Lepton Mixing from the Groups $\Delta(3n^2)$ and $\Delta(6n^2)$

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Feruglio/H/Meroni/Vitale, work in progress

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Outline

- Data on lepton mixing
- Idea
- Overview over groups $\Delta(3n^2)$ and $\Delta(6n^2)$
- Some results for $\Delta(3n^2)$
- Some results for $\Delta(6n^2)$
- Conclusions

Data on lepton mixing

Results of latest global fits ***(Gonzalez-Garcia et al. ('12))***

best fit and $1\,\sigma$ error	$3\,\sigma$ range
$\sin^2\theta_{12} = 0.30^{+0.013}_{-0.013}$	$0.27 \leq \sin^2\theta_{12} \leq 0.34$
$\sin^2\theta_{23} = \begin{cases} 0.41^{+0.037}_{-0.025} \\ 0.59^{+0.021}_{-0.022} \end{cases}$	$0.34 \leq \sin^2\theta_{23} \leq 0.67$
$\sin^2\theta_{13} = 0.023^{+0.0023}_{-0.0023}$	$0.016 \leq \sin^2\theta_{13} \leq 0.030$

Indication of a flavor symmetry G_f?

You might answer: yes, since

Tri-Bimaximal mixing (TB mixing) ***(Harrison/Perkins/Scott ('02), Xing ('02))***

$$||U_{PMNS}|| = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\sin^2\theta_{12} = \frac{1}{3}\,, \quad \sin^2\theta_{23} = \frac{1}{2}\,, \quad \sin^2\theta_{13} = 0$$

describes the data still to a certain extent well.

Indication of a flavor symmetry G_f?

You might answer: yes, since

$\Delta(96)$ Mixing ***(de Adelhart Toorop et al. ('11))***

$$||U_{PMNS}|| = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{2}(\sqrt{3}+1) & 1 & \frac{1}{2}(\sqrt{3}-1) \\ \frac{1}{2}(\sqrt{3}-1) & 1 & \frac{1}{2}(\sqrt{3}+1) \\ 1 & 1 & 1 \end{pmatrix}$$

$$\sin^2\theta_{12} \approx 0.349 \; , \;\; \sin^2\theta_{23} \approx \left\{ \begin{array}{c} 0.349 \\ 0.651 \end{array} \right. \; , \;\; \sin^2\theta_{13} \approx 0.045$$

describes the data to a certain extent well.

Indication of a flavor symmetry G_f?

You might answer: yes, since

$\Delta(384)$ Mixing ***(de Adelhart Toorop et al. ('11))***

$$||U_{PMNS}|| = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{2}\sqrt{4+\sqrt{2}+\sqrt{6}} & 1 & \frac{1}{2}\sqrt{4-\sqrt{2}-\sqrt{6}} \\ \frac{1}{2}\sqrt{4+\sqrt{2}-\sqrt{6}} & 1 & \frac{1}{2}\sqrt{4-\sqrt{2}+\sqrt{6}} \\ \sqrt{1-\frac{1}{\sqrt{2}}} & 1 & \sqrt{1+\frac{1}{\sqrt{2}}} \end{pmatrix}$$

$$\sin^2\theta_{12} \approx 0.337 \ , \ \ \sin^2\theta_{23} \approx \begin{cases} 0.424 \\ 0.576 \end{cases} \ , \ \ \sin^2\theta_{13} \approx 0.011$$

describes the data quite well.

Indication of a flavor symmetry G_f?

You could also answer: no, see e.g. ***de Gouvea/Murayama ('12)***
However, if you follow this line of thought, then you forget that in many models beyond the SM the symmetry G_f also helps to constrain the form of

- mass matrices of additional particles
  (e.g. soft terms in SUSY models)
- additional gauge interactions
  (e.g. in models with gauge-Higgs unification)

in flavor space.

Idea

Derivation of the lepton mixing from how G_f is broken

Interpretation as mismatch of embedding of different subgroups G_ν and G_e into G_f

$$G_f$$

$$\swarrow \qquad\qquad\qquad \searrow$$

neutrinos — charged leptons

G_ν — G_e

Idea

Derivation of the lepton mixing from how G_f is broken
Interpretation as mismatch of embedding of different subgroups G_ν and G_e into G_f

$$G_f$$

$$\swarrow \qquad\qquad \searrow$$

neutrinos	charged leptons
distinguish 3 generations	distinguish 3 generations

Idea

Derivation of the lepton mixing from how G_f is broken

Interpretation as mismatch of embedding of different subgroups G_ν and G_e into G_f

$$G_f$$

$$\swarrow \qquad\qquad \searrow$$

neutrinos

Majorana: $G_\nu = Z_2 \times Z_2$

Dirac: $G_\nu = Z_M$ with $M \geq 3$

charged leptons

$G_e = Z_N$ with $N \geq 3$

Idea

$$\begin{array}{ccc} & G_f & \\ \swarrow & & \searrow \\ G_\nu & & G_e \end{array}$$

