The University Of Sheffield.

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Thermal Field Theory to All Orders in Gradient Expansion

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Outline

Introduction

Motivation

- the density frontier: ultra-relativistic many-body dynamics
- early Universe:
  - baryon asymmetry of the Universe
  - electroweak phase transition
  - reheating/preheating
  - relic densities
- 'terrestrial:'
  - quark gluon plasma/glasma/color glass condensates

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Current Approaches

- (semi-classical) Boltzmann transport equations
  - effective resummation of finite-width effects

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- Kadanoff–Baym $\Rightarrow$ quantum Boltzmann equations
  - incorporation of off-shell effects
  - truncated gradient expansion in time derivative
  - separation of time scales and quasi-particle approximation
  - varying definitions of physical observables,
    e.g. particle number density

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- (semi-classical) Boltzmann transport equations
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  - incorporation of off-shell effects
  - truncated gradient expansion in time derivative
  - separation of time scales and quasi-particle approximation
  - varying definitions of physical observables, e.g. particle number density
- underlying perturbation series contain pinch singularities: $\delta^2(p^2 - m^2)$

Canonical Quantisation

Boundary Conditions

- No assumption of adiabatic hypothesis.
- QM pictures have a finite microscopic time of coincidence $\tilde{t}_i$:

$$\Phi_{\mathrm{S}}(\mathbf{x};\tilde{t}_i) \;=\; \Phi_{\mathrm{I}}(\tilde{t}_i,\mathbf{x};\tilde{t}_i) \;=\; \Phi_{\mathrm{H}}(\tilde{t}_i,\mathbf{x};\tilde{t}_i)$$

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$\Rightarrow$ interactions switched on at $\tilde{t}_i$

$\Rightarrow$ initial density matrix $\rho(\tilde{t}_i;\tilde{t}_i)$ specified fully in on-shell Fock states

$\Rightarrow$ finite lower bound on time integrals in path-integral action

Canonical Quantisation

Canonical Commutation Relations

- Interaction-picture creation and annihilation operators satisfy:

$$\big[a(\mathbf{p},\tilde{t};\tilde{t}_i),a^{\dagger}(\mathbf{p}',\tilde{t}';\tilde{t}_i)\big] \;=\; (2\pi)^3 2E(\mathbf{p})\delta^{(3)}(\mathbf{p}\,-\,\mathbf{p}')e^{-iE(\mathbf{p})(\tilde{t}\,-\,\tilde{t}')}$$

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- Ensemble Expectation Value (EEV) at macroscopic time $t = \tilde{t}_f - \tilde{t}_i$:

$$\langle \bullet \rangle_t \;=\; \frac{\mathrm{tr}\,\rho(\tilde{t}_f;\tilde{t}_i)\,\bullet}{\mathrm{tr}\,\rho(\tilde{t}_f;\tilde{t}_i)}$$

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- Most general EEVs permitted:

$$\begin{aligned}\langle a(\mathbf{p},\tilde{t}_f;\tilde{t}_i)a^\dagger(\mathbf{p}',\tilde{t}_f;\tilde{t}_i)\rangle_t \;&=\; (2\pi)^3 2E(\mathbf{p})\delta^{(3)}(\mathbf{p}\,-\,\mathbf{p}') \\ &\qquad +\;2E^{1/2}(\mathbf{p})E^{1/2}(\mathbf{p}')f(\mathbf{p},\mathbf{p}',t) \\ \langle a^\dagger(\mathbf{p}',\tilde{t}_f;\tilde{t}_i)a(\mathbf{p},\tilde{t}_f;\tilde{t}_i)\rangle_t \;&=\; 2E^{1/2}(\mathbf{p})E^{1/2}(\mathbf{p}')f(\mathbf{p},\mathbf{p}',t)\end{aligned}$$

