Role of Coarse-Graining Size for Local Thermal Equilibrium and non homogeneities in Initial Condition

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Collaboration with

Philipe de A. Mota - FIAS

Rafael D. Souza - UNICAMP

Jun Takahashi – UNICAMP

Tomoi Koide - UFRJ

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– Ph Mota, et al, EP A48, 1–12, 2012
– T Koide & T. K. JPhysA 45, 255204, 2012

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Common Statement :

Success of (almost ideal) hydrodynamic description in Relativistic Heavy Ion Collisions

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Expectations and hopes :

- Determination of Properties of Matter (EoS, Transport coefficieints)
- Comparison with Lattice QCD
- Determination of Initial State just after the Collision
- Key for the QCD dynamics…

Common Statement we hear frequently:

Success of (almost ideal) hydrodynamic description in Relativistic Heavy Ion Collisions

Local Thermal Equilibirum

Expectations and hopes :

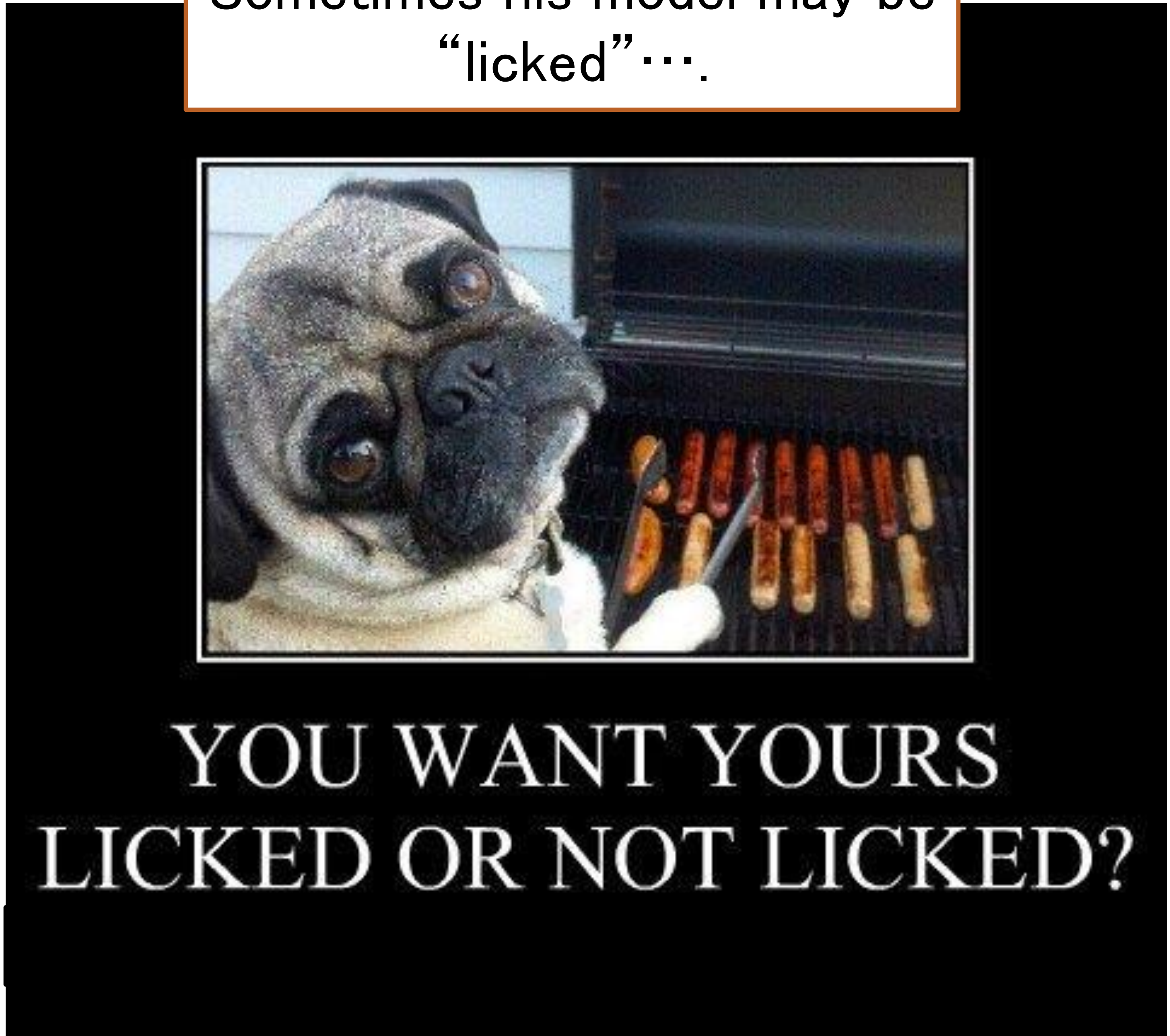
- Determination of Properties of Matter (EoS, Transport coefficieints)
- Comparison with Lattice QCD
- Determination of Initial State just after the Collision
- Key for the QCD dynamics…

Very Nice, should be pushed forward.

However,······

When a theorist cooks his model,..

Sometimes his model may be "licked"….



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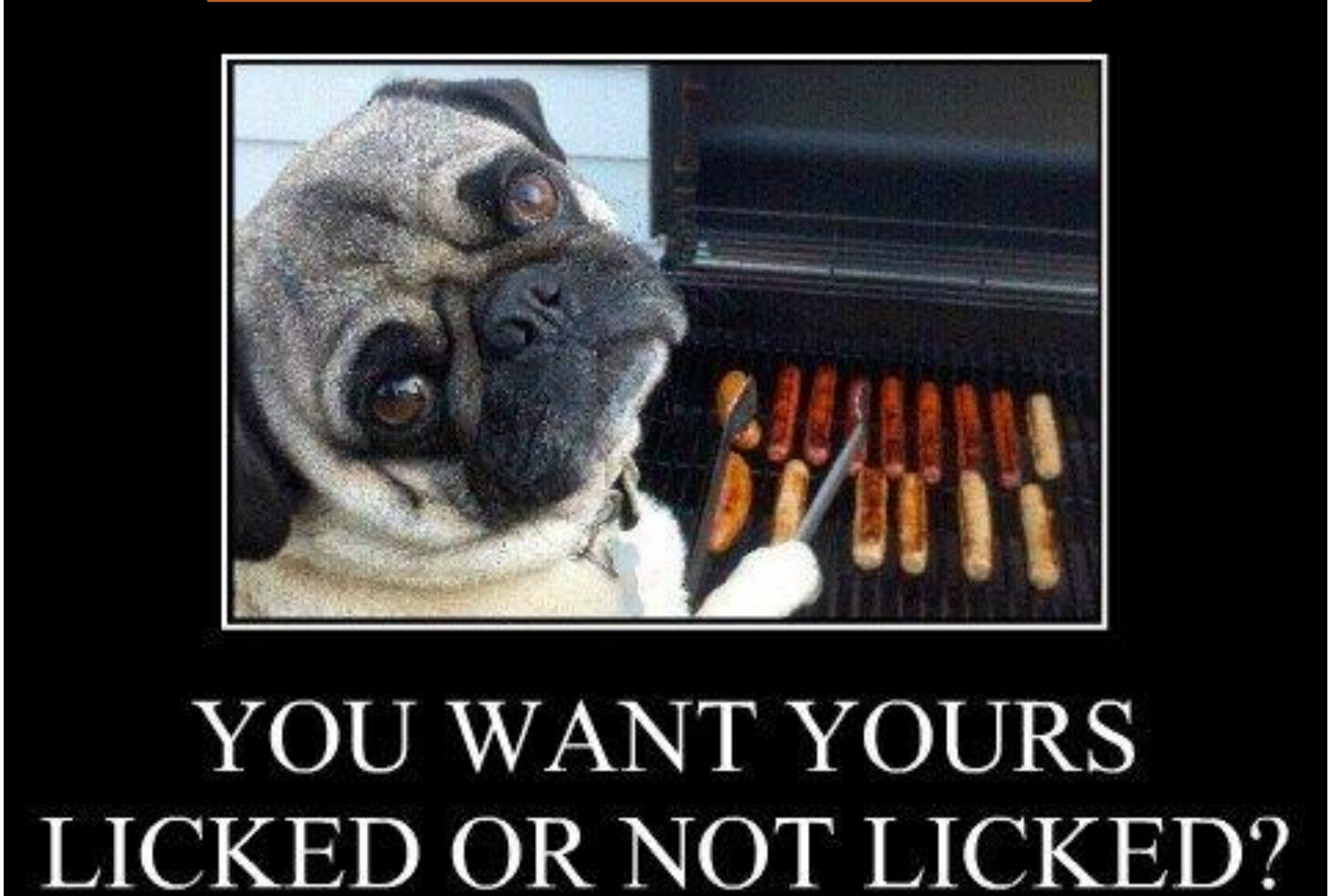


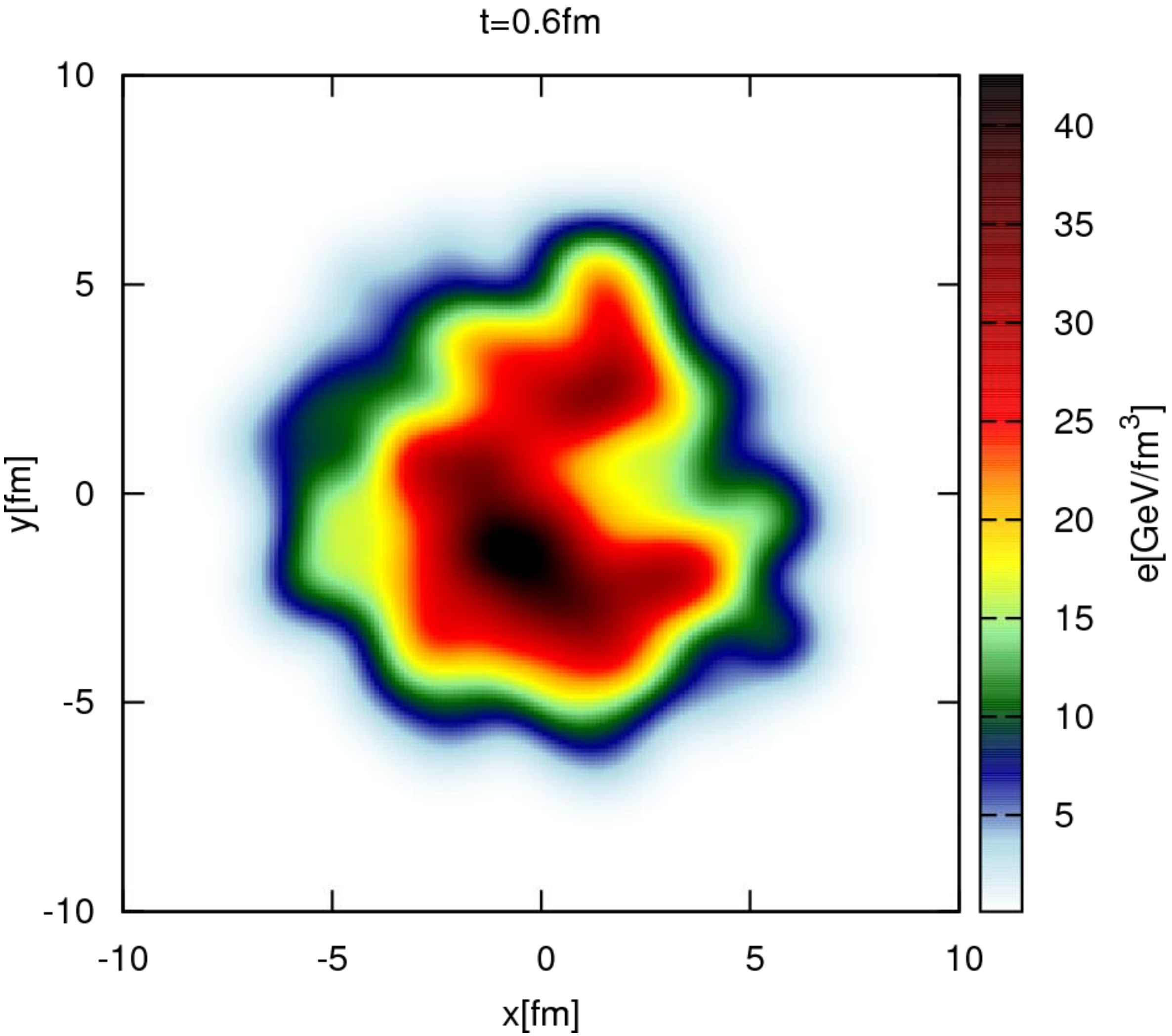
Just kidding.., They are ALL licked…

Relativistic Hydrodynamics as Covariant Local Classical Field Theory

- Local Thermal Equilibrium is sometimes considered as a necessary condition
- Not necessarily …. even *Conflicting*, if strictly local.

