}| = 0.974277 \pm 0.00020, \quad |V_{us}^{th}| = 0.22533 \pm 0.009,$$
$$|V_{ub}^{th}| = 0.00331 \pm 0.00019, \quad J^{th} = (2.63 \pm 0.33) \times 10^{-5},$$

$$|V_{ud}^{PDG}| = 0.97427 \pm 0.00015, \quad |V_{us}^{PDG}| = 0.2253 \pm 0.007, \quad |V_{ub}^{PDG}| = 0.00351 \pm 0.00015$$
$$J_q^{PDG} = 2.96 \pm 0.18$$

We find an excellent agreement between our theoretical V_{CKM}^{th} and V_{CKM}^{PDG} (2012).

The leptonic sector

To achieve a further reduction of the number of parameters, in the leptonic sector we introduce an additional discrete Z_2 symmetry

$-$	$+$
$H_I,\ \nu_{3R}$	$H_S,\ L_3,\ L_I,\ e_{3R},\ e_{IR},\ \nu_{IR}$

then,

$$Y_1^e = Y_3^e = Y_1^\nu = Y_5^\nu = 0$$

Hence, the leptonic mass matrices are

$$M_e = \begin{pmatrix} \mu_2^e & \mu_2^e & \mu_5^e \\ \mu_2^e & -\mu_2^e & \mu_5^e \\ \mu_4^e & \mu_4^e & 0 \end{pmatrix} \qquad \text{and} \qquad M_{\nu D} = \begin{pmatrix} \mu_2^\nu & \mu_2^\nu & 0 \\ \mu_2^\nu & -\mu_2^\nu & 0 \\ \mu_4^\nu & \mu_4^\nu & \mu_3^\nu \end{pmatrix}$$

The Mass Matrix of the charged leptons as function of its eigenvalues

The mass matrix of the charged leptons is

$$M_e \approx m_\tau \begin{pmatrix} \frac{1}{\sqrt{2}}\frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & \frac{1}{\sqrt{2}}\frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & \frac{1}{\sqrt{2}}\sqrt{\frac{1+x^2-\tilde{m}_\mu^2}{1+x^2}} \\ \frac{1}{\sqrt{2}}\frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & -\frac{1}{\sqrt{2}}\frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & \frac{1}{\sqrt{2}}\sqrt{\frac{1+x^2-\tilde{m}_\mu^2}{1+x^2}} \\ \frac{\tilde{m}_e(1+x^2)}{\sqrt{1+x^2-\tilde{m}_\mu^2}}e^{i\delta_e} & \frac{\tilde{m}_e(1+x^2)}{\sqrt{1+x^2-\tilde{m}_\mu^2}}e^{i\delta_e} & 0 \end{pmatrix}.$$

$x = m_e/m_\mu$, $\tilde{m}_\mu = m_\mu/m_\tau$ and $\tilde{m}_e = m_e/m_\tau$

This expression is accurate to order 10^{-9} in units of the τ mass

There are no free parameters in $\mathbf{M}_e$ other than the Dirac Phase δ_D!!

The Unitary Matrix U_{eL}

The unitary matrix U_{eL} is calculated from

$$U_{eL}^{\dagger} M_e M_{eL}^{\dagger} U_{eL} = diag\left(m_e^2, m_\mu^2, m_\tau^2\right)$$

We find

$$U_{eL} = \Phi_{eL} O_{eL}, \qquad \Phi_{eL} = diag\left[1, 1, e^{i\delta_D}\right]$$