Schwinger–Keldysh CTP Formalism

$$\mathcal{Z}[\rho, J_{\pm}, t] \;=\; \mathrm{tr}\left[\bar{\mathrm{T}}\, e^{-i\int_{\Omega_t}\mathrm{d}^4x\, J_-(x)\Phi_{\mathrm{H}}(x)}\right]\rho_{\mathrm{H}}(\tilde{t}_f; \tilde{t}_i)\left[\mathrm{T}\, e^{i\int_{\Omega_t}\mathrm{d}^4x\, J_+(x)\Phi_{\mathrm{H}}(x)}\right]$$

$$x_0 \;\in\; [\tilde{t}_i = -\tfrac{t}{2}, \tilde{t}_f = +\tfrac{t}{2}]$$

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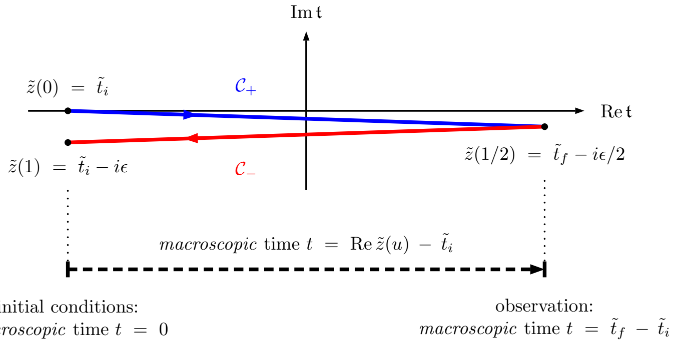


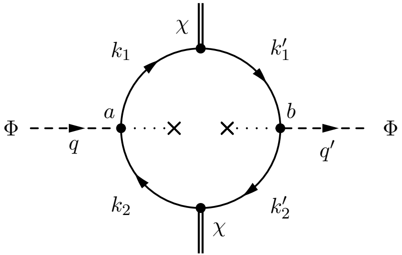
$\Rightarrow$ finite upper and lower bounds on time integrals in path-integral action.

Non-Homogeneous Free Propagators

Propagator	**Double-Momentum Representation**
Feynman (Dyson)	$i\Delta^0_{\mathrm{F(D)}}(p,p',\tilde{t}_f;\tilde{t}_i) = \frac{(-)i}{p^2 - M^2 + (-)i\epsilon}(2\pi)^4\delta^{(4)}(p-p')$ $+2\pi\lvert 2p_0\rvert^{1/2}\delta(p^2-M^2)\tilde{f}(p,p',t)\,e^{i(p_0-p_0')\tilde{t}_f}2\pi\lvert 2p_0'\rvert^{1/2}\delta(p'^2-M^2)$
+(−)ve-freq. Wightman	$i\Delta^0_{>(<)}(p,p',\tilde{t}_f;\tilde{t}_i) = 2\pi\theta(+(-)p_0)\delta(p^2-M^2)(2\pi)^4\delta^{(4)}(p-p')$ $+2\pi\lvert 2p_0\rvert^{1/2}\delta(p^2-M^2)\tilde{f}(p,p',t)\,e^{i(p_0-p_0')\tilde{t}_f}2\pi\lvert 2p_0'\rvert^{1/2}\delta(p'^2-M^2)$
Retarded (Advanced)	$i\Delta^0_{\mathrm{R(A)}}(p,p') = \frac{i}{(p_0+(-)i\epsilon)^2 - \mathbf{p}^2 - M^2}(2\pi)^4\delta^{(4)}(p-p')$
Pauli-Jordan	$i\Delta^0(p,p') = 2\pi\varepsilon(p_0)\delta(p^2-M^2)(2\pi)^4\delta^{(4)}(p-p')$
Hadamard	$i\Delta^0_1(p,p',\tilde{t}_f;\tilde{t}_i) = 2\pi\delta(p^2-M^2)(2\pi)^4\delta^{(4)}(p-p')$ $+2\pi\lvert 2p_0\rvert^{1/2}\delta(p^2-M^2)2\tilde{f}(p,p',t)\,e^{i(p_0-p_0')\tilde{t}_f}2\pi\lvert 2p_0'\rvert^{1/2}\delta(p'^2-M^2)$
Principal-part	$i\Delta^0_{\mathcal{P}}(p,p') = \mathcal{P}\frac{i}{p^2-M^2}(2\pi)^4\delta^{(4)}(p-p')$

Diagrammatics

$$\mathcal{L} \;=\; \tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi \;-\; \tfrac{1}{2}M^2\Phi^2 \;+\; \partial_\mu\chi^\dagger\partial^\mu\chi \;-\; m^2\chi^\dagger\chi \;-\; g\Phi\chi^\dagger\chi$$



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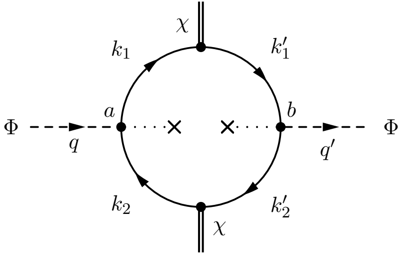