Structure of Relativistic Hydrodynamics

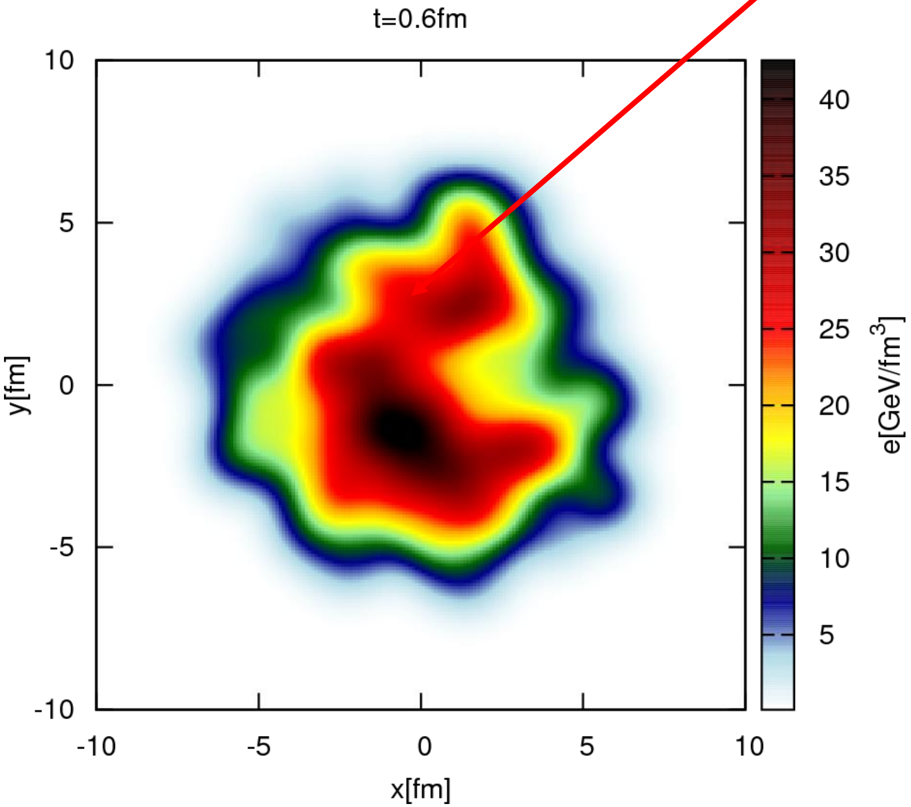
Structure of Relativistic Hydrodynamics



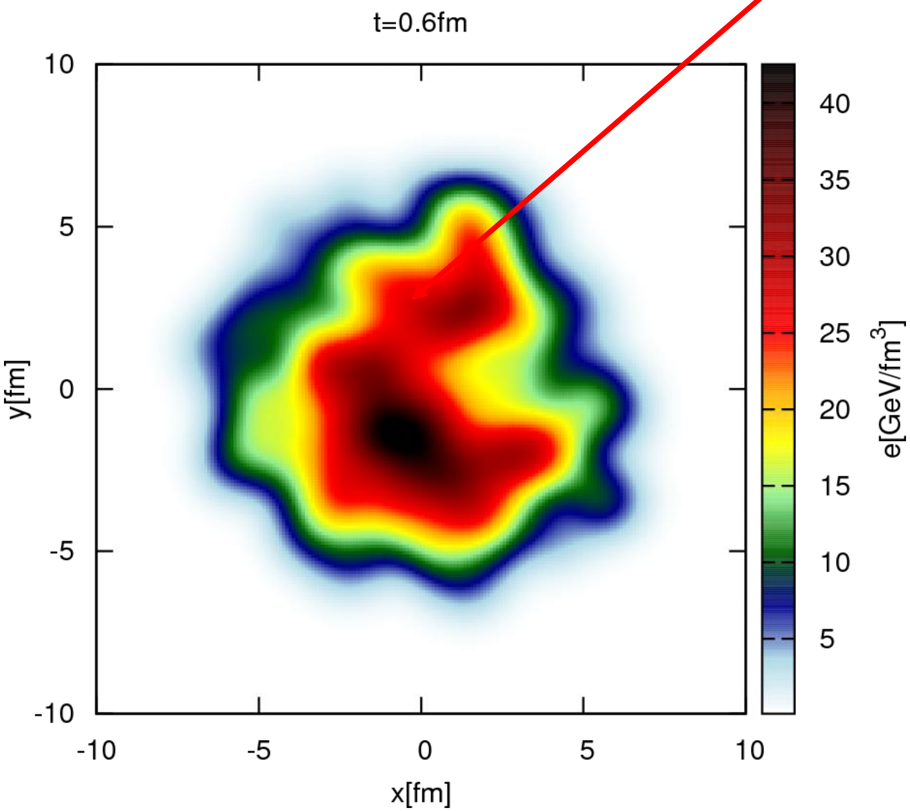
Structure of Relativistic Hydrodynamics

$$^{\forall}x = (t, \vec{r}),$$

$$^{\exists}T^{\mu\nu}(x),\ n^{\mu}(x)$$



Structure of Relativistic Hydrodynamics



$$^{\forall}x = (t, \vec{r}),$$

$$^{\exists}T^{\mu\nu}(x),\ n^{\mu}(x)$$

with

$$\partial_{\mu} T^{\mu\nu}(x) = 0,$$

$$\partial_{\mu} n^{\mu}(x) = 0.$$

Structure of Relativistic Hydrodynamics

14 unknowns

$$^{\forall}x = (t, \vec{r}),$$

$$^{\exists}T^{\mu\nu}(x),\ n^{\mu}(x)$$

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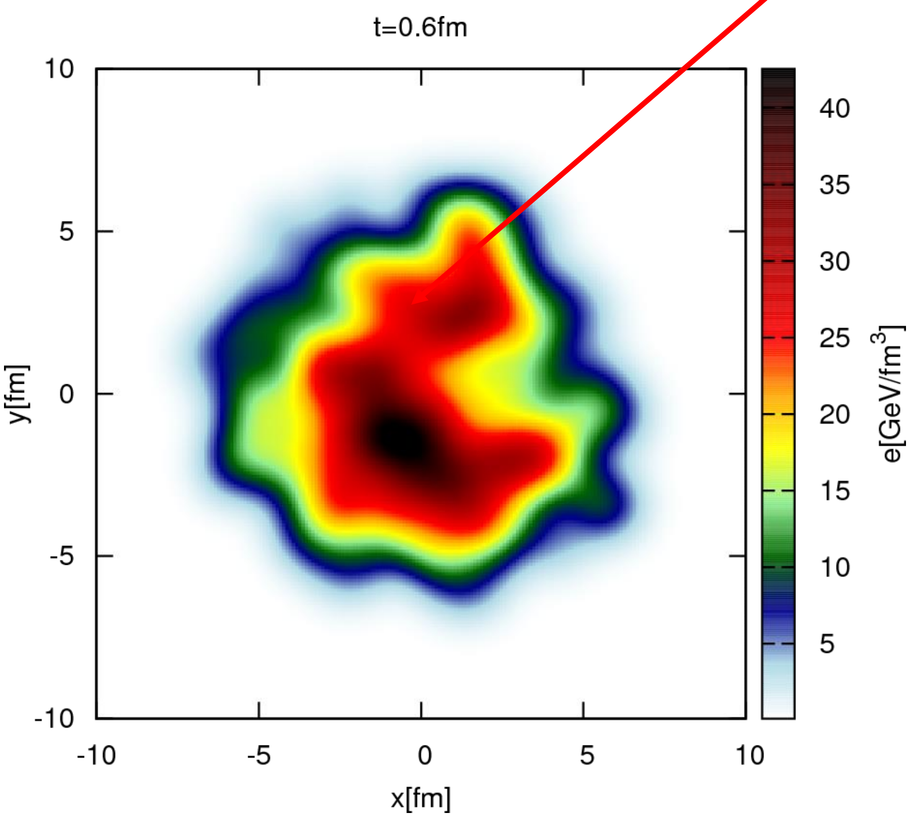
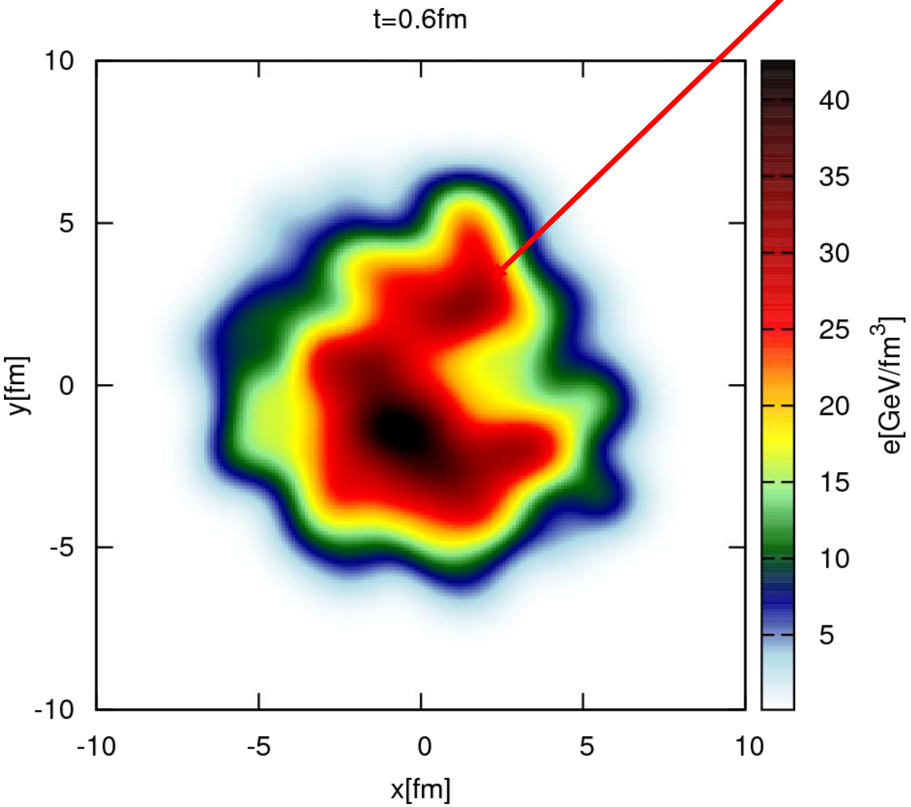
5 equations



Structure of Relativistic Hydrodynamics

Local Rest Frame (Landau)

$^{\forall}x,$

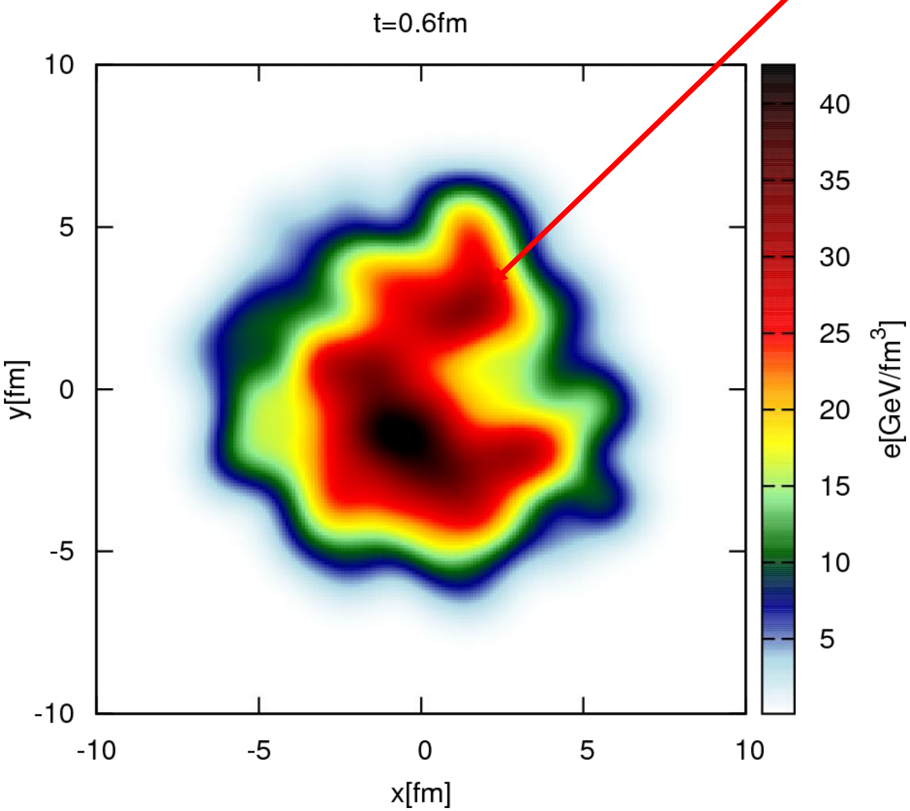


$$T^{\mu\nu}(x) \xrightarrow{\Lambda(u)} \begin{pmatrix} \varepsilon & 0 \\ 0 & T \end{pmatrix}$$

Structure of Relativistic Hydrodynamics

Local Rest Frame (Landau)

$\forall x,$



$$T^{\mu\nu}(x) \xrightarrow{\Lambda(u)} \begin{pmatrix} \varepsilon & 0 \\ 0 & T \end{pmatrix}$$

and

$$n^{\mu}(x) \xrightarrow{\Lambda(u)} \begin{pmatrix} n \\ \vec{j}_{Diff} \end{pmatrix}$$

Structure of Relativistic Hydrodynamics

Local Rest Frame (Landau)

$$^{\forall}x,$$

Additional conditions:

- T is isotropic
- No diffusion $\vec{j}_{Diff} = 0$

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14-5

Structure of Relativistic Hydrodynamics

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14−5 −3

Structure of Relativistic Hydrodynamics

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$$T^{\mu\nu}(x) \xrightarrow{\Lambda(u)} \begin{pmatrix} \varepsilon & 0 \\ 0 & T \end{pmatrix}$$

and

14−5 −3 −1 = 5

$$n^{\mu}(x) \xrightarrow{\Lambda(u)} \begin{pmatrix} n \\ \vec{j}_{Diff} \end{pmatrix}$$

- ε is strongly correlated with n

Structure of Relativistic Hydrodynamics

Ideal fluid case

$^{\forall} x,$

Additional conditions:

- T is isotropic
- No diffusion $\vec{j}_{Diff} = 0$

$$T^{\mu\nu}(x) \xrightarrow{\Lambda(u)} \begin{pmatrix} \varepsilon & 0 \\ 0 & T \end{pmatrix}$$

and

14−5 −3 −1 = 5 = No. of Eqs.!!

$$n^{\mu}(x) \xrightarrow{\Lambda(u)} \begin{pmatrix} n \\ \vec{j}_{Diff} \end{pmatrix}$$

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Structure of Relativistic Hydrodynamics

Local Thermal Equilibrium

$^{\forall}x,$

$$T^{\mu\nu}(x) \xrightarrow{\Lambda(u)} \begin{pmatrix} \varepsilon & 0 \\ 0 & T \end{pmatrix}$$

and

$$n^{\mu}(x) \xrightarrow{\Lambda(u)} \begin{pmatrix} n \\ \vec{j}_{Diff} \end{pmatrix}$$

Additional conditions:

- T is isotropic
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14−5 −3 −1 = 5

- ε is strongly correlated with n

Questions for Local Thermal Equilibrium

- It is a sufficient condition for Ideal Fluid dynamics. But is it a necessary condition?

- How local?

Can not be strictly local (compatibility with the thermodynamics).

- If not local, how the local covariant theory can emerge?

- How much can we say about the inhomogeneous nature of the initial conditions?

Questions for Local Thermal Equilibrium

- It is a sufficient condition for Ideal Fluid dynamics. But is it a necessary condition?
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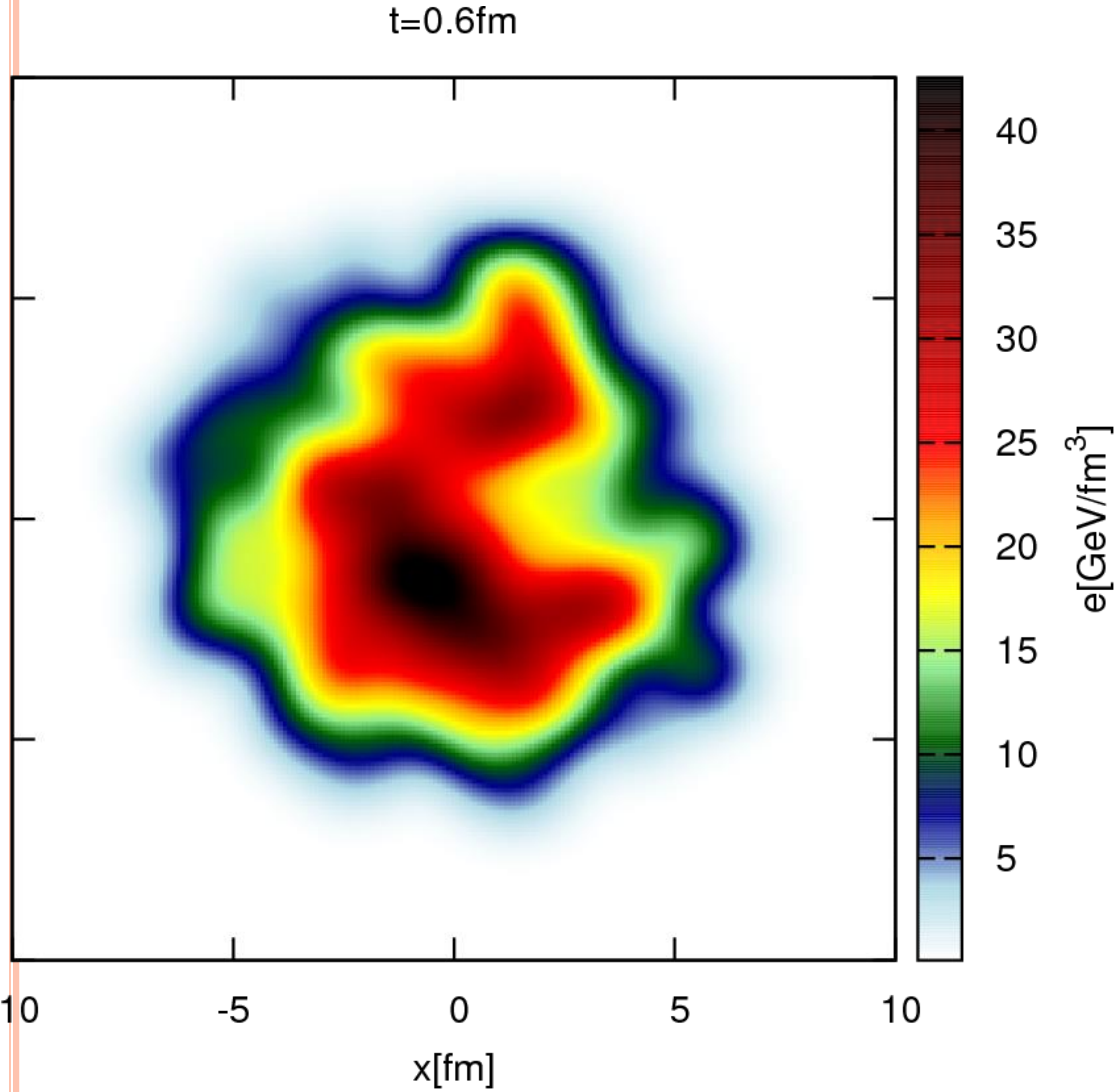
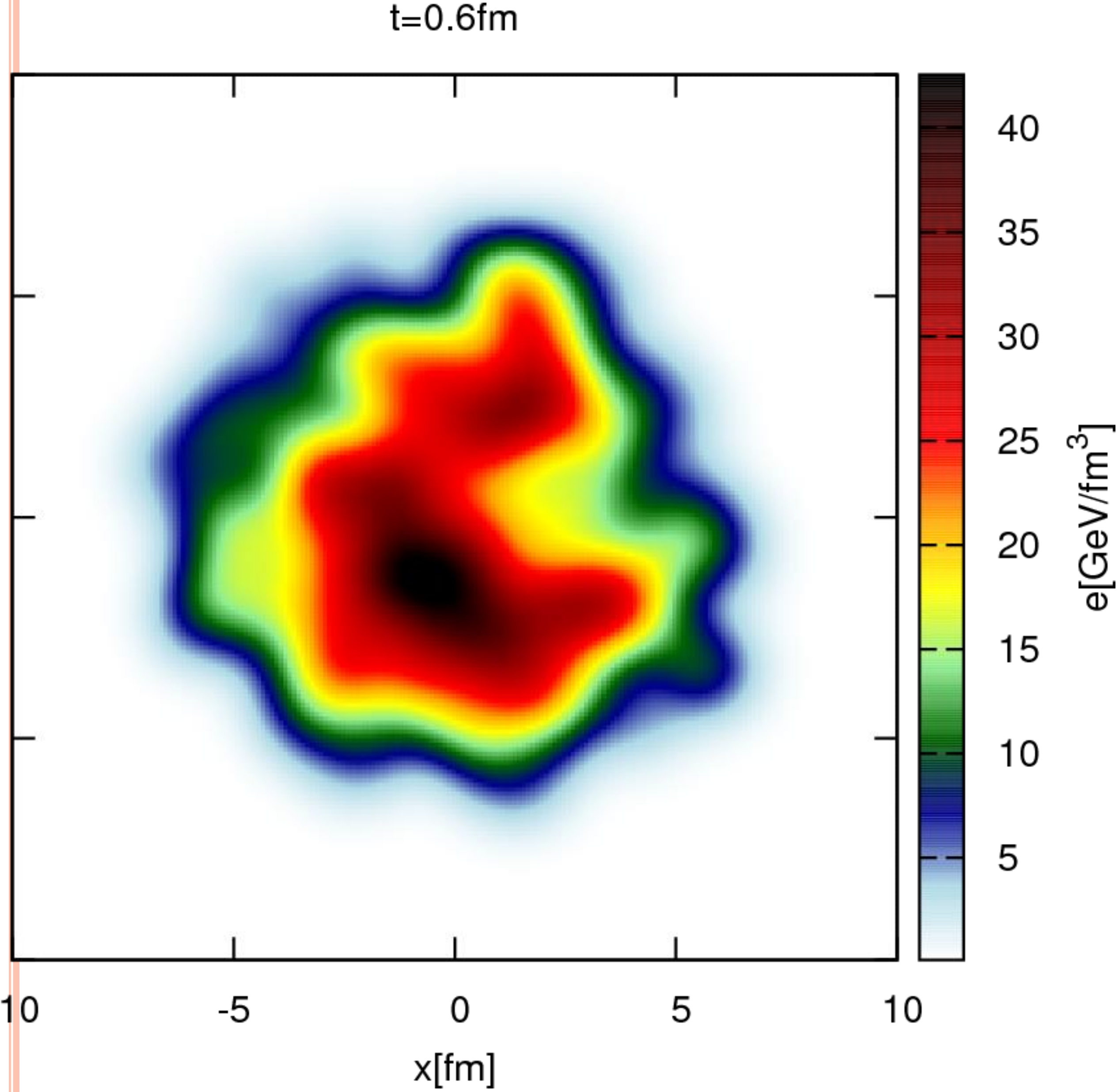
Relativity X Thermodynamics