and

$$\mathbf{O}_{eL} \approx \begin{pmatrix} \frac{1}{\sqrt{2}} x \frac{(1+2\tilde{m}_\mu^2+4x^2+\tilde{m}_\mu^4+2\tilde{m}_e^2)}{\sqrt{1+\tilde{m}_\mu^2+5x^2-\tilde{m}_\mu^4-\tilde{m}_\mu^6+\tilde{m}_e^2+12x^4}} & -\frac{1}{\sqrt{2}} \frac{(1-2\tilde{m}_\mu^2+\tilde{m}_\mu^4-2\tilde{m}_e^2)}{\sqrt{1-4\tilde{m}_\mu^2+x^2+6\tilde{m}_\mu^4-4\tilde{m}_\mu^6-5\tilde{m}_e^2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} x \frac{(1+4x^2-\tilde{m}_\mu^4-2\tilde{m}_e^2)}{\sqrt{1+\tilde{m}_\mu^2+5x^2-\tilde{m}_\mu^4-\tilde{m}_\mu^6+\tilde{m}_e^2+12x^4}} & \frac{1}{\sqrt{2}} \frac{(1-2\tilde{m}_\mu^2+\tilde{m}_\mu^4)}{\sqrt{1-4\tilde{m}_\mu^2+x^2+6\tilde{m}_\mu^4-4\tilde{m}_\mu^6-5\tilde{m}_e^2}} & \frac{1}{\sqrt{2}} \\ -\frac{\sqrt{1+2x^2-\tilde{m}_\mu^2-\tilde{m}_e^2}(1+\tilde{m}_\mu^2+x^2-2\tilde{m}_e^2)}{\sqrt{1+\tilde{m}_\mu^2+5x^2-\tilde{m}_\mu^4-\tilde{m}_\mu^6+\tilde{m}_e^2+12x^4}} & -x \frac{(1+x^2-\tilde{m}_\mu^2-2\tilde{m}_e^2)\sqrt{1+2x^2-\tilde{m}_\mu^2-\tilde{m}_e^2}}{\sqrt{1-4\tilde{m}_\mu^2+x^2+6\tilde{m}_\mu^4-4\tilde{m}_\mu^6-5\tilde{m}_e^2}} & \frac{\sqrt{1+x^2}\tilde{m}_e\tilde{m}_\mu}{\sqrt{1+x^2-\tilde{m}_\mu^2}} \end{pmatrix},$$

$x = m_e/m_\mu$, $\tilde{m}_\mu = m_\mu/m_\tau$ and $\tilde{m}_e = m_e/m_\tau$

The neutrino mass matrix I

The Majorana masses for ν_L are obtained from the see-saw mechanism

$$\mathbf{M}_\nu = \mathbf{M}_{\nu D} \tilde{\mathbf{M}}_R^{-1} \mathbf{M}_{\nu D}^T$$

with

$$\tilde{\mathbf{M}}_R = diag\left[M_1, M_2, M_3\right] \qquad M_1 \neq M_2 \neq M_3$$

and

$$\mathbf{M}_{\nu D} \quad = \quad \begin{pmatrix} \mu_2 & \mu_2 & 0 \\ \mu_2 & -\mu_2 & 0 \\ \mu_4 & \mu_4 & \mu_3 \end{pmatrix}$$

Then

$$M_\nu = \left(\begin{array}{ccc} \frac{2\mu_2^2}{\bar{M}} & \lambda\frac{2\mu_2^2}{\bar{M}} & \frac{2\mu_2\mu_4}{\bar{M}} \\ \lambda\frac{2\mu_2^2}{\bar{M}} & \frac{2\mu_2^2}{\bar{M}} & \lambda\frac{2\mu_2\mu_4}{\bar{M}} \\ \frac{2\mu_2\mu_4}{\bar{M}} & \lambda\frac{2\mu_2\mu_4}{\bar{M}} & \frac{2\mu_4^2}{\bar{M}} + \frac{\mu_3^2}{\bar{M}} \end{array} \right).$$

$$\frac{1}{\bar{M}} = \frac{1}{2}\left(\frac{1}{M_1} + \frac{1}{M_2}\right) \quad \text{and} \quad \lambda = \frac{M_2 - M_1}{M_1 + M_1}$$

The neutrino mass matrix II

When $\bar{M}$ is real, the phases in M_ν may be factorized out as

$$M_\nu = P_\nu \bar{M}_\nu P_\nu \tag{1}$$

where

$$P_\nu = e^{i\phi} diag\left[1, 1, e^{i\delta_\nu}\right] \tag{2}$$

Then, the real symmetric matrix $\bar{M}_\nu$ may be diagonalized by means of an orthogonal matrix O_ν

$$M_\nu = P_\nu O_\nu \left[diag(|m_{1\nu}|, |m_{2\nu}|, |m_{3\nu}|)\right] O^T P_\nu \tag{3}$$

Therefore, the unitary matrix

$$U_\nu = P_\nu O_\nu \tag{4}$$

diagonalizes the neutrino mass matrix M_ν.