1. time-dependent, energy-non-conserving vertices:

$$\sim\; -ig\frac{t}{2\pi}\mathrm{sinc}\left[\left(\sum_i p_i^0\right)\frac{t}{2}\right]\delta^{(3)}\left(\sum_i \mathbf{p}_i\right)$$

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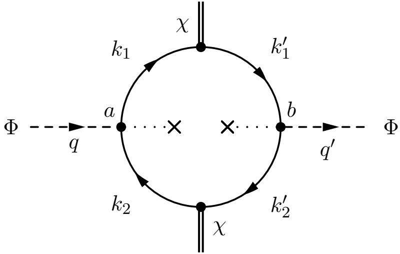

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2. momentum-non-conserving, non-homogeneous free propagators

Physically Meaningful Observables

- Construct from EEVs of field operators:

$$\langle \Phi(x;\tilde{t}_i)\Phi(y;\tilde{t}_i)\rangle_t \;=\; \frac{\operatorname{tr}\rho(\tilde{t}_f;\tilde{t}_i)\Phi(x;\tilde{t}_i)\Phi(y;\tilde{t}_i)}{\operatorname{tr}\rho(\tilde{t}_f;\tilde{t}_i)}$$

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- By writing the Noether charge in terms of a charge density, we may define the particle number density:

$$n(\mathbf{p},\mathbf{X},t) \;=\; \lim_{X_0\,\to\,t} 2\int\frac{\mathrm{d}p_0}{2\pi}\int\frac{\mathrm{d}^4P}{(2\pi)^4}\;e^{-iP\cdot X}\,\theta(p_0)p_0\,i\Delta_<(p+\tfrac{P}{2},p-\tfrac{P}{2},t;0)$$

Master Time Evolution Equations

Partially inverting the CTP Schwinger–Dyson equation:

$$\partial_t f(\mathbf{p}+\tfrac{\mathbf{P}}{2},\mathbf{p}-\tfrac{\mathbf{P}}{2},t) \; - \iint \frac{\mathrm{d}p_0}{2\pi}\,\frac{\mathrm{d}P_0}{2\pi}\; e^{-iP_0 t}\, 2\,\mathbf{p}\cdot\mathbf{P}\,\theta(p_0)\Delta_<(p+\tfrac{P}{2},p-\tfrac{P}{2},t;0)$$
$$+ \iint \frac{\mathrm{d}p_0}{2\pi}\,\frac{\mathrm{d}P_0}{2\pi}\; e^{-iP_0 t}\,\theta(p_0)\Big(\mathscr{F}(p+\tfrac{P}{2},p-\tfrac{P}{2},t;0) \; + \; \mathscr{F}^*(p-\tfrac{P}{2},p+\tfrac{P}{2},t;0)\Big)$$
$$= \iint \frac{\mathrm{d}p_0}{2\pi}\,\frac{\mathrm{d}P_0}{2\pi}\; e^{-iP_0 t}\,\theta(p_0)\Big(\mathscr{C}(p+\tfrac{P}{2},p-\tfrac{P}{2},t;0) \; + \; \mathscr{C}^*(p-\tfrac{P}{2},p+\tfrac{P}{2},t;0)\Big)$$

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Force and collision terms:

$$\mathscr{F}(p+\tfrac{P}{2},p-\tfrac{P}{2},t;0)\ \equiv\ -\int\frac{\mathrm{d}^4q}{(2\pi)^4}\ i\Pi_{\mathcal{P}}(p+\tfrac{P}{2},q,t;0)\,i\Delta_<(q,p-\tfrac{P}{2},t;0),$$
$$\mathscr{C}(p+\tfrac{P}{2},p-\tfrac{P}{2},t;0)\ \equiv\ \frac{1}{2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\ \Big[\,i\Pi_>(p+\tfrac{P}{2},q,t;0)\,i\Delta_<(q,p-\tfrac{P}{2},t;0)$$
$$-\ i\Pi_<(p+\tfrac{P}{2},q,t;0)\,\Big(i\Delta_>(q,p-\tfrac{P}{2},t;0)\ -\ 2i\Delta_{\mathcal{P}}(q,p-\tfrac{P}{2},t;0)\Big)\Big]$$

No nested Poisson brackets as in gradient expansion of Kadanoff–Baym equations.