- If not local, how the local covariant theory can emerge?
- How much can we say about the inhomogeneous nature of the initial conditions?

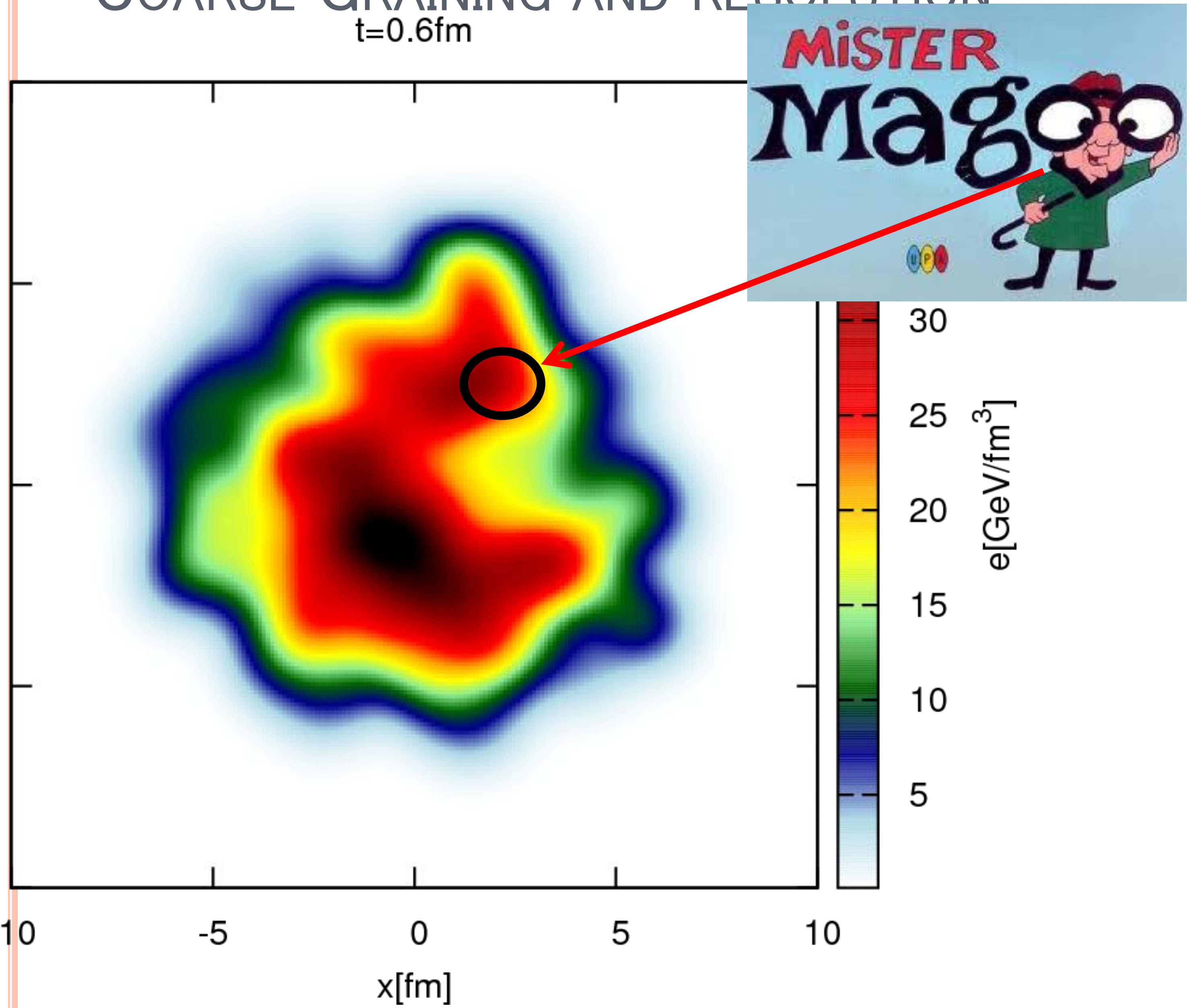
Coarse Graining and resolution



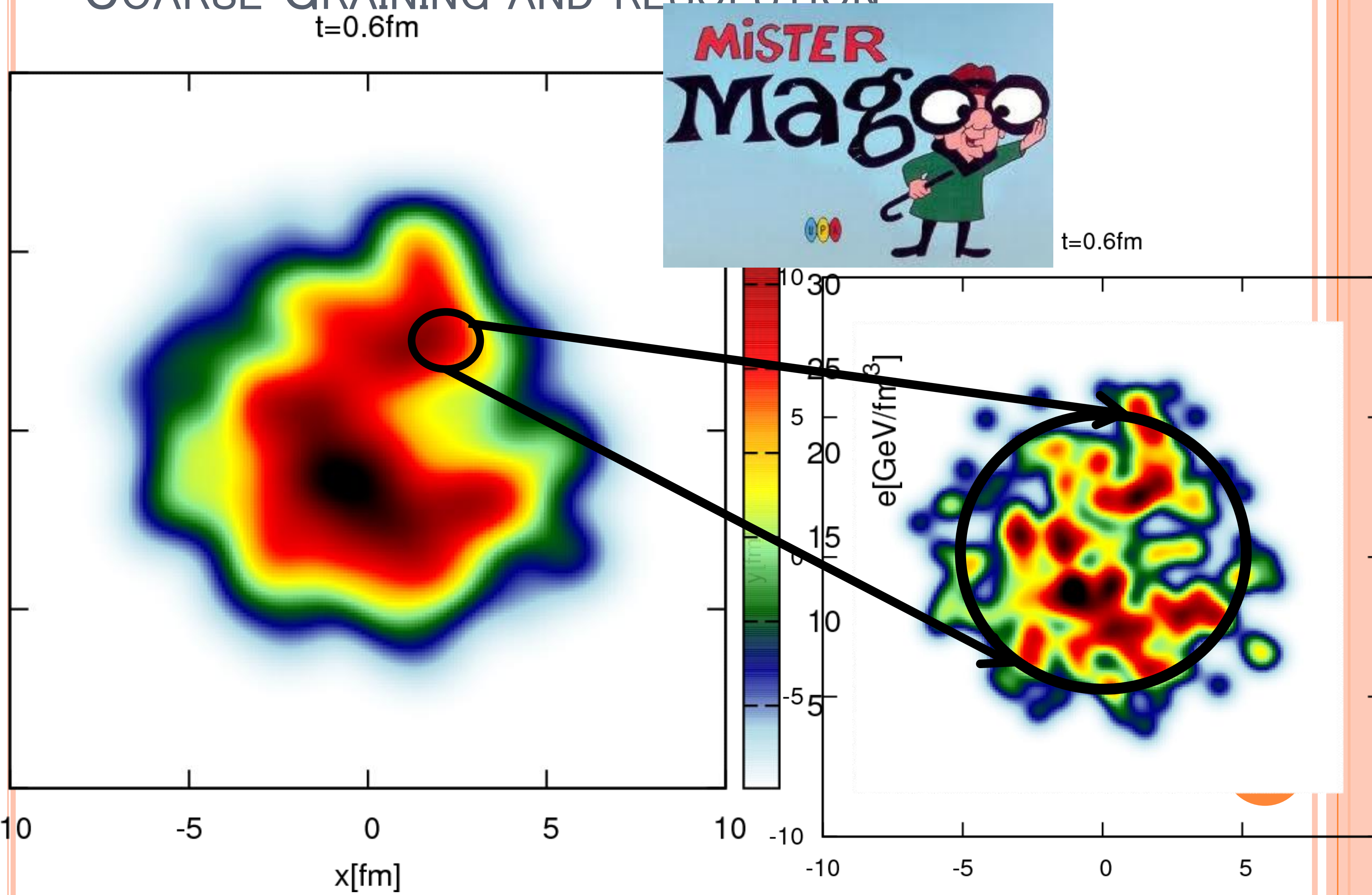
Coarse Graining and resolution



Coarse Graining and resolution



Coarse Graining and resolution



Example:

- Matter density expressed in terms of Lagrange Coordinates:

$$n^*(t,\vec{r}) = \int d^3\vec{R}\; n_0(\vec{R})\; \delta(\vec{r} - \vec{r}_R(t))$$

EXAMPLE:

- Matter density expressed in terms of Lagrange Coordinates:

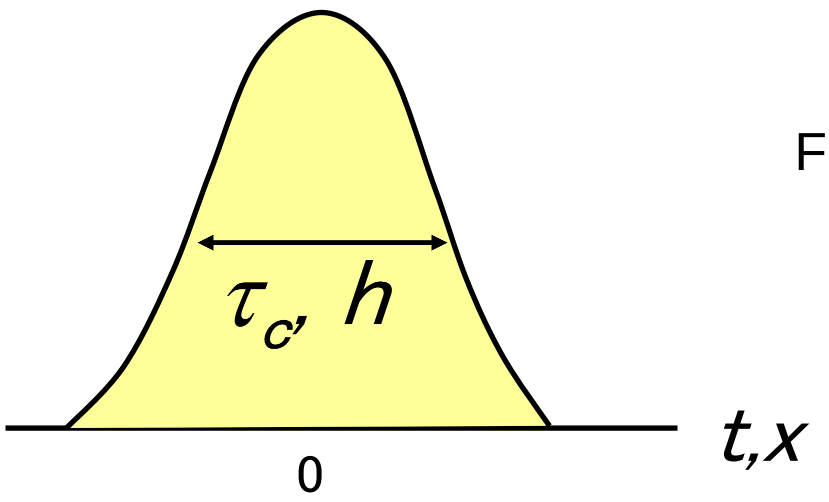
$$n^*(t,\vec{r}) = \int d^3\vec{R}\; n_0(\vec{R})\; \delta(\vec{r} - \vec{r}_R(t))$$

- When we don't have space and time resolution,

$$n^*(t,\vec{r}) \rightarrow$$

$$\int dt' d^3\vec{R}\; n_0(\vec{R})\; U_{\tau_c}(t'-t) W_h(\vec{r} - \vec{r}_R(t))$$

$$U_{\tau_c}(t), W_h(\mathbf{x}) \leftrightarrow \text{smoothing kernel}$$



For example, Gaussian

$$\int U(t)dt = \int W(x)dx = 1$$

$$n^*(t,\vec{r}) = \int dt' d^3\vec{R}\, n_0(\vec{R})\, U_{\tau_c} W_h$$

$$\vec{j}(t,\vec{r}) = \int dt' d^3\vec{R}\, n_0(\vec{R})\, U_{\tau_c} W_h \frac{d\vec{r}_M}{dt'}$$

$$U_{\tau_c} = U_{\tau_c}(t'-t)$$

$$W_h = W_h(\vec{r} - \vec{r}_R(t'))$$

$$n^*(t,\vec{r}) = \int dt' d^3\vec{R}\, n_0(\vec{R})\, U_{\tau_c} W_h$$

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$$U_{\tau_c} = U_{\tau_c}(t' - t)$$

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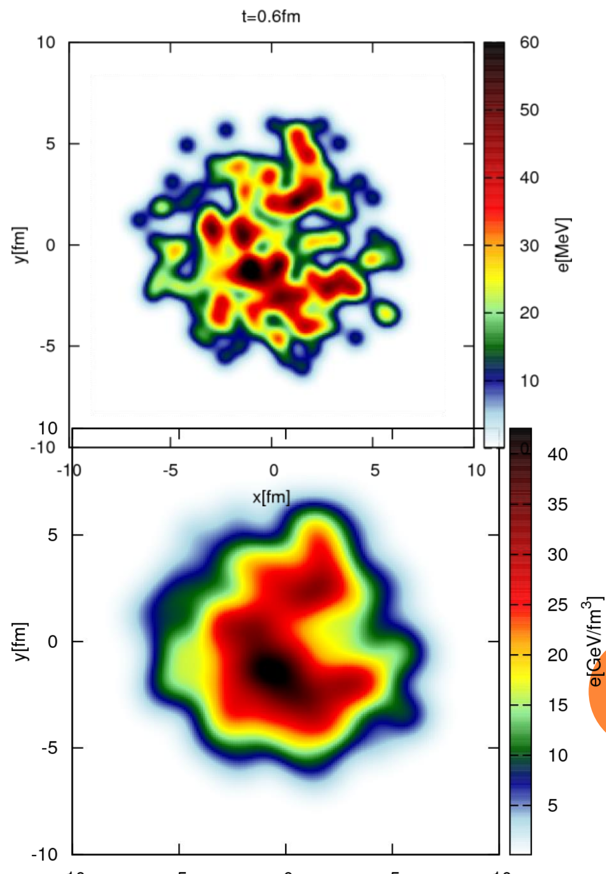