Further requirements

- Two/three non-trivial angles $\Rightarrow$ irred 3-dim rep of G_f
- Fix angles through G_ν, G_e $\Rightarrow$ 3 families are diff. under G_ν, G_e

Idea

- Neutrino sector, Majorana: $Z_2 \times Z_2$ preserved

  $\rightarrow$ neutrino mass matrix m_ν fulfills

  $$Z_i^T m_\nu Z_i = m_\nu \quad \text{with} \quad i = 1, 2$$

- Charged lepton sector: Z_N, $N \geq 3$, preserved

  $\rightarrow$ charged lepton mass matrix m_e fulfills

  $$Q_e^\dagger m_e^\dagger m_e Q_e = m_e^\dagger m_e$$

Idea

- Neutrino sector, Majorana: $Z_2 \times Z_2$ preserved

  $\rightarrow$ neutrino mass matrix m_ν fulfills

  $$\Omega_\nu^T m_\nu \Omega_\nu \quad \text{is diagonal}$$

  $$Z_i = \Omega_\nu Z_i^{diag} \Omega_\nu^\dagger \quad \text{with} \quad i = 1, 2$$

- Charged lepton sector: Z_N, $N \geq 3$, preserved

  $\rightarrow$ charged lepton mass matrix m_e fulfills

  $$\Omega_e^\dagger m_e^\dagger m_e \Omega_e \quad \text{is diagonal}$$

  $$Q_e = \Omega_e Q_e^{diag} \Omega_e^\dagger \quad \text{with} \quad \Omega_e \quad \text{unitary}$$

Idea

- Neutrino sector, Majorana: $Z_2 \times Z_2$ preserved

  $\rightarrow$ neutrino mass matrix m_ν fulfills

  $$\Omega_\nu^T m_\nu \Omega_\nu \quad \text{is diagonal}$$

- Charged lepton sector: Z_N, $N \geq 3$, preserved

  $\rightarrow$ charged lepton mass matrix m_e fulfills

  $$\Omega_e^\dagger m_e^\dagger m_e \Omega_e \quad \text{is diagonal}$$

- Conclusion: $\Omega_{\nu,e}$ diagonalize m_ν and $m_e^\dagger m_e$

$$U_{PMNS} = \Omega_e^\dagger \Omega_\nu$$

Idea

$$U_{PMNS} = \Omega_e^\dagger \Omega_\nu$$

- 3 unphysical phases are removed by $\Omega_e \to \Omega_e K_e$
- Neutrino masses are made real and positive through $\Omega_\nu \to \Omega_\nu K_\nu$
- Permutations of columns of Ω_e, Ω_ν are possible: $\Omega_{e,\nu} \to \Omega_{e,\nu} P_{e,\nu}$

$$\Downarrow$$

Predictions:

Mixing angles up to exchange of rows/columns

Dirac phase δ_{CP} up to π

Majorana phases undetermined

Overview over groups $\Delta(3n^2)$

- Series of subgroups of $SU(3)$: infinite, in general non-abelian $n \geq 2$

  (Miller et al. ('16), Fairbairn et al. ('64), Luhn et al. ('07), Grimus/Ludl ('11))

- $\Delta(3n^2)$ used in the literature: $\Delta(12) \simeq A_4$, $\Delta(27)$, $\Delta(48)$, $\Delta(75)$
- Notice $\Delta(3n^2) \simeq (Z_n \times Z_n) \rtimes Z_3$
- Presentation of $\Delta(3n^2)$ in terms of three generators a, c, d

$$a^3 = e \; , \;\; c^n = e \; , \;\; d^n = e \; , \;\; cd = dc \; ,$$
$$aca^{-1} = c^{-1} d^{-1} \; , \;\; ada^{-1} = c$$

- All elements can be written as $g = a^\alpha c^\gamma d^\delta$

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- Presentation of $\Delta(3n^2)$ in terms of three generators a, c, d
  $(\eta = e^{2\pi i/n})$

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \; , \; c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \eta^{-1} \end{pmatrix} \; , \; d = \begin{pmatrix} \eta^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \eta \end{pmatrix}$$

Overview over groups $\Delta(6n^2)$

- Series of subgroups of $SU(3)$: infinite, in general non-abelian

  (Miller et al. ('16), Fairbairn et al. ('64), Escobar/Luhn ('08), Grimus/Ludl ('11))