Simple Example

Time-Dependent Width

- $\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}M^2\Phi^2 + \partial_\mu\chi^\dagger\partial^\mu\chi - m^2\chi^\dagger\chi - g\Phi\chi^\dagger\chi$
- $t < 0$: Φ's and χ's in non-interacting equilibria at same temperature
- $t = 0$: interactions switched on

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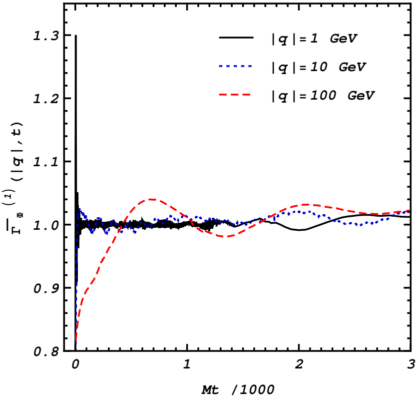

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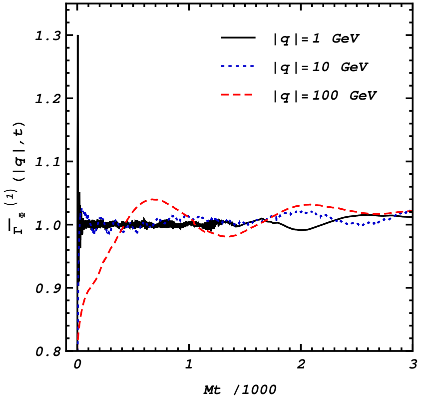
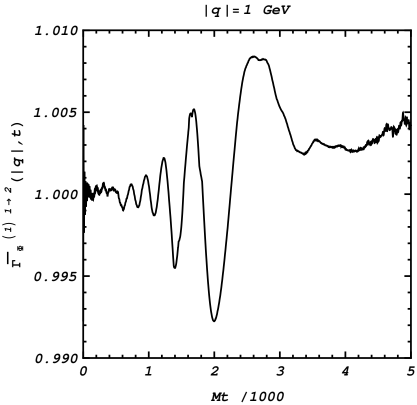
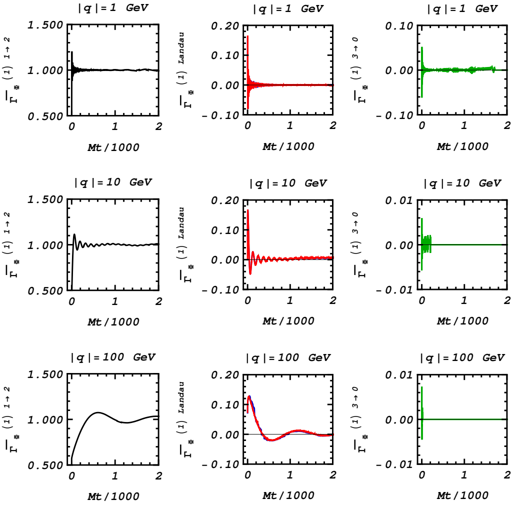
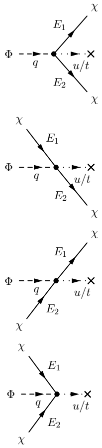




Simple Example

Evanescent Processes





$1 \to 2$ decay (left)

$2 \to 1$ Landau damping (center)

$3 \to 0$ total annihilation (right)

Simple Example

Time Evolution Equations

Truncating the master time evolution equations in a loopwise sense:

$$
\begin{aligned}
\partial_t f_\Phi(|\mathbf{p}|,t) \;=\; & -\frac{g^2}{2}\sum_{\{\alpha\}=\pm 1}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2E_\Phi(\mathbf{p})}\,\frac{1}{2E_\chi(\mathbf{k})}\,\frac{1}{2E_\chi(\mathbf{p}-\mathbf{k})} \\
& \times\;\frac{t}{2\pi}\,\mathrm{sinc}\left[\Big(\alpha E_\Phi(\mathbf{p})-\alpha_1E_\chi(\mathbf{k})-\alpha_2E_\chi(\mathbf{p}-\mathbf{k})\Big)\frac{t}{2}\right] \\
& \times\;\left\{\pi+2\mathrm{Si}\left[\Big(\alpha E_\Phi(\mathbf{p})+\alpha_1E_\chi(\mathbf{k})+\alpha_2E_\chi(\mathbf{p}-\mathbf{k})\Big)\frac{t}{2}\right]\right\} \\
& \times\;\Big\{\big[\theta(-\alpha)+f_\Phi(|\mathbf{p}|,t)\big]\Big[\theta(\alpha_1)\big(1+f_\chi(|\mathbf{k}|,t)\big)+\theta(-\alpha_1)f_\chi^C(|\mathbf{k}|,t)\Big] \\
& \qquad\times\;\Big[\theta(\alpha_2)\big(1+f_\chi^C(|\mathbf{p}-\mathbf{k}|,t)\big)+\theta(-\alpha_2)f_\chi(|\mathbf{p}-\mathbf{k}|,t)\Big] \\
& -\;\big[\theta(\alpha)+f_\Phi(|\mathbf{p}|,t)\big]\Big[\theta(\alpha_1)f_\chi(|\mathbf{k}|,t)+\theta(-\alpha_1)\big(1+f_\chi^C(|\mathbf{k}|,t)\big)\Big] \\
& \qquad\times\;\Big[\theta(\alpha_2)f_\chi^C(|\mathbf{p}-\mathbf{k}|,t)+\theta(-\alpha_2)\big(1+f_\chi(|\mathbf{p}-\mathbf{k}|,t)\big)\Big]\Big\}
\end{aligned}
$$

Still valid to all orders in gradient expansion.

Conclusions

- Obtain master time evolution equations valid to all orders in gradient expansion and to all orders in perturbation theory.

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- Underlying non-equilibrium field theory free of pinch singularities.
- Non-homogeneous free propagators and time-dependent vertices break space-time translational invariance from tree-level.
- Early-time dynamics consistently describe energy-violating processes, leading to non-Markovian evolution of memory effects.

Backup Slides

Particle Number Density

- Charge operator:

$$\mathcal{Q}(x_0;\tilde{t}_i) \;=\; i\int \mathrm{d}^3\mathbf{x}\,\Big[\Phi^\dagger(x;\tilde{t}_i)\pi^\dagger(x;\tilde{t}_i) - \pi(x;\tilde{t}_i)\Phi(x;\tilde{t}_i)\Big]$$
$$\stackrel{?}{\equiv}\;\int \mathrm{d}^3\mathbf{X}\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\mathcal{Q}(\mathbf{p},\mathbf{X},X_0;\tilde{t}_i)$$

- Insert unity and symmetrise in x and y:

$$1 = \int \mathrm{d}^4y\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\delta(x_0-y_0)$$

- Charge-density operator:

$$\mathcal{Q}(\mathbf{p},\mathbf{X},X_0;\tilde{t}_i) \;=\; \frac{i}{2}\int \mathrm{d}^4R\,e^{-i\mathbf{p}\cdot\mathbf{R}}\,\delta(R_0)$$
$$\times\Big[\Phi^\dagger(X-\tfrac{R}{2};\tilde{t}_i)\pi^\dagger(X+\tfrac{R}{2};\tilde{t}_i) \;-\; \pi(X-\tfrac{R}{2};\tilde{t}_i)\Phi(X+\tfrac{R}{2};\tilde{t}_i) \;+\; (R\to -R)\Big]$$

Backup Slides

Particle Number Density

- Take EEV in equal-time limit:

$$\langle \mathcal{Q}(\mathbf{p}, \mathbf{X}, \tilde{t}_f; \tilde{t}_i) \rangle_t \; = \; \lim_{X_0 \to \tilde{t}_f} i \int \mathrm{d}^4 R \, e^{-i\mathbf{p}\cdot\mathbf{R}}$$

$$\times \;\; \delta(R_0) \, \partial_{R_0} \Big[i\Delta_<(R, X, \tilde{t}_f; \tilde{t}_i) \; - \; i\Delta_<(-R, X, \tilde{t}_f; \tilde{t}_i) \Big]$$

- Separate particles ($+$ve freq.) and anti-particles ($-$ve freq.):

$$\delta(R_0) \; = \; \frac{i}{2\pi} \lim_{\epsilon \to 0^+} \left[\frac{1}{R_0 + i\epsilon} \; - \; \frac{1}{R_0 - i\epsilon} \right]$$