$$n^*(t,\vec{r}) = \int dt'\, d^3\vec{R}\, n_0(\vec{R})\, U_{\tau_c} W_h$$

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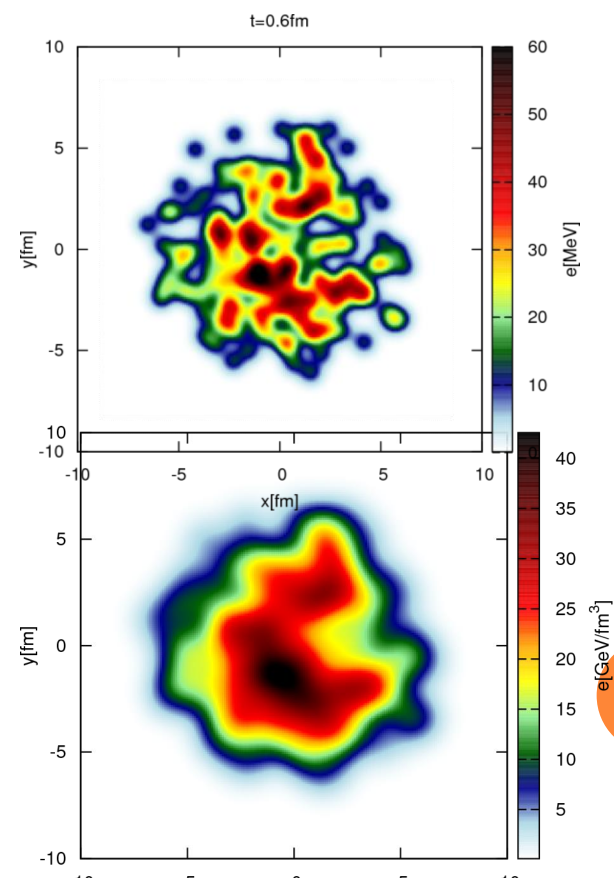
Not exactly local .. but

$$\partial_t n^*(t,\vec{r}) + \nabla \cdot \vec{j}(t,\vec{r}) = 0$$



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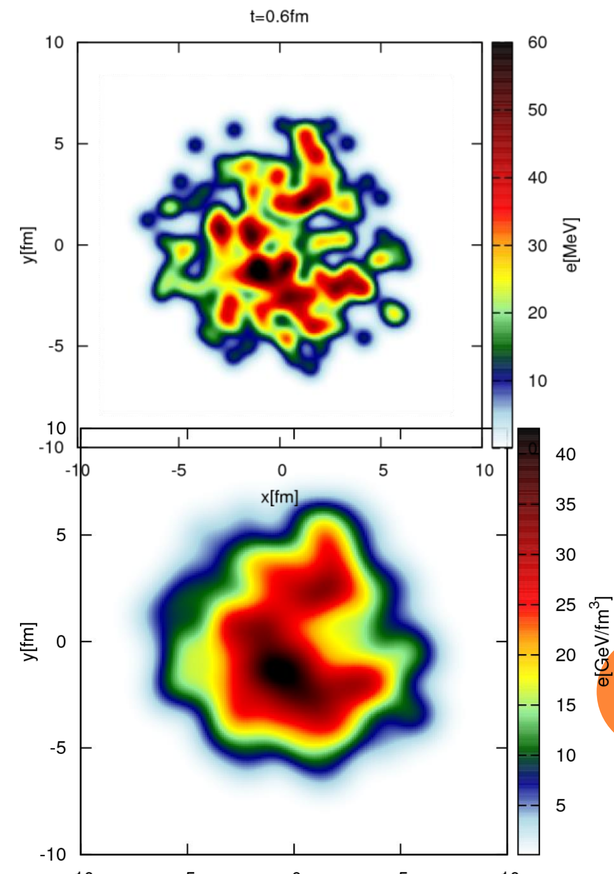
Even we can write

$$n^\mu = (n^*, \vec{j}),$$

$$\partial_\mu n^\mu = 0.$$



We can do this also for $T^{\mu\nu}(x)$

$$T^{\mu\nu}(x) = \int dt' d^3\vec{x}' \; U_{\tau_c} W_h \, T_M{}^{\mu\nu}(t, \vec{x}')$$

Define
$$n(t,\vec{r}) = \sqrt{n_\mu n^\mu},$$
$$u^\mu(t,\vec{r}) = n^\mu / n,$$
$$\varepsilon(t,\vec{r}) = u_\mu u_\nu T^{\mu\nu},$$

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Physical meaning of ε and n:

"Proper" energy and number densities measured in the local rest frame defined with the coarse-grained quantities.

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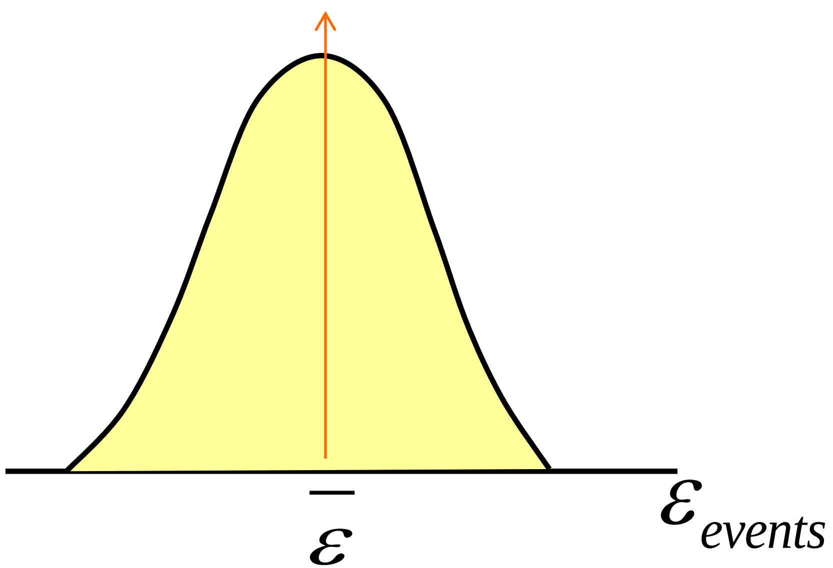
Reminder:

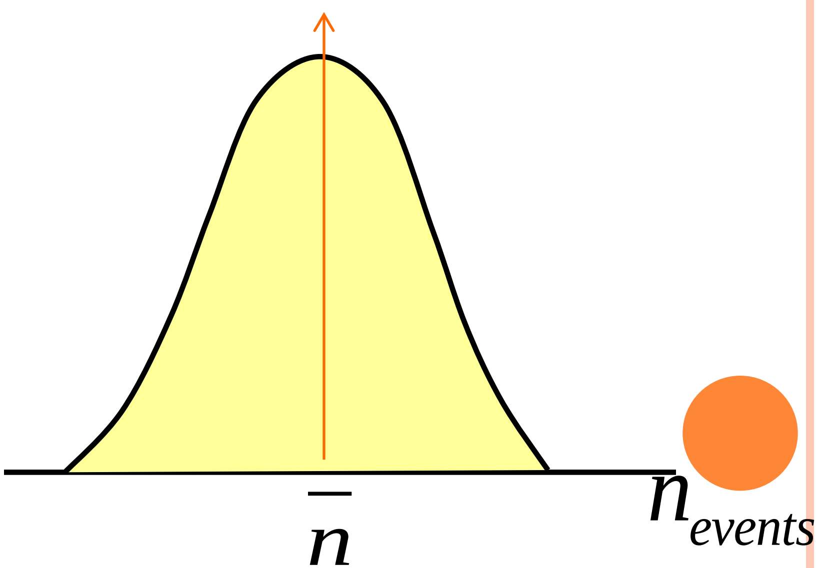
For a given coarse-grained profile $n^\mu(t_0,\vec{r})$
there are many events in microscopic level,
that is, there exists a big statistical ensemble.

Say, Ω, such an ensemble that,

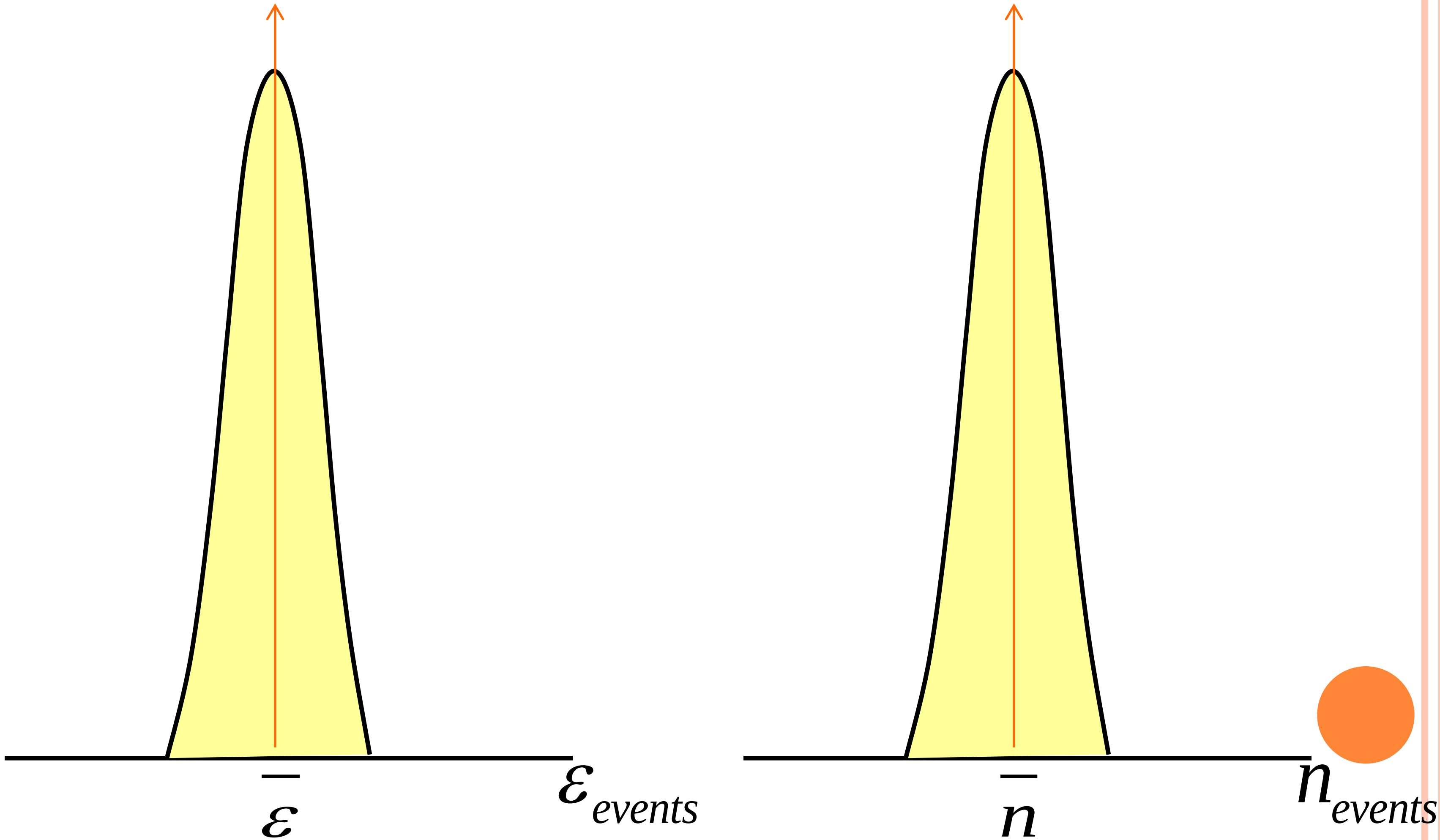
$$\Omega = \left\{ events \;\middle|\; n^{\mu}(t_0, \vec{r}) = n_0{}^{\mu}(\vec{r}) \right\}.$$

Densities at a given space and time point, $(t, \vec{r})$ ε and n fluctuate event by event in this ensemble, Ω.



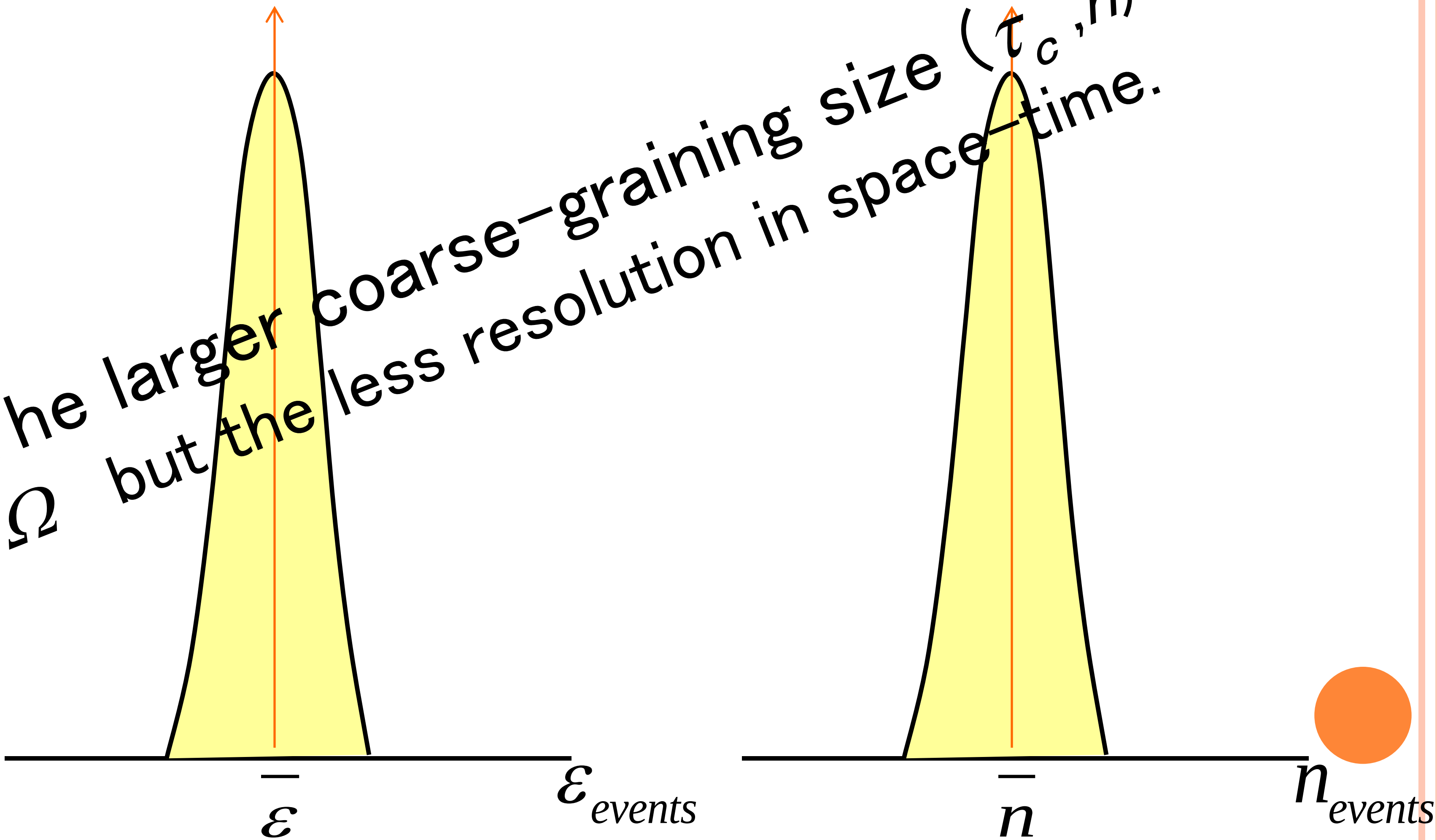


For larger Ω, the width may become small (central limit)



For larger Ω, the width may become small (central limit)

The larger coarse-graining size (τ_c, h) larger Ω but the less resolution in space-time.



Hydrodynamic modeling via Variational Principle

1. $\bar{\varepsilon}$ and $\bar{n}$ are strongly correlated so that

$$\bar{\varepsilon} = \bar{\varepsilon}(\bar{n})$$

2. Dynamics in terms of coarse-grained variable, $\bar{n}^{\mu}$ is determined by the action,

$$I = -\int d^4x \, \bar{\varepsilon}(\bar{n}(x))$$

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(continuum generalization of the Lagrangian for a particle)

$$L = -m\sqrt{1-\vec{v}^{\,2}}$$

Hydrodynamic modeling via Variational Principle

$$\delta I = -\delta \int d^4x\, \bar{\varepsilon}(\bar{n}(x)) = 0$$

with respect to

$$\bar{n}^{\mu} = (\bar{n}^{*},\ \bar{n}^{*}\vec{v})$$

subject to the constraint

$$\bar{n}_{\mu}\bar{n}^{\mu} = \bar{n}^{2}$$

H-T. Elze, Y. Hama, T. K and J. Rafelski,.J. PhysG:25(9):1935, 1999

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leads

Relativistic Euler Eqs.

$$\partial_{\mu}\left\{(\bar{\varepsilon}+P)u^{\mu}u^{\nu} - Pg^{\mu\nu}\right\} = 0, \qquad P = \frac{d\bar{\varepsilon}}{dn}\bar{n} - \bar{\varepsilon},$$

H-T. Elze, Y. Hama, T. K and J. Rafelski,.J. PhysG:25(9):1935, 1999

Hydrodynamic modeling via Variational Principle

When the fluctuation is not negligible;

$$\delta I = -\delta \int d^4x\, \varepsilon(n(x)) = 0$$

for stochastic variable leads to

Navier–Stokes Eqs. for a viscous fluid,
in non-relativistic limit !