- $\Delta(6n^2)$ used in the literature: $\Delta(6) \simeq S_3$, $\Delta(24) \simeq S_4$, $\Delta(54)$, $\Delta(96)$
- Notice $\Delta(6n^2) \simeq (Z_n \times Z_n) \rtimes S_3$
- Presentation of $\Delta(6n^2)$ in terms of four generators a, b, c, d

$$b^2 = e \; , \;\; (ab)^2 = e \; ,$$
$$bcb^{-1} = d^{-1} \; , \;\; bdb^{-1} = c^{-1}$$

- All elements can be written as $g = a^\alpha b^\beta c^\gamma d^\delta$

Overview over groups $\boldsymbol{\Delta(6n^2)}$

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- Notice $\Delta(6n^2) \simeq (Z_n \times Z_n) \rtimes S_3$
- Presentation of $\Delta(6n^2)$ in terms of four generators a, b, c, d

$$a \; , \quad c \; , \quad d \; , \quad b = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Systematic study

- Structure of $\Delta(3n^2)$ and $\Delta(6n^2)$ is "simple" $\rightarrow$ systematic study
- Approach:
  - use faithful 3-dim rep of a group for left-handed leptons
  - take all combinations of G_e and G_ν in G_f
  - reduce to limited number of categories of pairs $\{G_e, G_\nu\}$
  - require that three generations are distinguished by G_e and G_ν
  - consider only cases in which G_f is "generated" from G_e and G_ν
  - additional constraint for Majorana neutrinos: $G_\nu = Z_2 \times Z_2$
  - use fact that all elements g are represented by $g = a^\alpha c^\gamma d^\delta$ or $g = a^\alpha b^\beta c^\gamma d^\delta$

Some results for $\boldsymbol{\Delta(3n^2)}$

- Form of semi-direct product of $\Delta(3n^2)$ shows that only one case for Majorana neutrinos exists
- Only case $n = 2$ (group A_4) leads to "independent" pattern
- Known result (G_e has to be Z_3): "magic matrix"

(Cabibbo ('78), Wolfenstein ('78), Ma/Rajasekaran ('01), Lam ('11), de Adelhart Toorop et al. ('11))

$$||U_{PMNS}|| = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\sin^2\theta_{12} = \frac{1}{2} \ , \ \ \sin^2\theta_{23} = \frac{1}{2} \ , \ \ \sin^2\theta_{13} = \frac{1}{3} \ , \ \ |\delta_{CP}| = \pi/2$$

Some results for $\boldsymbol{\Delta(3n^2)}$

- Thus these groups are mainly interesting for Dirac neutrinos (and quarks)
- Next bigger group: $\Delta(27)$, $n = 3$:
  again the form of the PMNS matrix is the magic matrix
- Cases which need to be distinguished: n even, n odd and $3|n$
- Interestingly, many of the patterns we can find have non-trivial δ_{CP}

Some results for $\Delta(6n^2)$

- Much more appropriate for Majorana neutrinos, if n is even, since one of the factors of the semi-direct product is S_3
- Known results for small n are encouraging:
  - $n = 2$: $\Delta(24) \simeq S_4$: TB mixing or bi-maximal mixing

    (Lam ('07,'11), de Adelhart Toorop et al. ('11))
  - $n = 4$: $\Delta(96)$: ***(de Adelhart Toorop et al. ('11))***

$$||U_{PMNS}|| = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{2}(\sqrt{3}+1) & 1 & \frac{1}{2}(\sqrt{3}-1) \\ \frac{1}{2}(\sqrt{3}-1) & 1 & \frac{1}{2}(\sqrt{3}+1) \\ 1 & 1 & 1 \end{pmatrix}$$

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  - $n = 8$: $\Delta(384)$: ***(de Adelhart Toorop et al. ('11))***

$$||U_{PMNS}|| = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{2}\sqrt{4+\sqrt{2}+\sqrt{6}} & 1 & \frac{1}{2}\sqrt{4-\sqrt{2}-\sqrt{6}} \\ \frac{1}{2}\sqrt{4+\sqrt{2}-\sqrt{6}} & 1 & \frac{1}{2}\sqrt{4-\sqrt{2}+\sqrt{6}} \\ \sqrt{1-\frac{1}{\sqrt{2}}} & 1 & \sqrt{1+\frac{1}{\sqrt{2}}} \end{pmatrix}$$

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    (Lam ('07,'11), de Adelhart Toorop et al. ('11))
  - $n = 10$: $\Delta(600)$: ***(Feruglio et al. (in progress), Lam ('12))***

$$||U_{PMNS}|| \approx \begin{pmatrix} 0.799 & 0.577 & 0.170 \\ 0.546 & 0.577 & 0.607 \\ 0.252 & 0.577 & 0.777 \end{pmatrix}$$