- $+$ve freq. part of $i\Delta_<(R, X, \tilde{t}_f; \tilde{t}_i)$ and
  $-$ve freq. part of $i\Delta_<(-R, X, \tilde{t}_f; \tilde{t}_i)$
  $\Rightarrow$ particle number density

Backup Slides

Particle Number Density

- Fourier transform w.r.t. R and shift $\tilde{t}_f \rightarrow \tilde{t}_f - \tilde{t}_i = t$:
  $\Rightarrow$ particle number density:

$$n(\mathbf{p}, \mathbf{X}, t) = \lim_{X_0 \rightarrow t} \int \frac{\mathrm{d}\, p_0}{2\pi}\, p_0 \times \Big[\theta(p_0) i\Delta_<(p, X, t; 0) - \theta(-p_0) i\Delta_<(-p, X, t; 0)\Big]$$

- Also counts off-shell contributions.
- Inserting equilibrium propagators:

$$n(\mathbf{p}, \mathbf{X}, t) = f_{\mathrm{B}}\big(E(\mathbf{p})\big) = \frac{1}{e^{\beta E(\mathbf{p})} - 1}$$

Backup Slides

Pinch Singularities: $\delta^2(p^2 - M^2)$

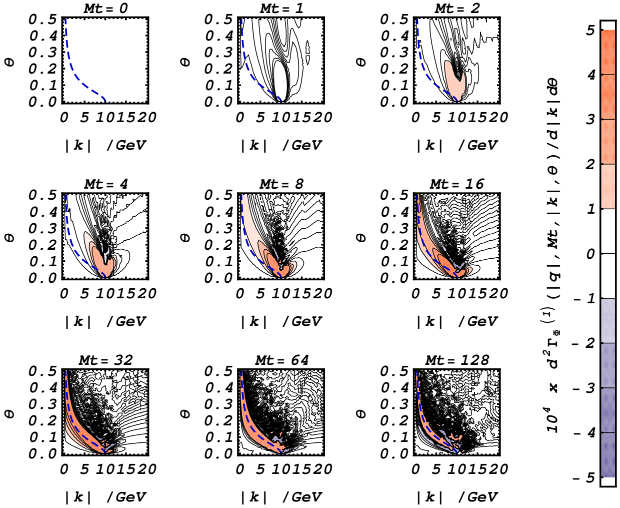
- early times: $\delta^2(p^2 - M^2) \rightarrow \delta(p^2 - M^2)\delta_t(p_0 - p_0')\delta(p'^2 - M^2)$
- intermediate times:
  - pinch singularities grow: $t\delta(p^2 - M^2)$
  - equilibration occurs: $f(t) - f_{eq} = \delta f(t) = \delta f(0)e^{-\Gamma t}$
- late times: $f \rightarrow f_{eq}$ and pinch singularities cancel

$\Leftarrow$ finite time domain

$\Leftarrow$ f's in free propagators evaluated at time of observation

Backup Slides

Phase-Space Evolution



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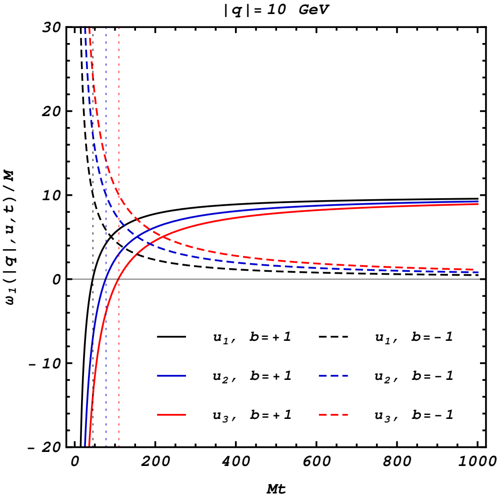
Non-Markovian Oscillations



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Kadanoff–Baym Equations

Kinetic equation:

$$q \cdot \partial_X \Delta_{\gtrless}(q, X) \; - \int \frac{\mathrm{d}^4 Q}{(2\pi)^4} \, (2\pi)^4 \delta_t^{(4)}(Q) \, \sin\Big(\, Q \cdot X \, + \, \diamondsuit^{-}_{q,X} \, + \, 2\diamondsuit^{+}_{Q,X} \Big)$$