T. Koide and T. K, .J. PhysA: 45(25):255204

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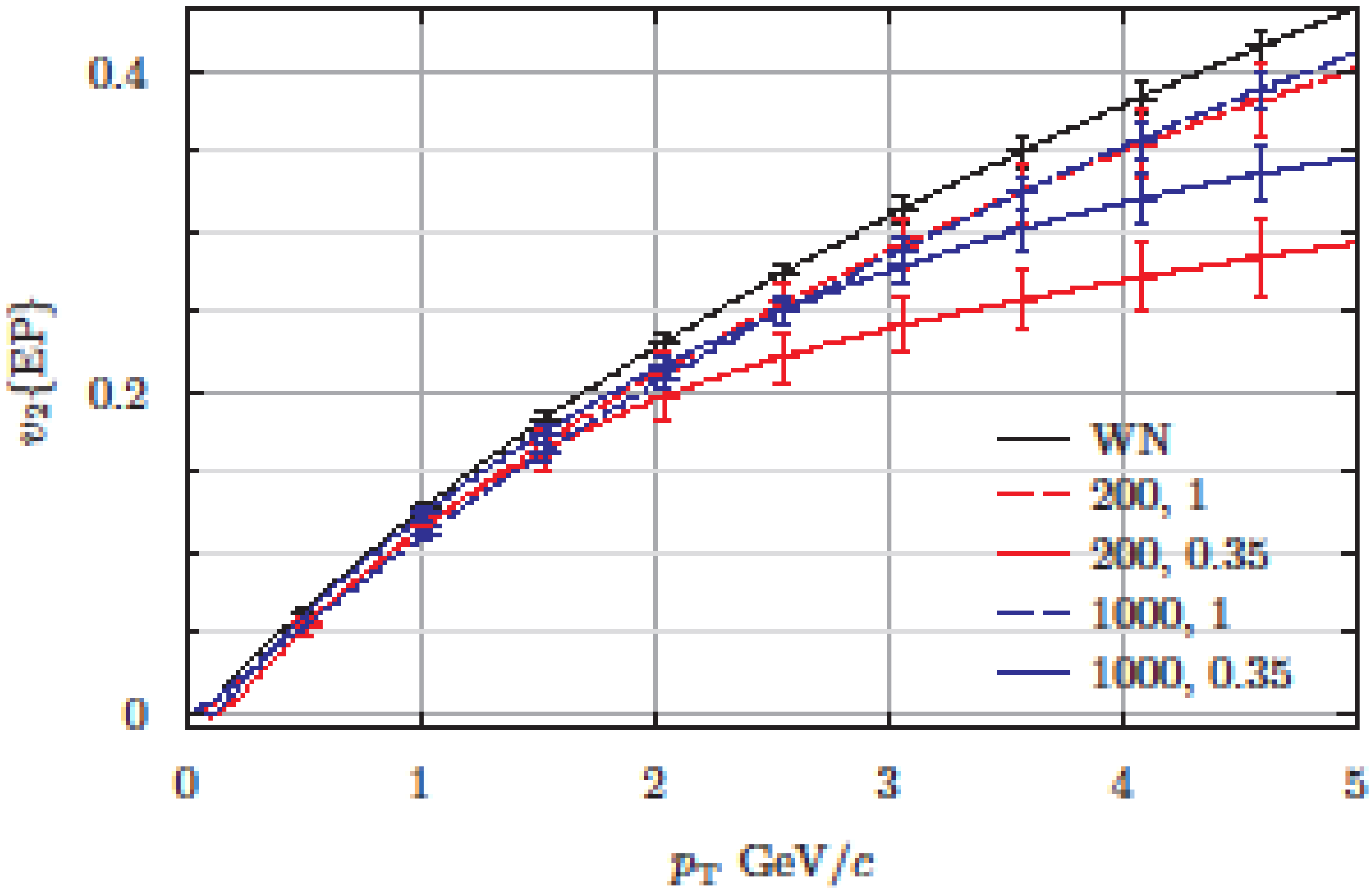
In fact, fluctuations in initial conditions gives a similar effect as viscosity

T. Koide and T. K, .J. PhysA: 45(25):255204

Event averaged v_2



R. Andrade, et al,, Phys. Rev. Lett., 97:202302

Ph. Mota et al., Nuclear Physics A, 862:188 , 2011

Now we have problem…..

- Once arrived to the relativistic Euler equation, we cannot tell the coarse-graining scale.
- Transport coefficients, or even effective EoS may depend on this scale.
- Some observables may not be sensitive to this scale. If we see only these, we would conclude that the ideal hydro works well…

Important to study

- Find observables that are sensitive to the coarse graining scale via genuine hydro signal
- Event-by-Event hydro

Genuine (Local) Hydrodynamic Signal

- Time evolution of hydrodynamic profile.

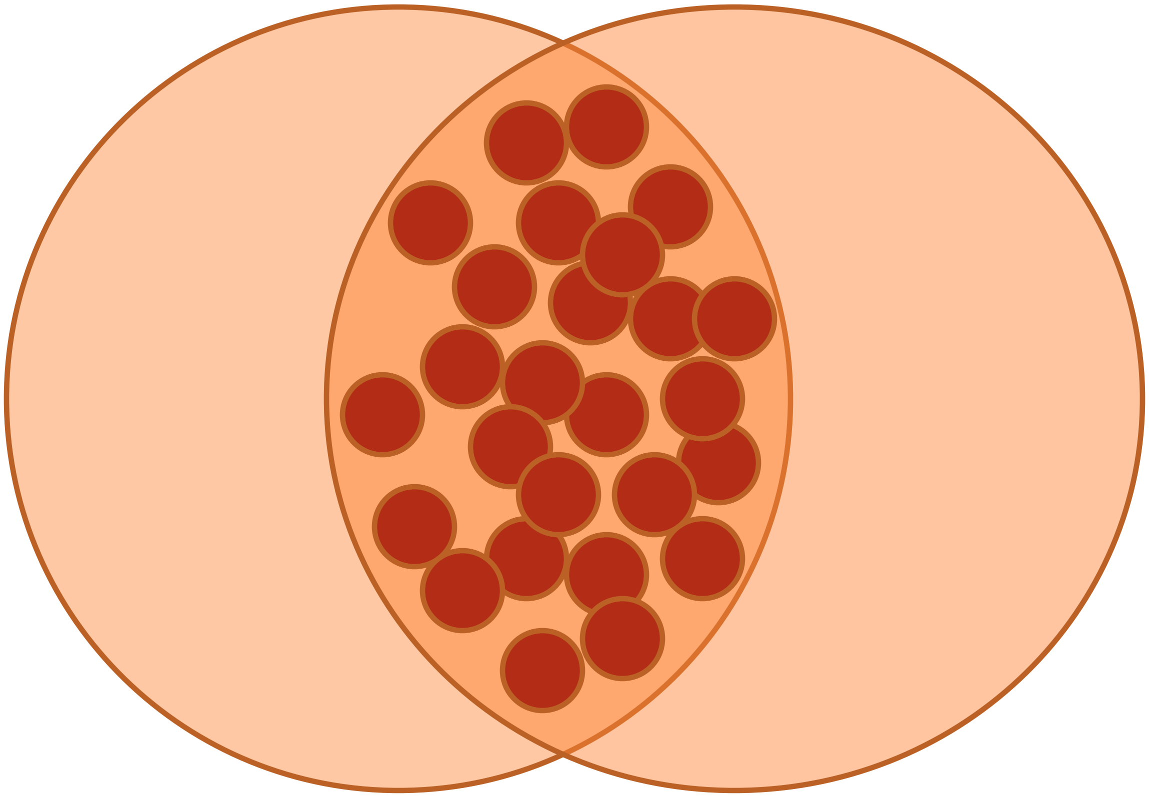
Genuine (Local) Hydrodynamic Signal

- Time evolution of hydrodynamic profile.

– Not observable in heavy ion collisions (may be shock wave and its thickness, or Kelvin-Helmholtz instability (L. P. Csernai, D. D. Strottman, and Cs. Anderlik. Phys. Rev. C, 85:054901)

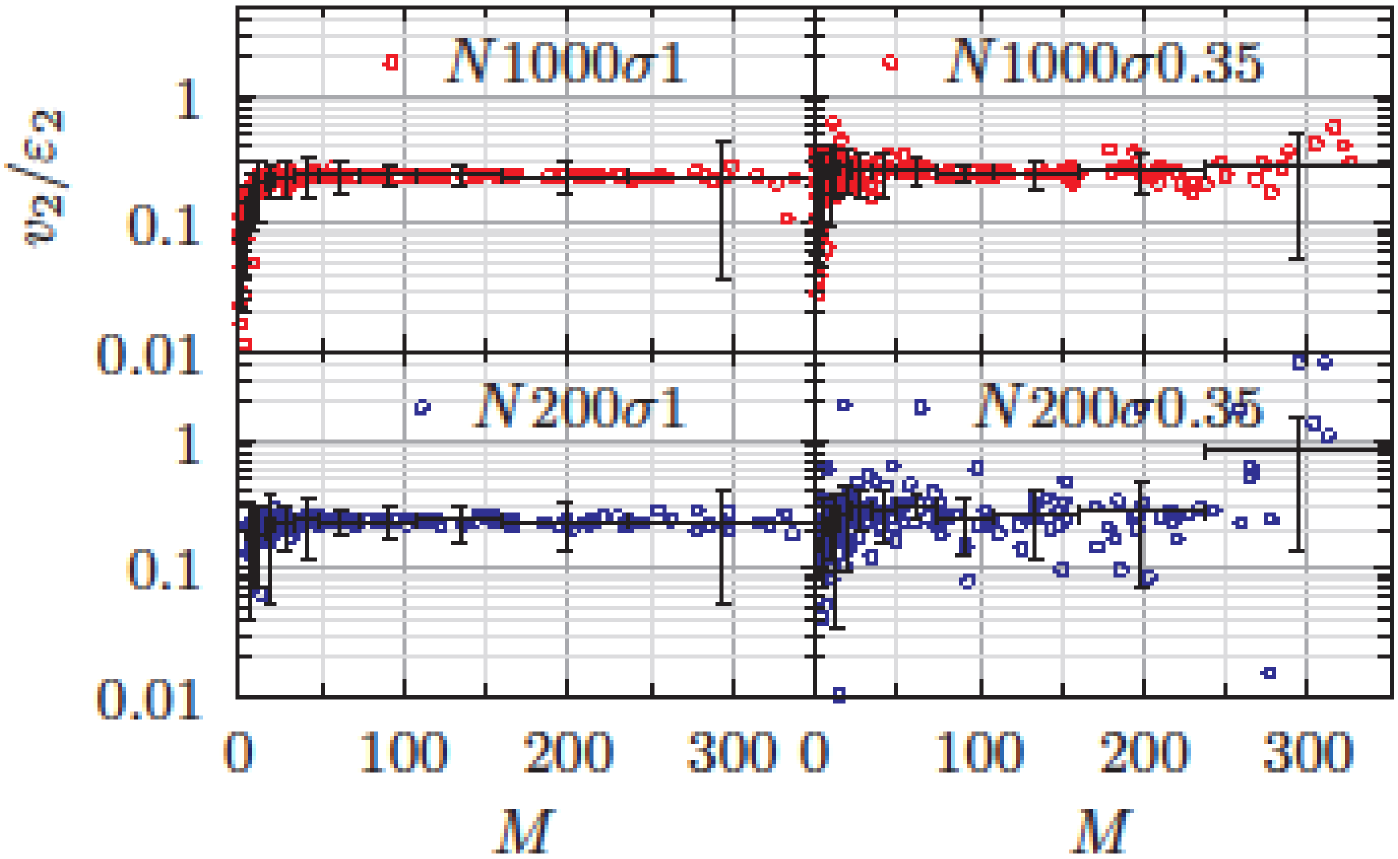
Necessity for Systematic Studies on the effects of granularities in the initial conditions

- Multi–flux tube inspired model



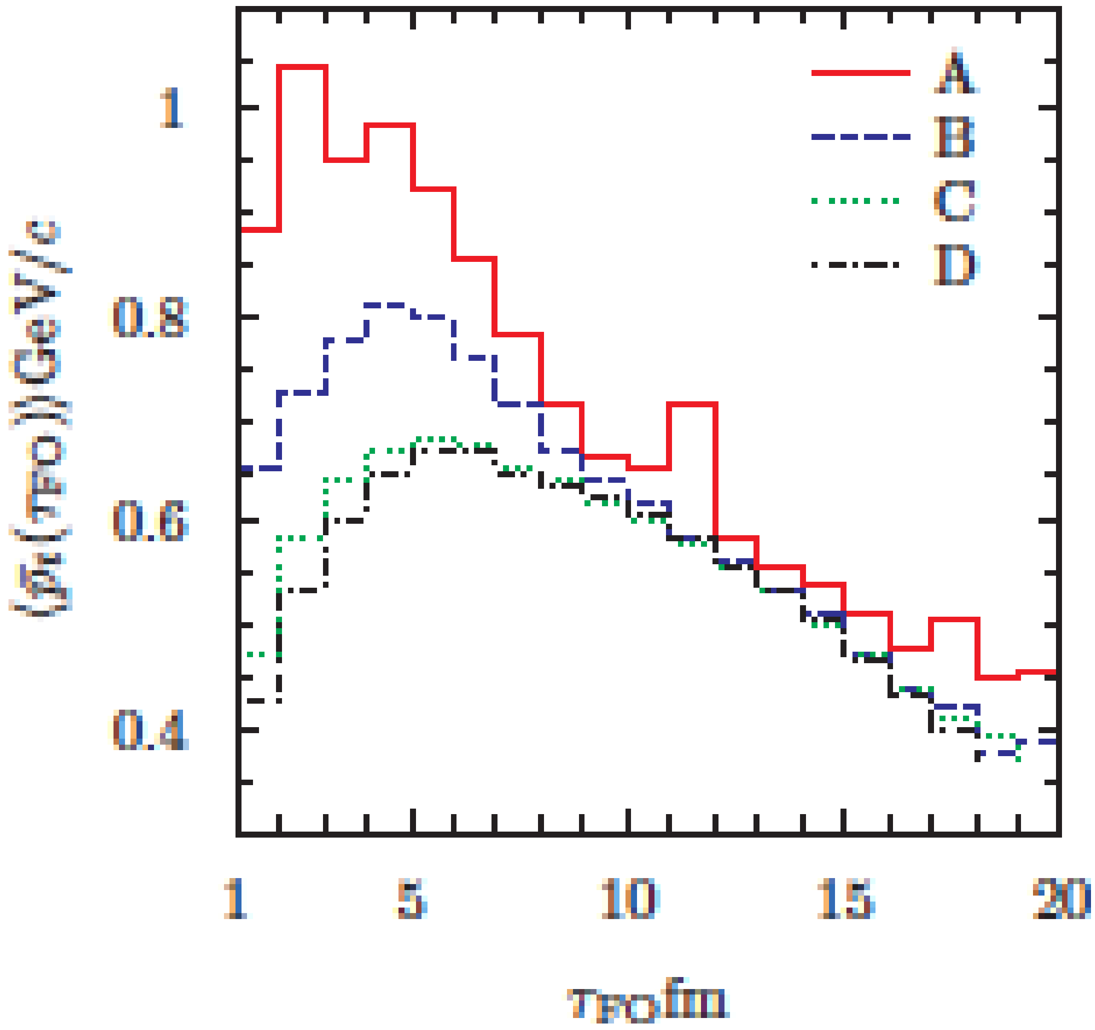
Gaussian with the width σ and the energy $\varepsilon_0 = \varepsilon_T / N$

Sensitivity of v_2 / e_2



Event averaged v_2 / e_2 is not sensitive to the granularity, although almost looses the EbE correlation for high granularity