The neutrino mass matrix III

The reduced neutrino mass matrix $\bar{M}_\nu$ is reparametrized in terms of the neutrino masses $|m_i|$ and two free parameters μ_o and d, as

$$\bar{M}_\nu = \begin{pmatrix} \mu_o + \frac{1}{2}d & \frac{1}{2}d & \frac{1}{\sqrt{2}}(a+c) \\ \frac{1}{2}d & \mu_o + \frac{1}{2}d & -\frac{1}{\sqrt{2}}(a-c) \\ \frac{1}{\sqrt{2}}(a+c) & -\frac{1}{\sqrt{2}}(a-c) & b+\mu_o \end{pmatrix}$$

where

$$a^2 = -\frac{(|m_1|-\mu_o)(|m_2|-\mu_o)(|m_3|-\mu_o)}{d}, \qquad \mu_o = \frac{2|\mu_2^2|}{\bar{M}}(1-\lambda)$$

$$b = |m_1| + |m_2| + |m_3| - d - 3\mu_o \qquad d = \lambda\frac{2|\mu_2|^2}{\bar{M}}$$

$$c^2 = \frac{1}{d}\big[(d-|m_1|)(d-|m_2|)(d-|m_3|)\big]$$

The Orthogonal Matrix O_ν

The real orthogonal matrix that diagonalizes $\bar{M}_\nu$ and M_ν is obtained from the eigenvectors of $\bar{M}_\nu$

$$O_\nu = \Big[|m_1>, |m_2>, |m_3>\Big]$$

$$|m_i> \;=\; \begin{pmatrix} \left[\frac{-\sigma_{i+1}\sigma_{i+2}(\sigma_i - d)}{d(\sigma_i - \sigma_{i+1})(\sigma_i - \sigma_{i+2})}\right]^{1/2} \\ \\ \left[\frac{\sigma_i(\sigma_i - d)}{(\sigma_i - \sigma_{i+1})(\sigma_i - \sigma_{i+2})}\right]^{1/2} \\ \\ \left[\frac{\sigma_i(d - \sigma_{i+1})(d - \sigma_{i+2})}{d(\sigma_i - \sigma_{i+1})(\sigma_i - \sigma_{i+2})}\right]^{1/2} \end{pmatrix}, \quad i = 1, 2, 3, ..mod(3)$$

where $\sigma_i = (m_i - \mu_o)$.

The unitary matrix U_ν that diagonalizes M_ν is

$$U_\nu = P_\nu O_\nu$$

The neutrino mixing matrix I

$$V_{PMNS}^{th} = U_{eL}^{\dagger} U_{\nu}.$$

The theoretical mixing matrix V_{PMNS}^{th} is

$$V_{PMNS}^{th} = O_{eL}^{T} P_{\nu-e} O_{\nu} K$$

where $P_{\nu-e}$ is a diagonal matix of phases

$$P_{\nu-e} = diag\left[1, 1, e^{i(\delta_{\nu}-\delta_{e})}\right]$$

Hence

$$\left(V_{PMNS}^{th}\right)_{ij} = \sum_{r=1}^{3} \left(O_{eL}\right)_{ri} \left(P_{\nu-e}\right)_{r} \left(O_{\nu}\right)_{rj}$$

From a comparison of V_{PMNS}^{PDG} with V_{PMNS}^{th}, we obtain the neutrino mixing angles as functions of the lepton masses.