$$\Big(\{\Pi_{\gtrless}(q + \tfrac{Q}{2}, X)\}\{\Delta_{\mathcal{P}}(q - \tfrac{Q}{2}, X)\} \, + \, \{\Pi_{\mathcal{P}}(q + \tfrac{Q}{2}, X)\}\{\Delta_{\gtrless}(q - \tfrac{Q}{2}, X)\} \Big)$$

$$= \; \frac{i}{2} \int \frac{\mathrm{d}^4 Q}{(2\pi)^4} \, (2\pi)^4 \delta_t^{(4)}(Q) \, \cos\Big(\, Q \cdot X \, + \, \diamondsuit^{-}_{q,X} \, + \, 2\diamondsuit^{+}_{Q,X} \Big)$$

$$\Big(\{\Pi_{>}(q + \tfrac{Q}{2}, X)\}\{\Delta_{<}(q - \tfrac{Q}{2}, X)\} \, - \, \{\Pi_{<}(q + \tfrac{Q}{2}, X)\}\{\Delta_{>}(q - \tfrac{Q}{2}, X)\} \Big)$$

Diamond operators:

$$\diamondsuit^{\pm}_{p,X}\{A\}\{B\} \; = \; \frac{1}{2}\{A, \; B\}^{\pm}_{p,X} \; \equiv \; \frac{1}{2}\Big(\frac{\partial A}{\partial p^{\mu}} \, \frac{\partial B}{\partial X_{\mu}} \, \pm \, \frac{\partial A}{\partial X^{\mu}} \, \frac{\partial B}{\partial p_{\mu}} \Big)$$

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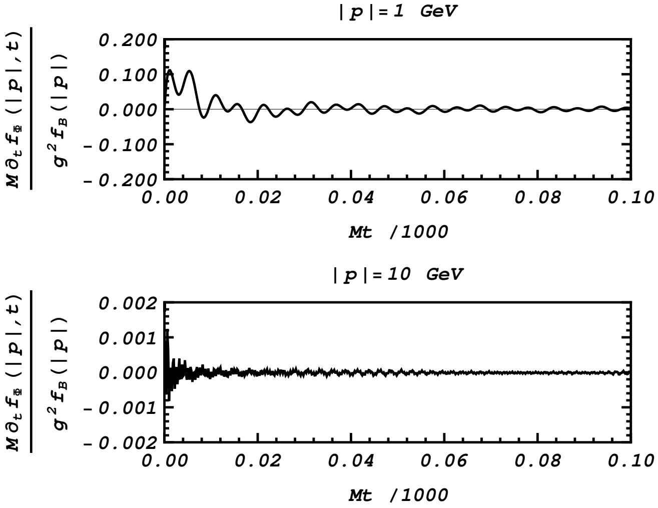
Not Just a Complicated Zero



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Inclusion of Thermal Masses

- Local self-energy $-\lambda\big[\chi^\dagger\chi\big]^2$:

$$\Pi_\chi^{\mathrm{loc}(1)}(p,p',\tilde{t}_f;\tilde{t}_i) \ = \ -(2\pi)^4\delta_t^{(4)}(p\ -\ p')e^{i(p_0\ -\ p_0')\tilde{t}_f}m_{\mathrm{th}}^2(\tilde{t}_f;\tilde{t}_i)$$

- Thermal mass:

$$m_{\mathrm{th}}^2(\tilde{t}_f;\tilde{t}_i) \ = \ \frac{\lambda}{2}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{1}{\sqrt{2E_\chi(\mathbf{k})}}\int\frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3}\frac{1}{\sqrt{2E_\chi(\mathbf{k}')}} \Big[f_\chi(\mathbf{k},\mathbf{k}',t)e^{i[E(\mathbf{k})-E(\mathbf{k}')]\tilde{t}_f}+f_\chi^{C*}(-\mathbf{k},-\mathbf{k}',t)e^{-i[E(\mathbf{k})-E(\mathbf{k}')]\tilde{t}_f}\Big]$$

- Quasi-particle approximation: $m^2 \ \rightarrow \ m_{\mathrm{th}}^2(\tilde{t}_f;\tilde{t}_i)$
- Coupling to system of ODEs (spatially homogeneous case):

$$\partial_t m_{\mathrm{th}}(t) = \frac{\lambda}{2m_{\mathrm{th}}(t)}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{1}{2E_\chi(\mathbf{k})}\frac{1}{2}\Big[\partial_t f_\chi(|\mathbf{k}|,t)+\partial_t f_\chi^C(|\mathbf{k}|,t)\Big]$$