Some results for $\boldsymbol{\Delta(6n^2)}$

- Much more appropriate for Majorana neutrinos, if n is even, since one of the factors of the semi-direct product is S_3
- Known results for small n are encouraging:
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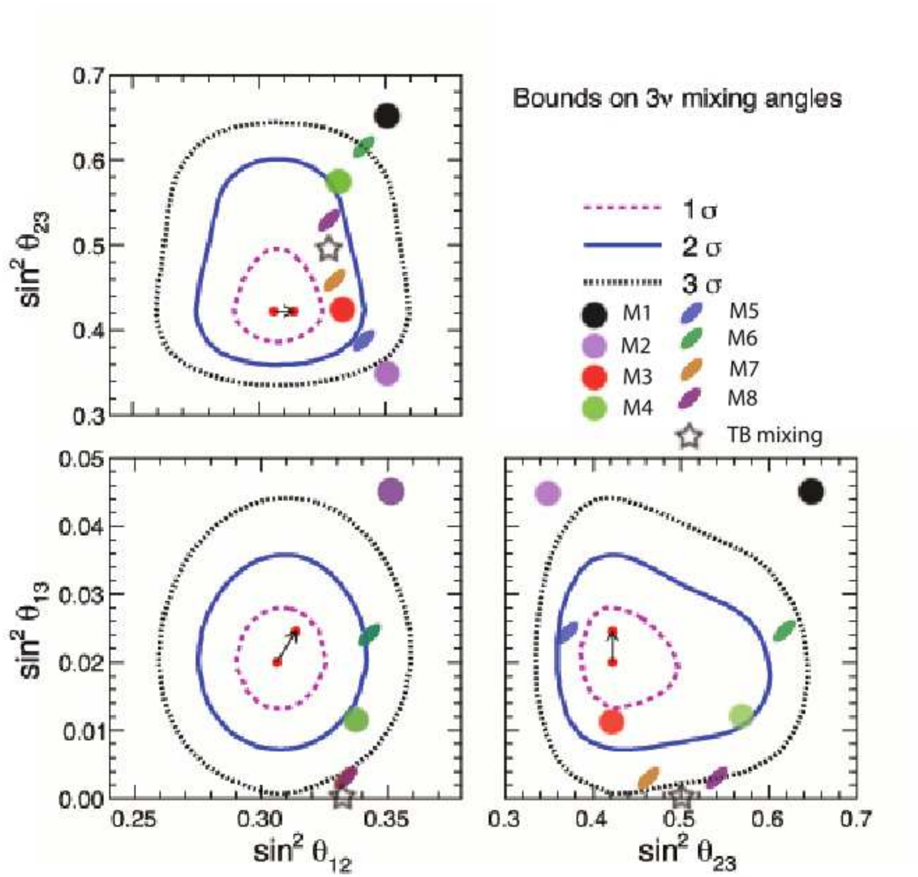
    (Lam ('07,'11), de Adelhart Toorop et al. ('11))
  - $n = 10$: $\Delta(600)$: ***(Feruglio et al. (in progress), Lam ('12))***

$$\sin^2\theta_{12} \approx 0.343\;, \quad \sin^2\theta_{23} \approx \begin{cases} 0.379 \\ 0.621 \end{cases}\;, \quad \sin^2\theta_{13} \approx 0.029$$

Some results for $\Delta(6n^2)$

- Much more appropriate for Majorana neutrinos, if n is even, since one of the factors of the semi-direct product is S_3
- Known results for small n are encouraging
- General results can be understood as "deviation" from TB mixing

Some results for $\Delta(6n^2)$



Some results for $\boldsymbol{\Delta(6n^2)}$

- Much more appropriate for Majorana neutrinos, if n is even, since one of the factors of the semi-direct product is S_3
- Known results for small n are encouraging
- General results can be understood as "deviation" from TB mixing
- Two interesting points:
  - Solar mixing angle subject to $\sin^2\theta_{12} \geq 1/3$
  - No non-trivial Dirac phase δ_{CP}
- Patterns for Majorana neutrinos work also for Dirac neutrinos and can be generated with smaller n as well

Conclusions

- Idea to relate lepton mixing to the non-trivial breaking of flavor symmetry is interesting
- Groups $\Delta(3n^2)$ and $\Delta(6n^2)$ allow for systematic study of mixing
- Preliminary results:
  - new patterns from $\Delta(3n^2)$ for Dirac neutrinos (and quarks)
  - $\Delta(6n^2)$ much more suitable for Majorana neutrinos
    viable patterns with θ_{13} small for $n = 4$, $n = 8$, $n = 10$ found
- Goal: exhaust all possible non-abelian discrete $SU(3)$ subgroups

Thank you for your attention.