The neutrino mixing matrix II

From a comparison of V_{PMNS}^{th} with V_{PMNS}^{PDG}, we obtain the neutrino mixing angles as function of the lepton masses

$$\sin^2\theta_{12}^{\nu} = \frac{|(V_{PMNS})_{12}|^2}{1 - |(V_{PMNS})_{13}|^2}$$

$$\sin^2\theta_{23}^{\nu} = \frac{|(V_{PMNS})_{23}|^2}{1 - |(V_{PMNS})_{13}|^2}$$

$$\sin^2\theta_{13}^{\nu} = |(V_{PMNS})_{13}|^2$$

Neutrino Mixing Angles I

The solar angle θ_{12} is strongly dependent on the neutrino masses but depends only very weakly on the charged lepton masses

$$\tan\theta_{12}^2 = \frac{(\Delta m_{12}^2 + \Delta m_{13}^2 + |m_{\nu_3}|^2)^{1/2} - |m_{\nu_3}|}{(\Delta m_{13}^2 + |m_{\nu_3}|^2 \cos^2\phi_\nu)^{1/2} + |m_{\nu_3}|}.$$

the numerical value of $\tan^2\theta_{12}$ fixes the origen and scale of the neutrino masses.

The atmospheric mixing angle θ_{23} depends mostly on the charged lepton masses

$$\sin\theta_{23} \approx \frac{1}{\sqrt{2}} \frac{1 - 2\tilde{m}_\mu^2 + \tilde{m}_\mu^4}{\sqrt{1 - 4\tilde{m}_\mu^2 + x^2 + 6\tilde{m}_\mu^4}} = 0.7071$$

$$x = m_e/m_\mu = 4.84 \times 10^{-3}, \quad \tilde{m}_\mu = m_\mu/m_\tau = 5.95 \times 10^{-2}$$

Neutrino Mixing Angles II

The reactor mixing angle θ_{13} is mostly determined by the interplay of the S_3 symmetry and the mass spliting of the right-handed neutrinos in the see saw mechanism plus a very small contribution from the charged leptons,

$$\sin\theta_{13} \approx \frac{2(\lambda\mu)m_{\nu_3}}{m_{\nu_1}-m_{\nu_2}}\left(1-\sqrt{\frac{(m_{\nu_2}-m_{\nu_3})}{m_{\nu_3}-m_{\nu_1}}}\right)\left(\cos\eta-\sqrt{(1-\frac{m_{\nu_1}}{m_{\nu_3}})(\frac{m_{\nu_2}}{m_{\nu_3}}-\frac{m_{\nu_1}}{m_{\nu_3}})}\sin\eta\right)$$

$$+ \frac{1}{\sqrt{2}}\frac{m_e}{m_\mu}\frac{\left(1+4(\frac{m_e}{m_\mu})^2-(\frac{m_\mu}{m_\tau})^4\right)}{\sqrt{1+(\frac{m_\mu}{m_\tau})^2+5(\frac{m_e}{m_\mu})^2-(\frac{m_\mu}{m_\tau})^4}}$$

where

$$\cos\eta = \sqrt{\frac{m_{\nu_2}-m_{\nu_3}}{m_{\nu_2}-m_{\nu_1}}}, \qquad \sin\eta = \sqrt{\frac{m_{\nu_3}-m_{\nu_1}}{m_{\nu_2}-m_{\nu_1}}}$$

We get

$$\sin\theta_{13}^{th} \approx 0.17 \quad \text{with} \quad (\lambda\mu)\approx 0.02$$

Neutrino Mixing Angles: Theory vs Experiment

The most recent experimental values of the neutrino mixing angles θ_{13} and θ_{23} (D.V. Forero, M. Tórtola and J.W.F. Valle, arXiv: 1205.4018 v4 [hep-ph] 24 Oct 2012)

$$\begin{aligned} \bar{\theta}_{13}^{o\ exp} &= 9.02 \pm 1.2 \rightarrow \quad \sin^2 \bar{\theta}_{13}^{o\ exp} = 0.0246^{+0.0029}_{-0.0027}; \\ \bar{\theta}_{23}^{o\ exp} &= 50.8 \pm 1.5 \qquad \sin^2 \bar{\theta}_{23}^{o\ exp} = 0.6 \pm 0.029 \end{aligned}$$