Average p_T as function of freezeout time

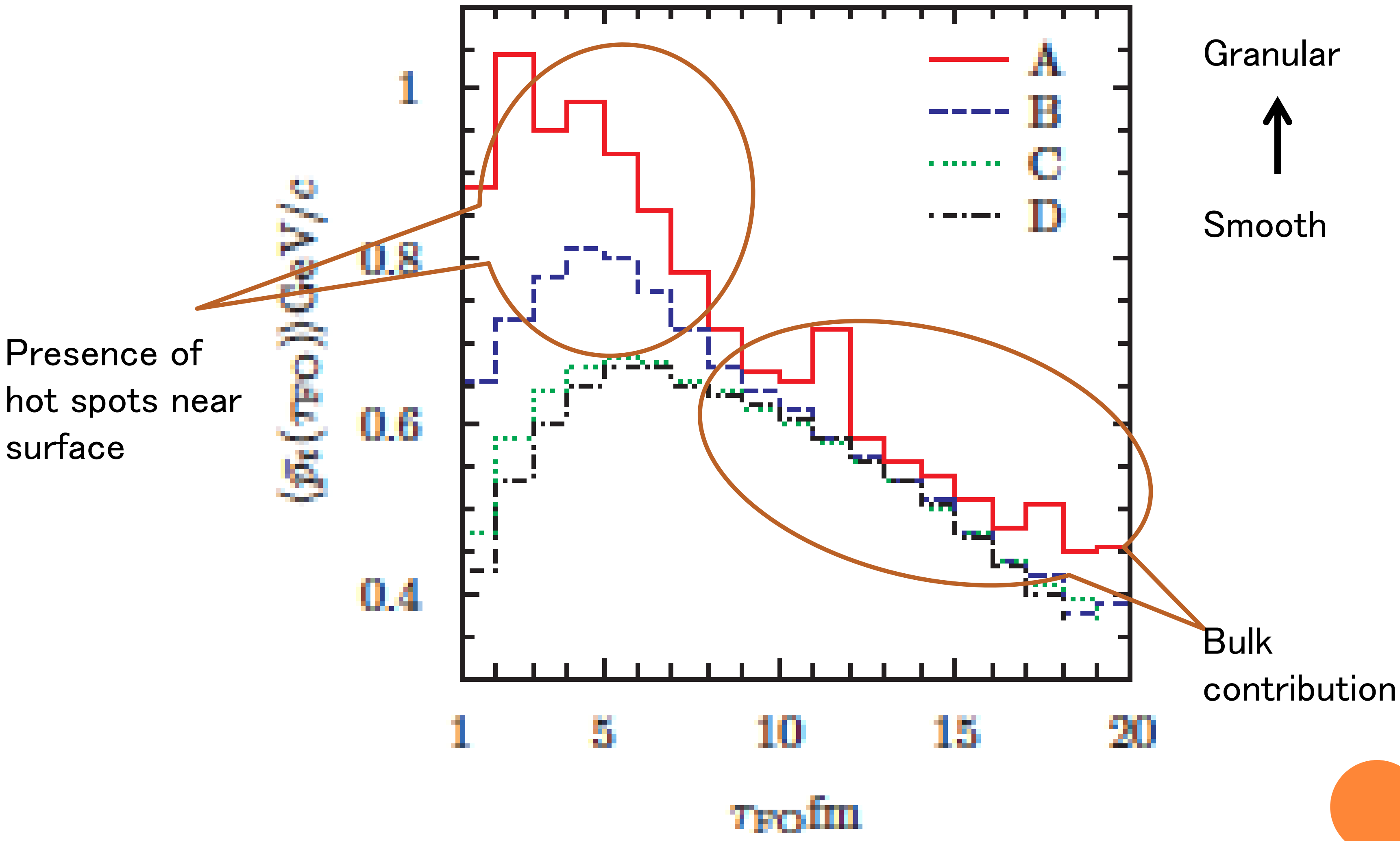


Granular

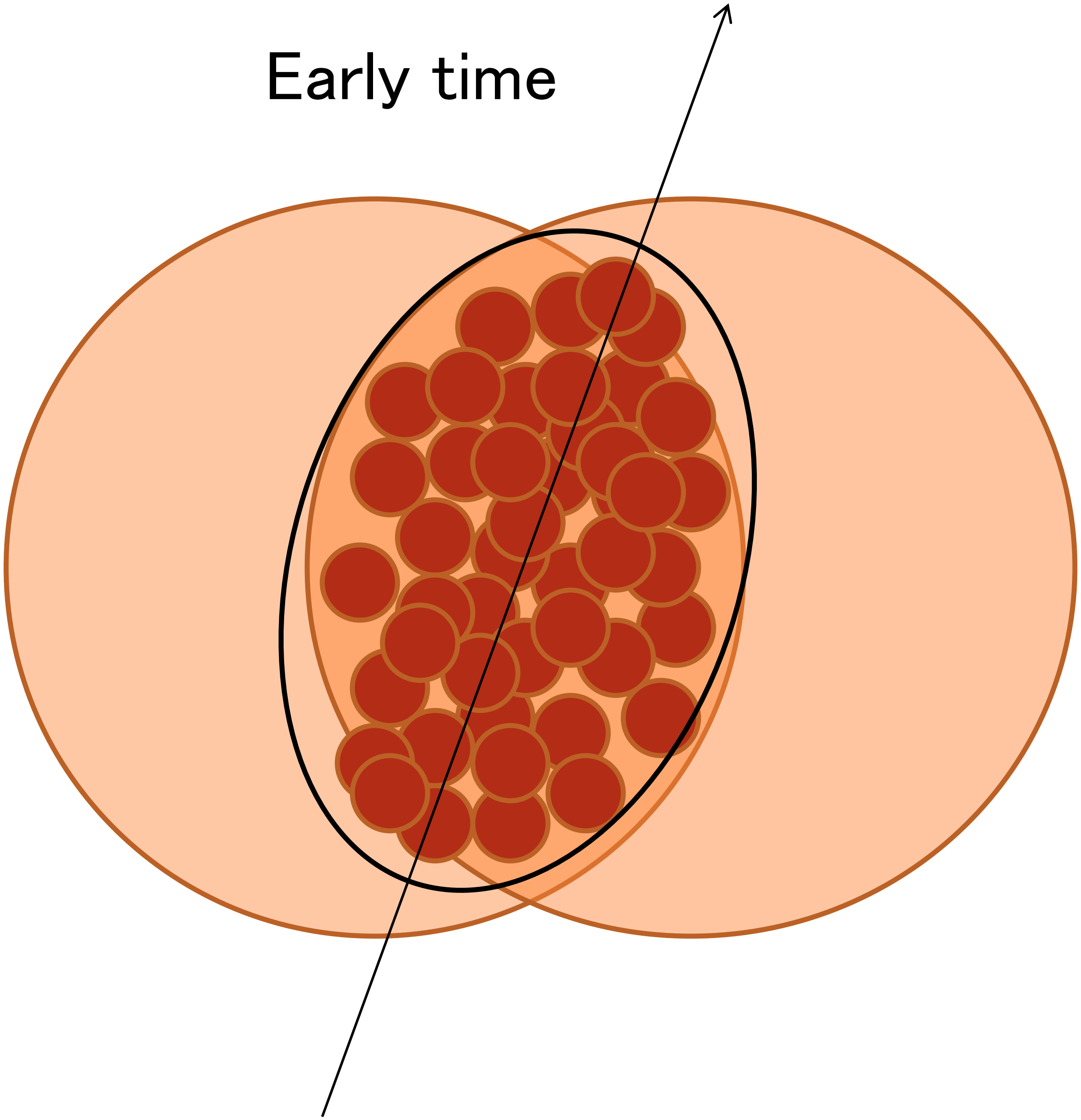
↑

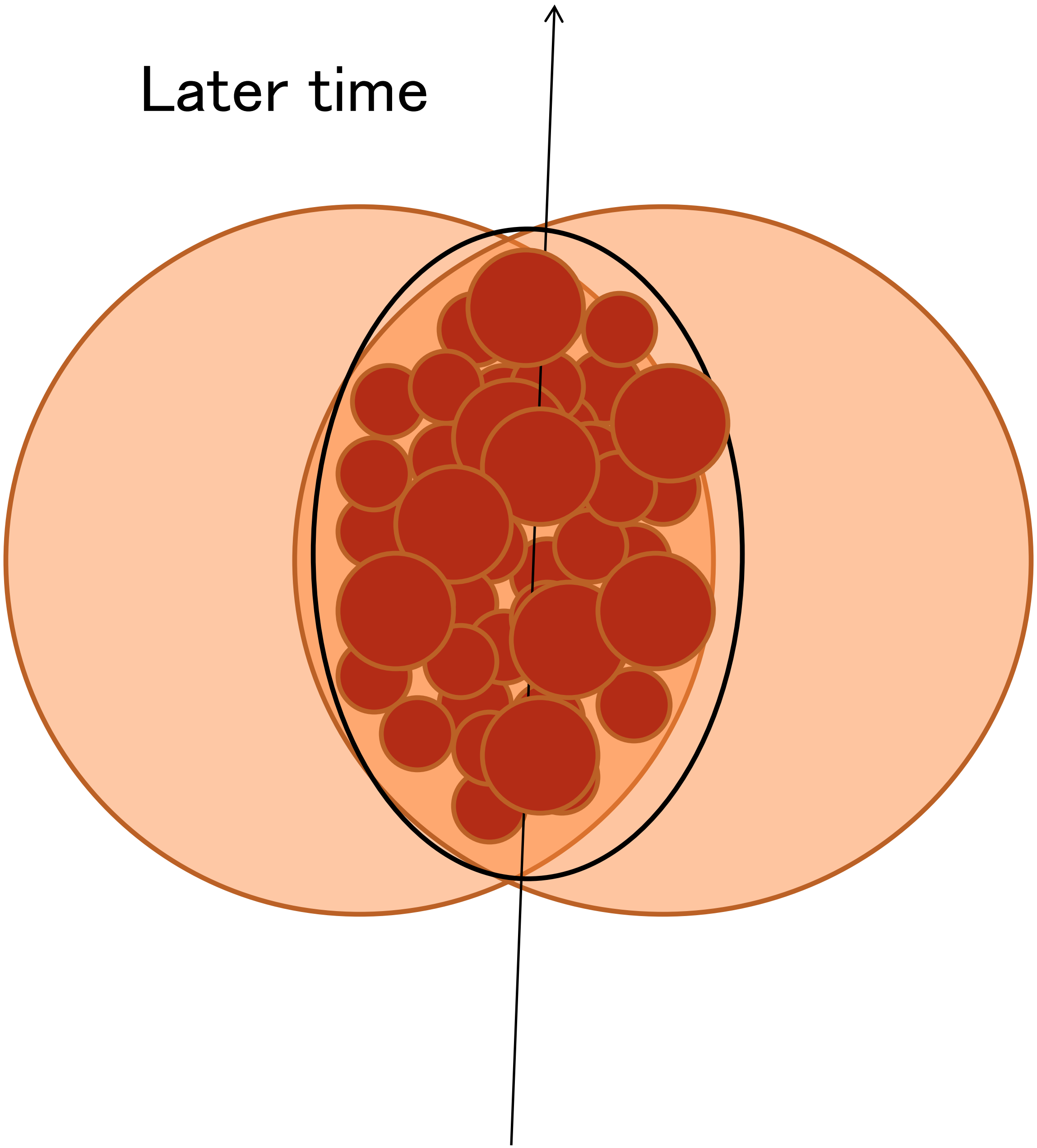
Smooth

Average p_T as function of freezeout time

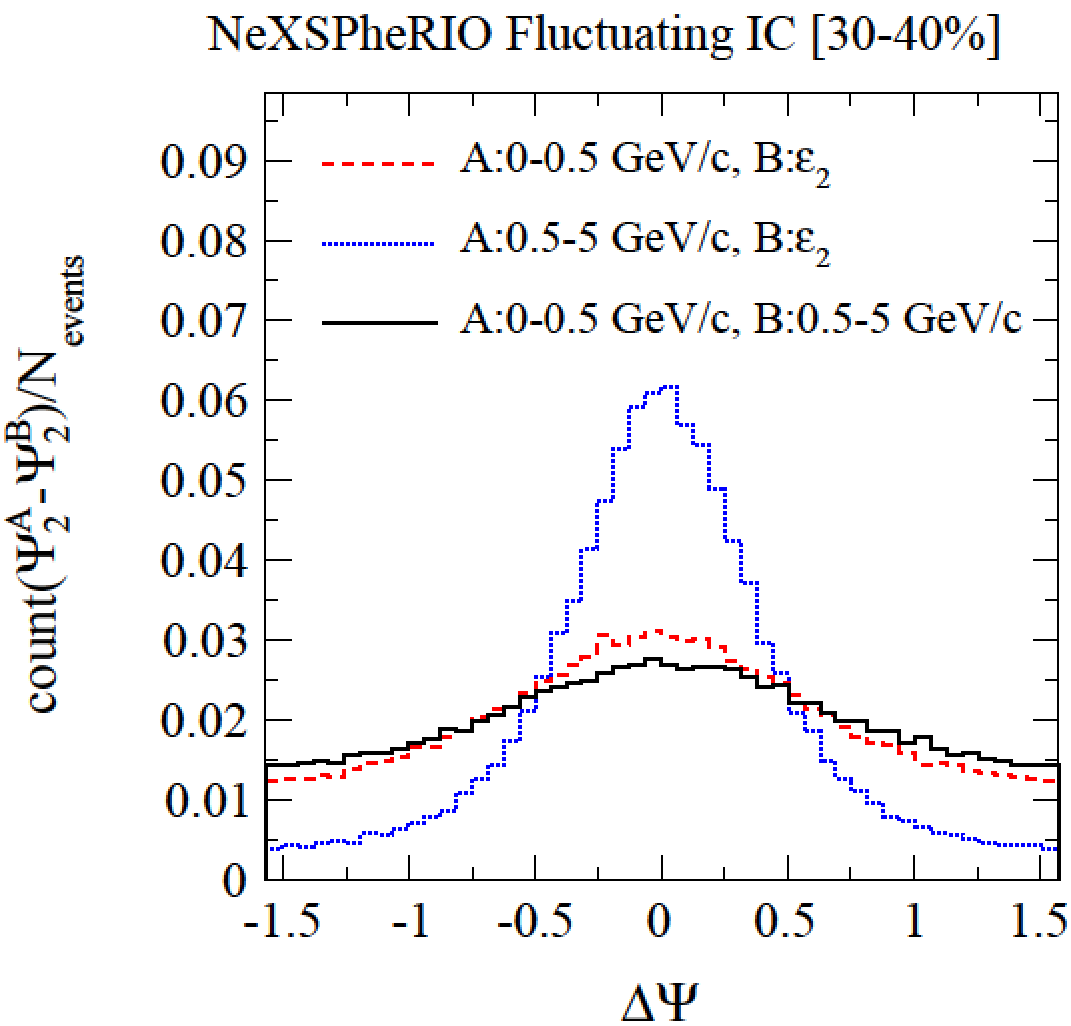


Early time

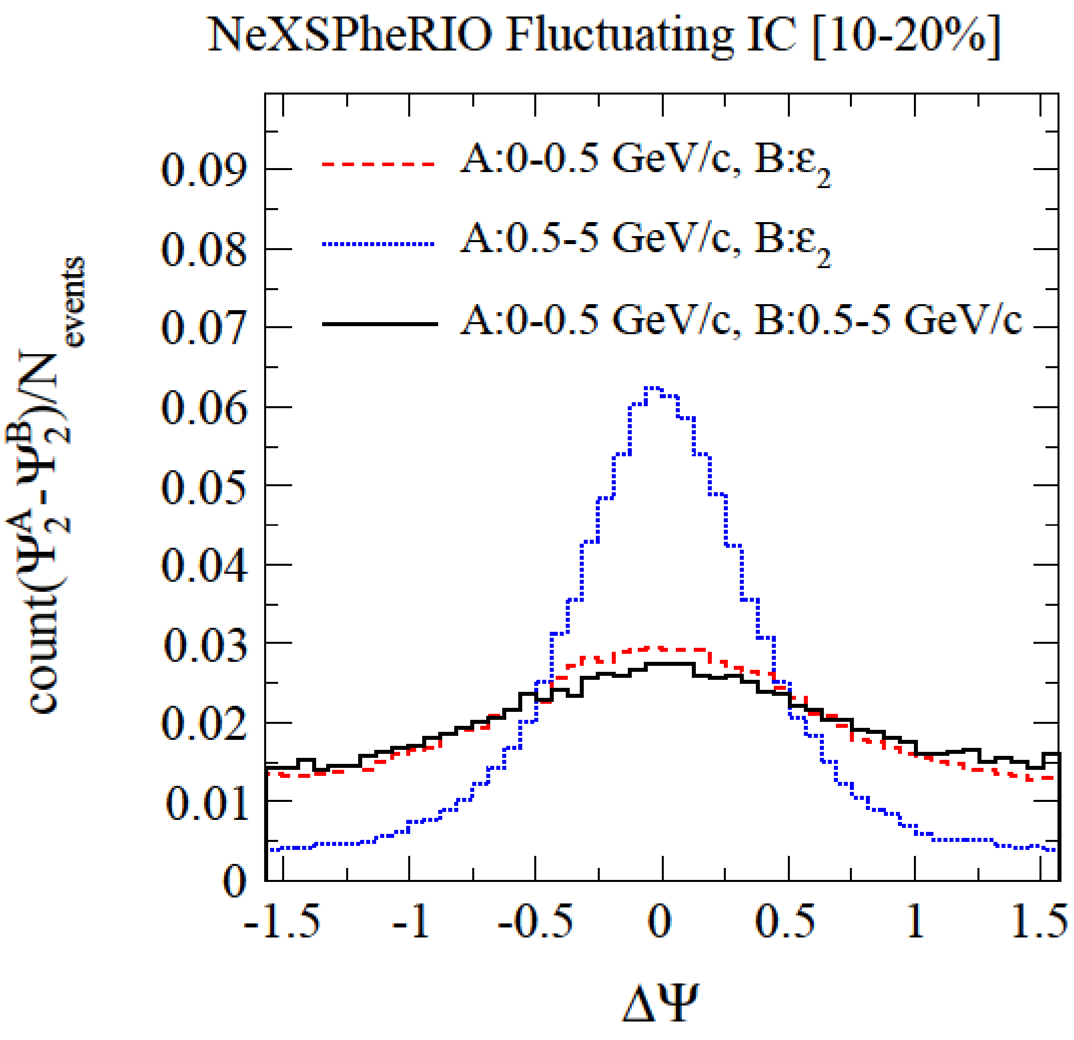



Later time

Take a look on the NeXSPheRIO[1)] case



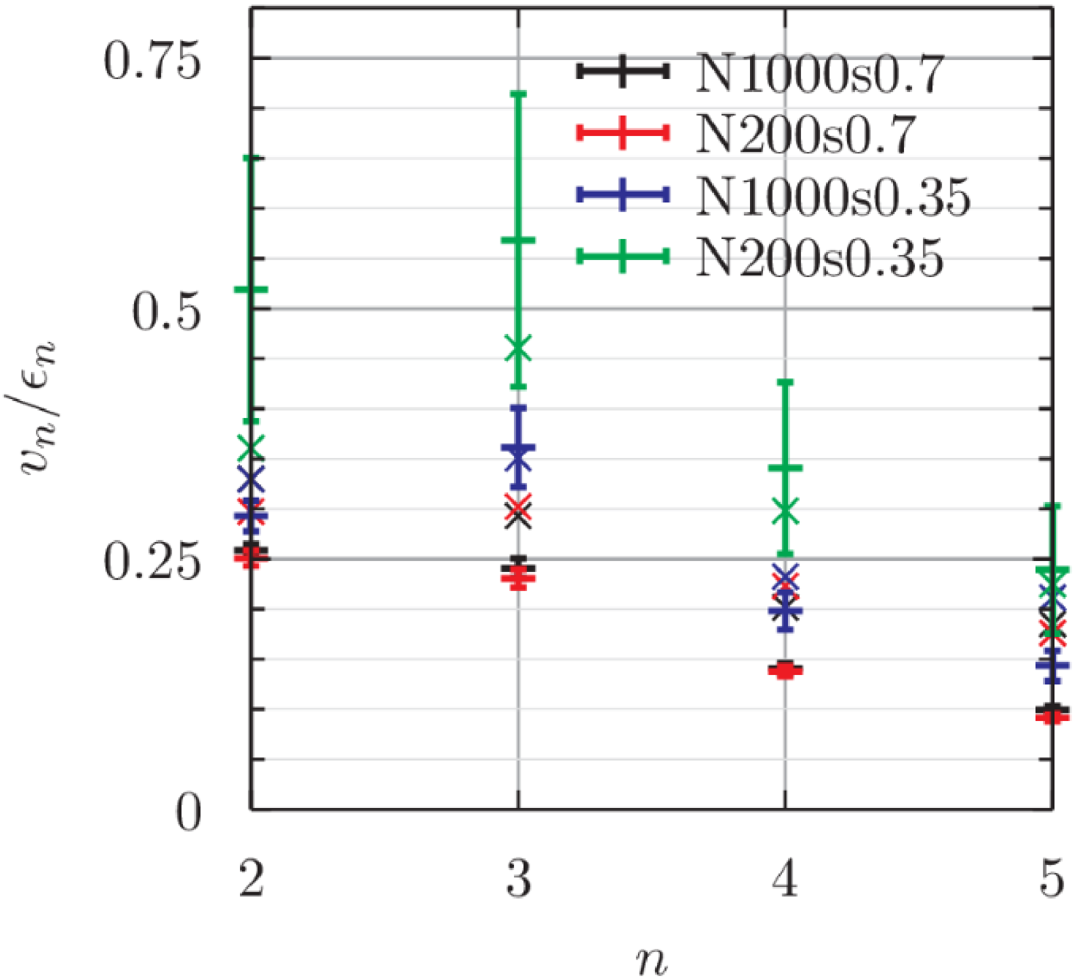
1) See J. Takahashi's talk



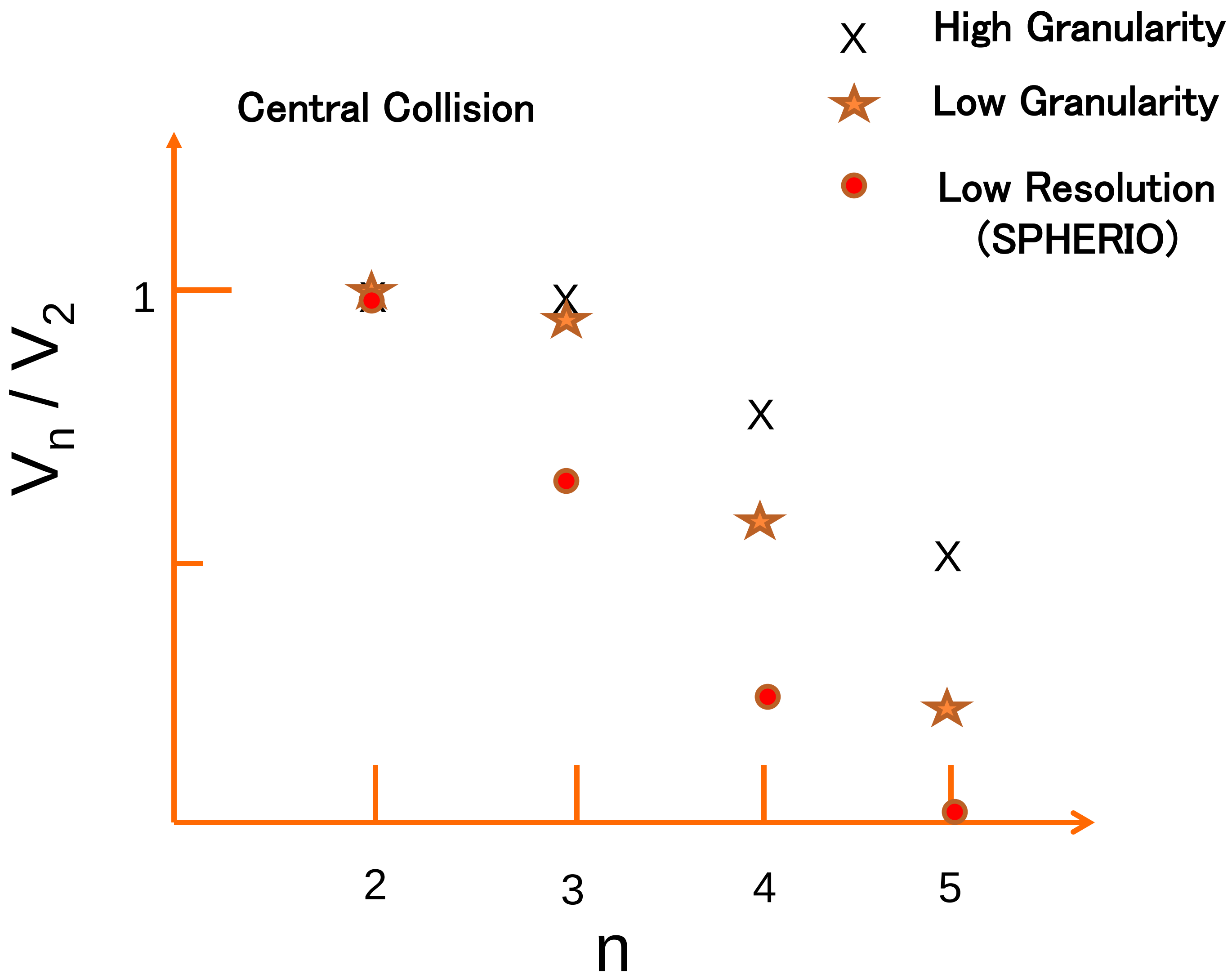
NeXSPheRIO Fluctuating IC [10-20%]
A:0-0.5 GeV/c, B:ε2
A:0.5-5 GeV/c, B:ε2
A:0-0.5 GeV/c, B:0.5-5 GeV/c