Our theoretical predictions (A.Mondragón, M. Mondragón and E. Peinado Phys. Rev. D **76**, 076003 (2007), F. González Canales and A. Mondragón,J. Phys: Conference Series **387** (2012) 012008 and F. González Canales, A. Mondragón and M. Mondragón, Fortschr. Phys. (2012)/DOI 10.1002/prop.201200121)

$$\begin{aligned} \theta_{13}^{o\ th} &= 9.6 \pm 0.07 \qquad \sin^2 \theta_{13}^{o\ th} = 0.028 \pm 0.002 \\ \theta_{23}^{o\ th} &= 44.97 \pm 1.2 \qquad \sin^2 \theta_{23}^{o\ th} = 0.50 \pm 0.02 \end{aligned}$$

in very good agreement with the latest experimental values !!!

The neutrino mass spectrum I

In the present model, the experimental restriction

$$|\Delta m_{21}^2| < |\Delta m_{23}^2|$$

implies an inverted neutrino mass spectrum $\quad m_{\nu_3} \ < \ m_{\nu_1}, m_{\nu_2}$

From our previous expressions for $\tan\theta_{12}$

$$|m_{\nu_3}| = \frac{\sqrt{\Delta m_{13}^2}}{2\cos\phi_\nu \tan\theta_{12}} \frac{1 - \tan^4\theta_{12} + r^2}{\sqrt{1+\tan^2\theta_{12}}\sqrt{1+\tan^2\theta_{12}+r^2}},$$

where $\quad r = \Delta m_{21}^2/\Delta m_{23}^2$.

Then, the mass $|m_{\nu_3}|$ is approximately given by

$$|m_{\nu_3}| \approx \frac{1}{2}\frac{\sqrt{\Delta m_{13}^2}}{\tan\theta_{12}}(1 - \tan^2\theta_{12})$$

Neutrino mass spectrum II

- We wrote the neutrino mass differences, $m_{\nu_i} - m_{\nu_j}$, in terms of the differences of the squared masses $\Delta^2_{ij} = m^2_{\nu_i} - m^2_{\nu_j}$ and one of the neutrino masses, say m_{ν_3}.
- The mass m_{ν_2} was taken as a free parameter in the fitting of our formula for $\tan\theta_{12}$ to the experimental value
- with

$$\Delta m^2_{21} = 7.6 \times 10^{-5} eV^2 \qquad \Delta m^2_{13} = 2.4 \times 10^{-3} eV^2$$

and

$$\tan\theta_{12} = 0.696$$

we get

$$|m_{\nu_3}| \approx 0.019\ eV \quad \Longrightarrow \quad |m_{\nu_2}| \approx 0.053\ eV \qquad \text{and} \qquad |m_{\nu_1}| \approx 0.052\ eV$$

- The neutrino mass spectrum has an inverted hierarchy of masses

FCNC I

In the Standard Model the FCNC at tree level are suppressed by the GIM mechanism.

Models with more than one Higgs $SU(2)$ doublet have tree level FCNC due to the exchange of scalar fields. The mass matrix written in terms of the Yukawa couplings is

$$\mathcal{M}_Y^e = Y_w^{E1} H_1^0 + Y_w^{E2} H_2^0,$$

FCNC processes:

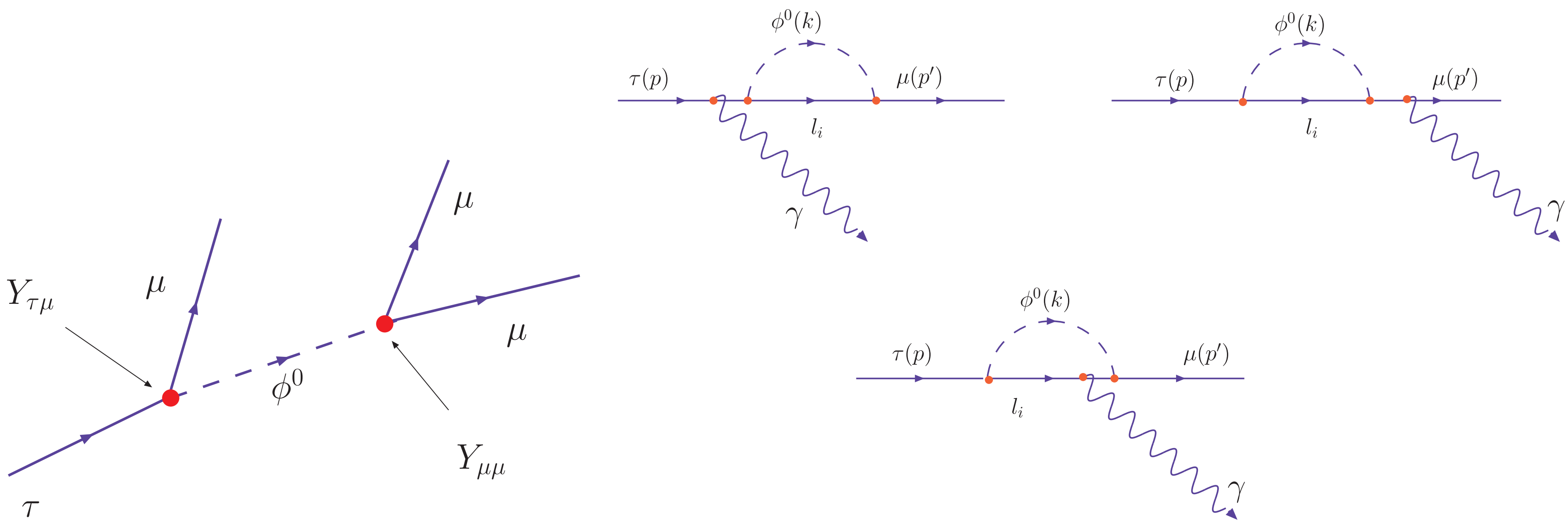


Figure 1: The diagram in the left contributes to the process $\tau^- \to 3\mu$. The three diagrams in the right contribute to the process $\tau \to \mu\gamma$.

The Yukawa matrices

The Yukawa matrices in the weak basis are

$$Y_w^{E1} = \frac{m_\tau}{v_1}\begin{pmatrix} 0 & \frac{1}{\sqrt{2}}\frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & \frac{1}{\sqrt{2}}\sqrt{\frac{1+x^2-\tilde{m}_\mu^2}{1+x^2}} \\ \frac{1}{\sqrt{2}}\frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & 0 & 0 \\ \frac{\tilde{m}_e(1+x^2)}{\sqrt{1+x^2-\tilde{m}_\mu^2}}e^{i\delta_e} & 0 & 0 \end{pmatrix}$$

and

$$Y_w^{E2} = \frac{m_\tau}{v_2}\begin{pmatrix} \frac{1}{\sqrt{2}}\frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}}\frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & \frac{1}{\sqrt{2}}\sqrt{\frac{1+x^2-\tilde{m}_\mu^2}{1+x^2}} \\ 0 & \frac{\tilde{m}_e(1+x^2)}{\sqrt{1+x^2-\tilde{m}_\mu^2}}e^{i\delta_e} & 0 \end{pmatrix}.$$

Yukawa matrices in the mass representation

The Yukawa matrices in the mass basis defined by

$$\tilde{Y}_m^{EI} = U_{eL}^{\dagger} Y_w^{EI} U_{eR}$$

$$\tilde{Y}_m^{E1} \approx \frac{m_\tau}{v_1} \begin{pmatrix} 2\tilde{m}_e & -\frac{1}{2}\tilde{m}_e & \frac{1}{2}x \\ -\tilde{m}_\mu & \frac{1}{2}\tilde{m}_\mu & -\frac{1}{2} \\ \frac{1}{2}\tilde{m}_\mu x^2 & -\frac{1}{2}\tilde{m}_\mu & \frac{1}{2} \end{pmatrix}_m ,$$

and

$$\tilde{Y}_m^{E2} \approx \frac{m_\tau}{v_2} \begin{pmatrix} -\tilde{m}_e & \frac{1}{2}\tilde{m}_e & -\frac{1}{2}x \\ \tilde{m}_\mu & \frac{1}{2}\tilde{m}_\mu & \frac{1}{2} \\ -\frac{1}{2}\tilde{m}_\mu x^2 & \frac{1}{2}\tilde{m}_\mu & \frac{1}{2} \end{pmatrix}_m ,$$

all off diagonal terms give rise to FCNC processes!!