count(Ψ2A-Ψ2B)/Nevents
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
-1.5
-1
-0.5
0
0.5
1
1.5
ΔΨ

n-dependence of event averaged v_n/ε_n

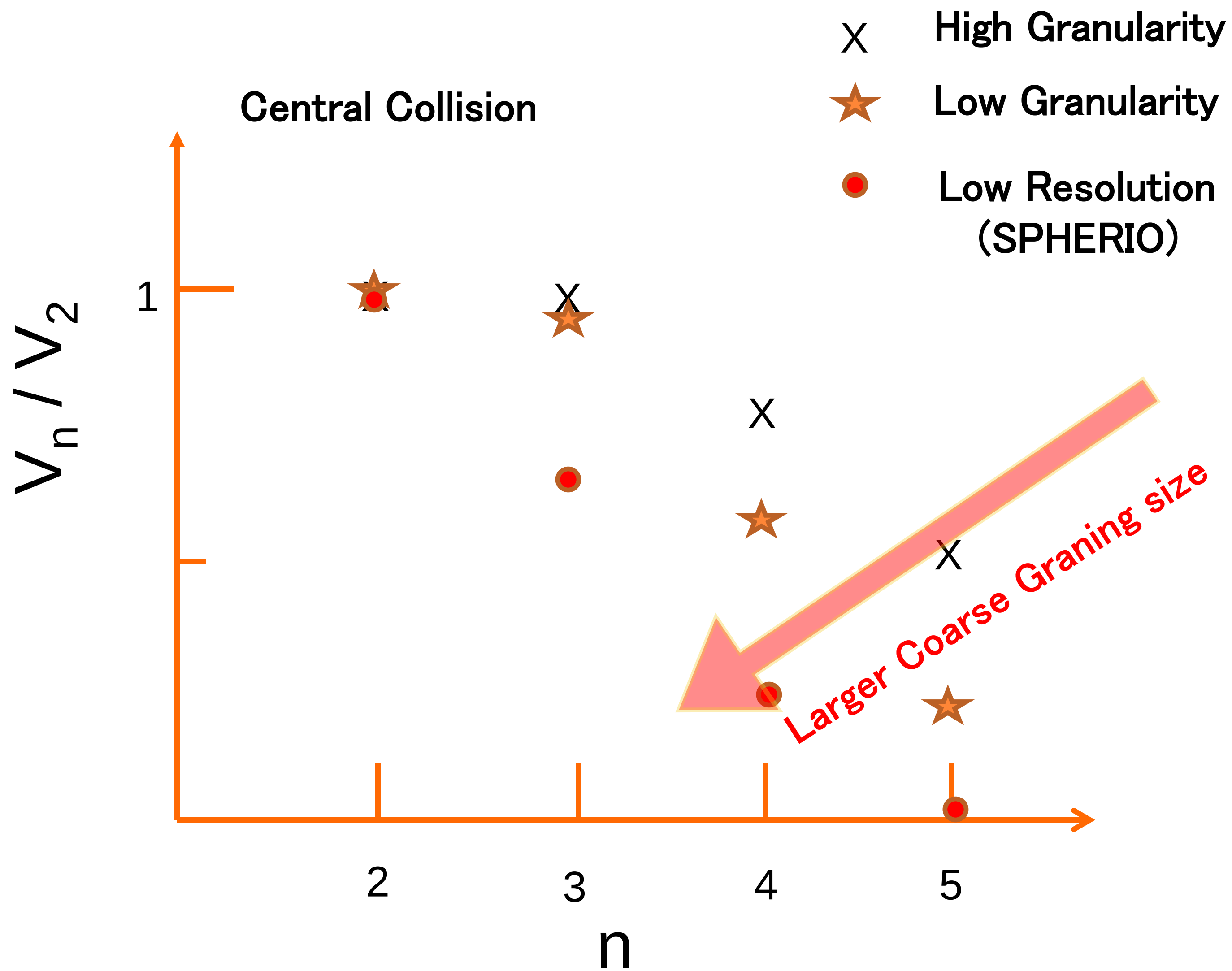
crosses:$p_t = 0.5 - 5$GeV and 10-50%; bars:10-50%



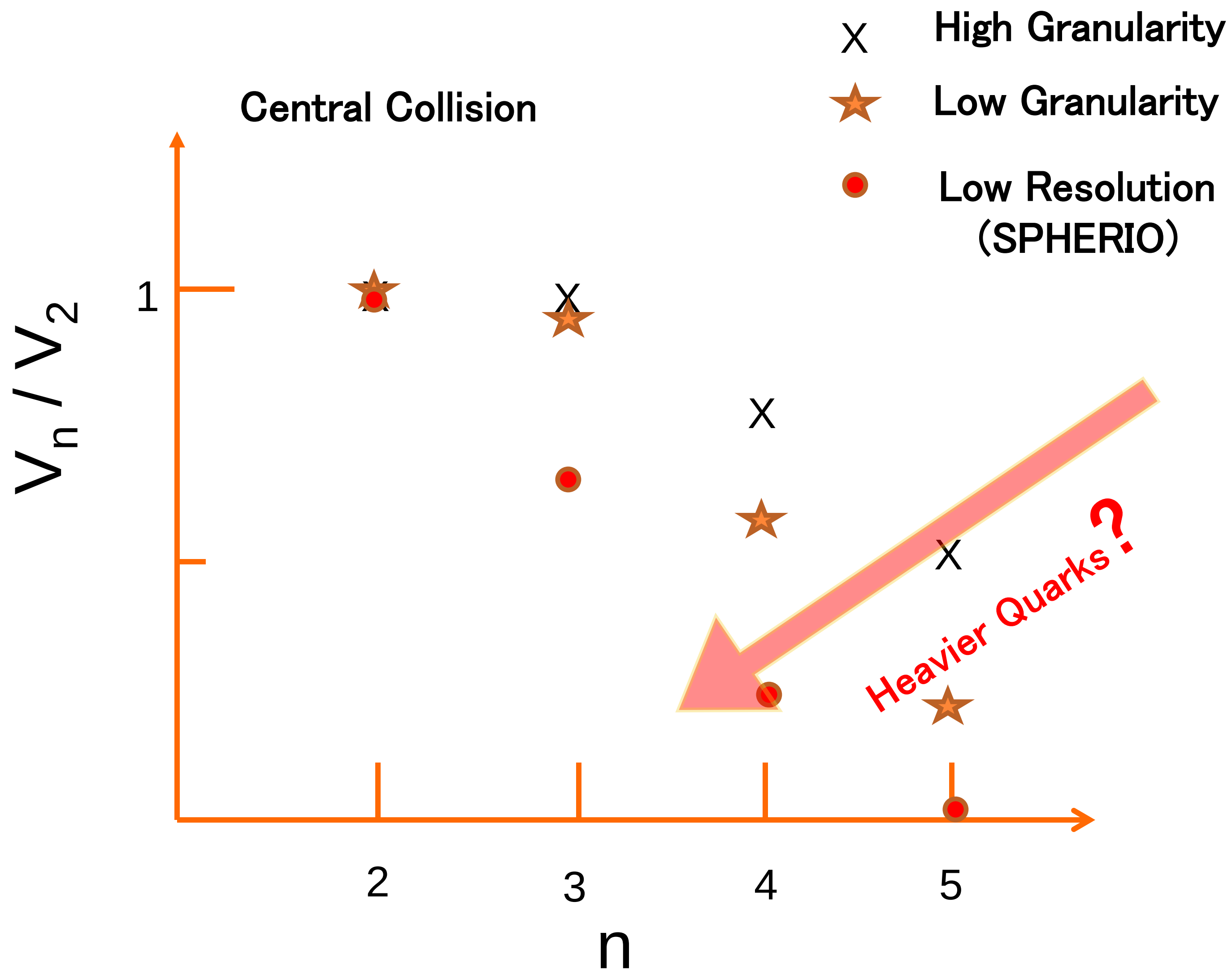
Effects of Coarse Graining in Flow



Effects of Coarse Graining in Flow



Effects of Coarse Graining in Flow



Part II
Dissipative Hydro in Variational Principle

- Variational Method -> Lagrangian Sytem

  ↓

  Conservative (Normal)

- Can we deal with dissipative dynamics via Variational Principle?