Branching ratios

We define the partial branching ratio (only leptonic decays)

$$Br(\tau \to \mu e^+ e^-) = \frac{\Gamma(\tau \to \mu e^+ e^-)}{\Gamma(\tau \to e\nu\bar{\nu}) + \Gamma(\tau \to \mu\nu\bar{\nu})}, \quad \Gamma(\tau \to \mu e^+ e^-) \approx \frac{m_\tau^5}{3 \times 2^{10}\pi^3} \frac{(Y_{\tau\mu}^{1,2} Y_{ee'}^{1,2})^2}{M_{H_{1,2}}^4}$$

thus

$$Br(\tau \to \mu e^+ e^-) \approx \frac{9}{4} \left(\frac{m_e m_\mu}{m_\tau^2}\right)^2 \left(\frac{m_\tau}{M_{H_{1,2}}}\right)^4,$$

Similar computations lead to

$$Br(\tau \to e\gamma) \approx \frac{3\alpha}{8\pi} \left(\frac{m_\mu}{M_H}\right)^4,$$

$$Br(\tau \to \mu\gamma) \approx \frac{3\alpha}{128\pi} \left(\frac{m_\mu}{m_\tau}\right)^2 \left(\frac{m_\tau}{M_H}\right)^4,$$

$$Br(\tau \to 3\mu) \approx \frac{9}{64} \left(\frac{m_\mu}{M_H}\right)^4,$$

$$Br(\mu \to 3e) \approx 18 \left(\frac{m_e m_\mu}{m_\tau^2}\right)^2 \left(\frac{m_\tau}{M_H}\right)^4,$$

$$Br(\mu \to e\gamma) \approx \frac{27\alpha}{64\pi} \left(\frac{m_e}{m_\mu}\right)^4 \left(\frac{m_\tau}{M_H}\right)^4.$$

Numerical results

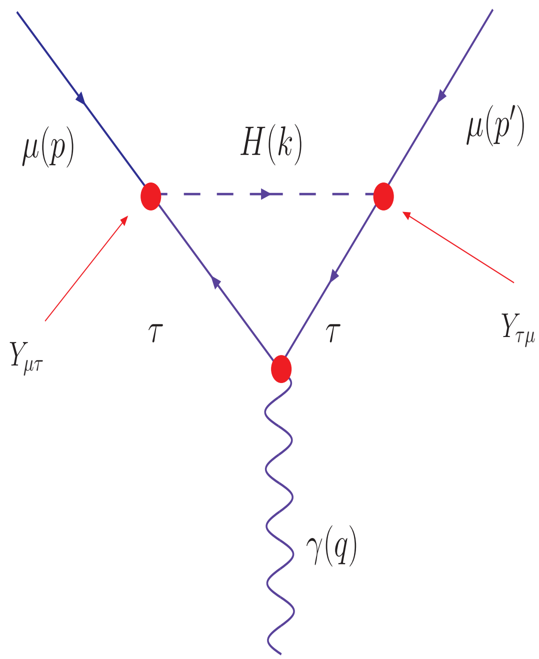
Table 2: Leptonic processes via FCNC

FCNC processes	Theoretical BR	Experimental upper bound BR	References
$\tau \to 3\mu$	8.43×10^{-14}	5.3×10^{-8}	B. Aubert *et. al.* (2007)
$\tau \to \mu e^+ e^-$	3.15×10^{-17}	8×10^{-8}	B. Aubert *et. al.* (2007)
$\tau \to \mu\gamma$	9.24×10^{-15}	6.8×10^{-8}	B. Aubert *et. al.*(2005)
$\tau \to e\gamma$	5.22×10^{-16}	1.1×10^{-11}	B. Aubert *et. al.* (2006)
$\mu \to 3e$	2.53×10^{-16}	1×10^{-12}	U. Bellgardt *et al.* (1998)
$\mu \to e\gamma$	2.42×10^{-20}	1.2×10^{-11}	M. L. Brooks *et al.* (1999)

Small FCNC processes mediating non-standard quark-neutrino interactions could be important in the theoretical description of the **gravitational core collapse and shock generation** in the explosion stage of a supernova

Muon Anomalous Magnetic Moment

The anomalous magnetic moment of the muon is related to the gyroscopic ratio by



$$a_\mu = \frac{\mu_\mu}{\mu_B} - 1 = \frac{1}{2}(g_\mu - 2)$$

In models with more than one Higgs $SU(2)$ doublet, the exchange of flavour changing neutral scalars also contribute to the anomalous magnetic moment of the muon

$$\delta a_\mu^{(H)} = \frac{Y_{\mu\tau} Y_{\tau\mu}}{16\pi^2} \frac{m_\mu m_\tau}{M_H^2} \left(log\left(\frac{M_H^2}{m_\tau^2}\right) - \frac{3}{2} \right)$$

From our results: $Y_{\mu\tau} Y_{\tau\mu} = \frac{m_\mu m_\tau}{4 v_1 v_2}$

$$\delta a_\mu^{(H)} = \frac{m_\tau^2}{(246\ GeV)^2} \frac{(2 + \tan^2\beta)}{32\pi^2} \frac{m_\mu^2}{M_H^2} \left(log\left(\frac{M_H^2}{m_\tau^2}\right) - \frac{3}{2} \right), \quad \tan\beta = \frac{v_s}{v_1}$$

From the experimental upper bound on $(\mu \to 3e)$, we get $\tan\beta \leq 14$, Hence

$$\delta a_\mu = 1.7 \times 10^{-10}$$

Contribution to the anomaly of the muon's magnetic moment

The difference between the experimental value and the Standard Model prediction for the anomaly is

$$\Delta a_\mu = a_\mu^{exp} - a_\mu^{SM} = (28.7 \pm 9.1) \times 10^{-10}$$

$\Delta a_\mu \sim 3\sigma$ (three standard deviations) !!

But, the uncertainty in the computation of higher order hadronic effects is large

$$\delta a_\mu^{LBL}(3, had) \approx 1.59 \times 10^{-9}; \quad \delta a_\mu^{VP}(3, had) \approx -1.82 \times 10^{-9}$$

$$\frac{\delta a_\mu^{(H)}}{\Delta a_\mu} \approx \frac{1.7}{28} \approx 6\% \quad \text{and} \quad \delta a_\mu^{(H)} < \delta a_\mu(3, had)$$

The contribution of the exchange of flavour changing scalars to the anomaly of the muon's magnetic moment, $\delta a_\mu^{(H)}$, is small but not negligible, and it is compatible with the best, state of the art, measurements and theoretical predictions.

Summary

- By introducing three $SU(2)_L$ Higgs doublet fields, in the theory, we extended the concept of flavour and generations to the Higgs sector and formulated a minimal S_3-invariant Extension of the Standard Model.

- The mass matrices of quarks and leptons are reparametrized in terms of their eigenvalues and the mixing matrices are computed.

- In the case of quarks, the agreement between our V_{CKM}^{th} (with only two free parameters) and V^{PDG} is excellent

- In the caser of neutrinos, the solar mixing angle, θ_{12}, fixes the scale and origin of the neutrino mass spectrum which has an inverted mass hierarchy with values

$$|m_{\nu_2}| \approx 0.053eV, \qquad |m_{\nu_1}| \approx 0.052eV, \qquad |m_{\nu_3}| \approx 0.019eV$$

- In the leptonic sector, the $S_3 \times Z_2$ symmetry implies a non vanishing and sizeble reactor mixing angle, $\theta_{13}^{\circ th} = 9.6 \pm 0.07$ in excelent agreement with the most recent experimental results.

- The fit of $\sin^2\theta_{13}^{th}$ to the $\sin^2\theta_{13}^{exp}$ breaks the mass degeneracy of the right handed neutrinos