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Dissipative Hydro in Variational Principle

- Variational Method -> Lagrangian Sytem
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  Conservative (Normal)

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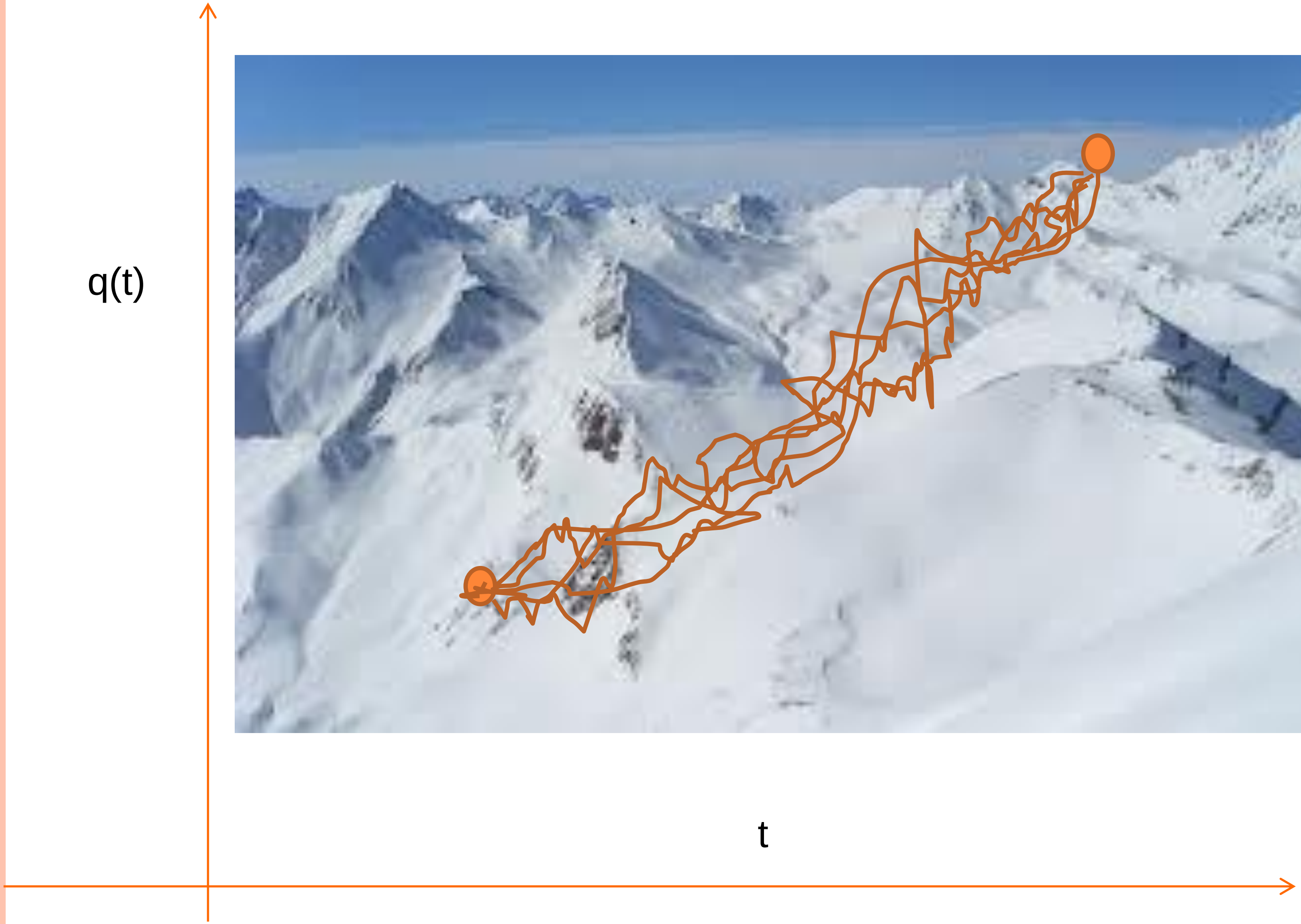
Include NOISES….

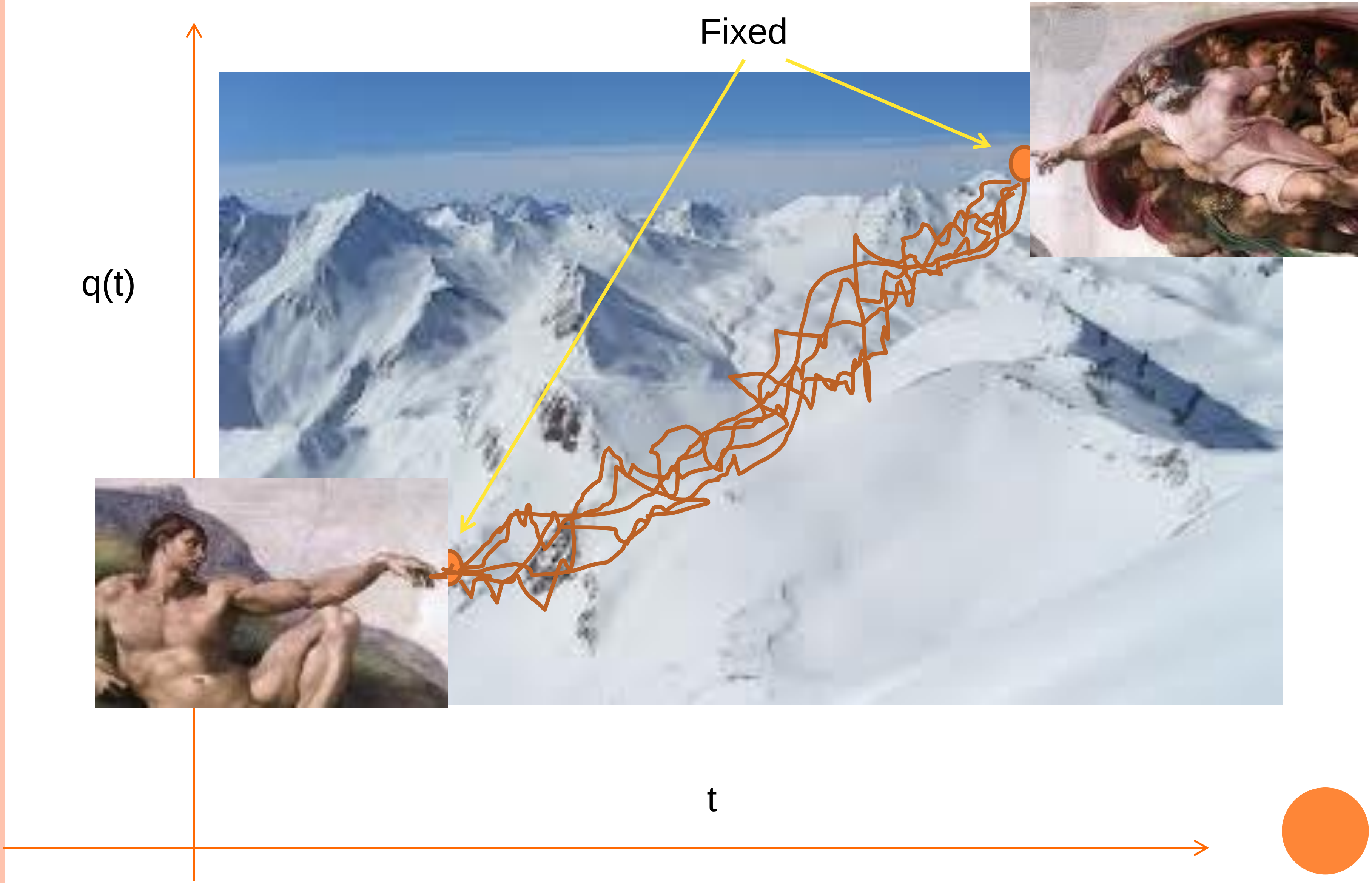
$$\frac{d}{dt}X(t)=V(t)$$

$$\frac{d}{dt}X(t)=V(t)+\xi(t)$$

Variational Formulation with Noises?

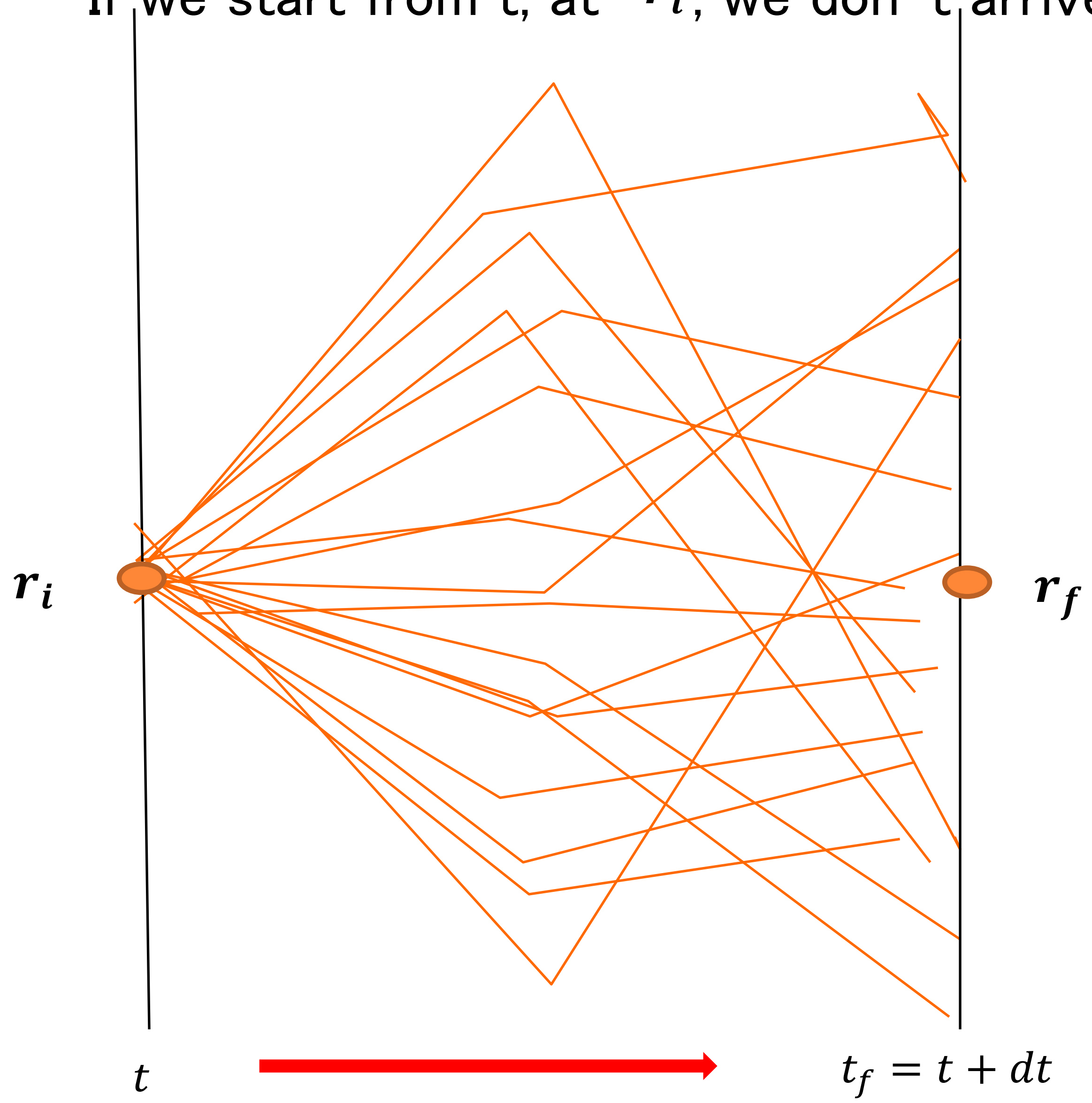


VARIATIONAL FORMULATION WITH NOISES?

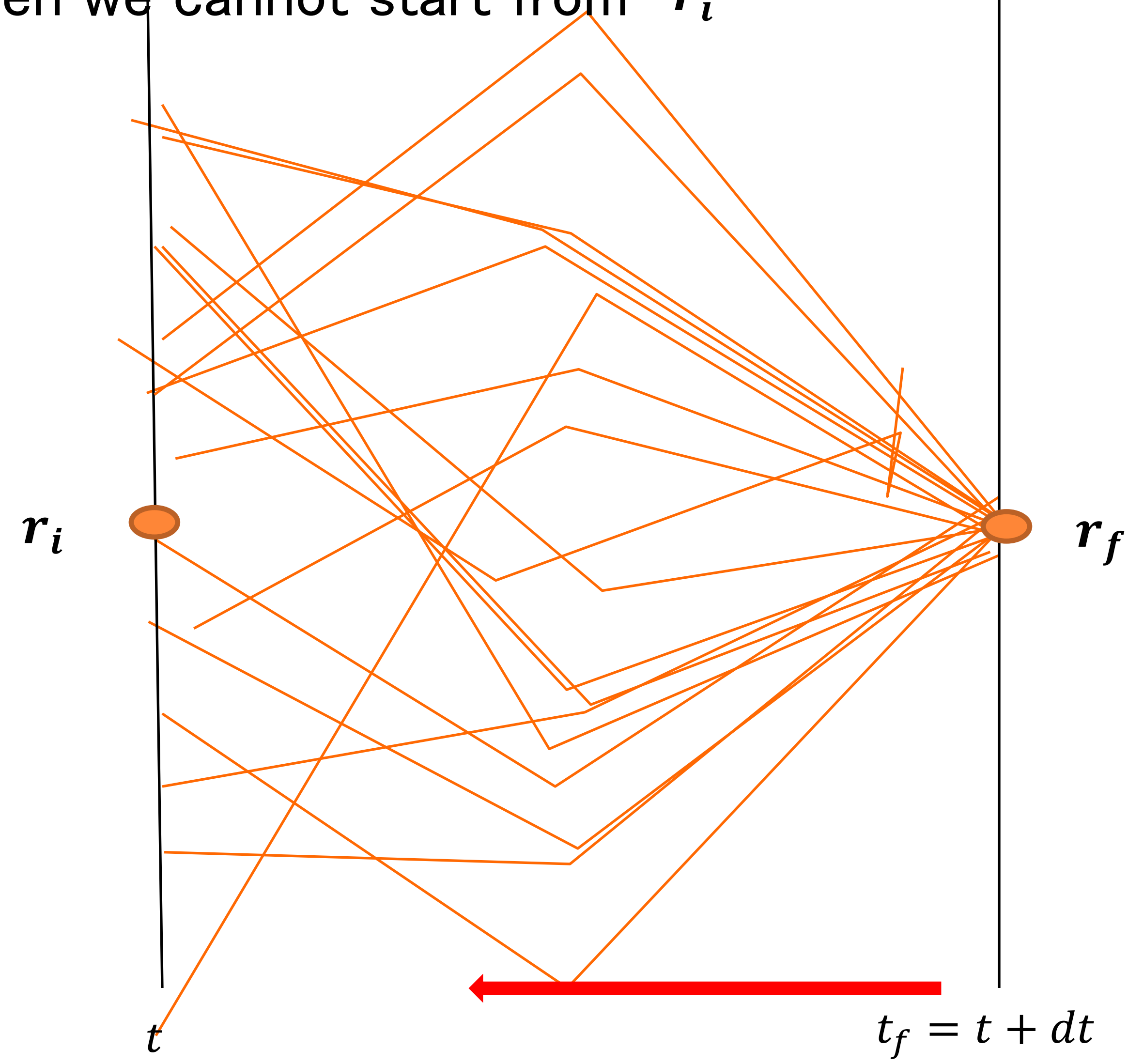


What is the problem for the variational approach when the trajectory of fluid elements are stochastic ??

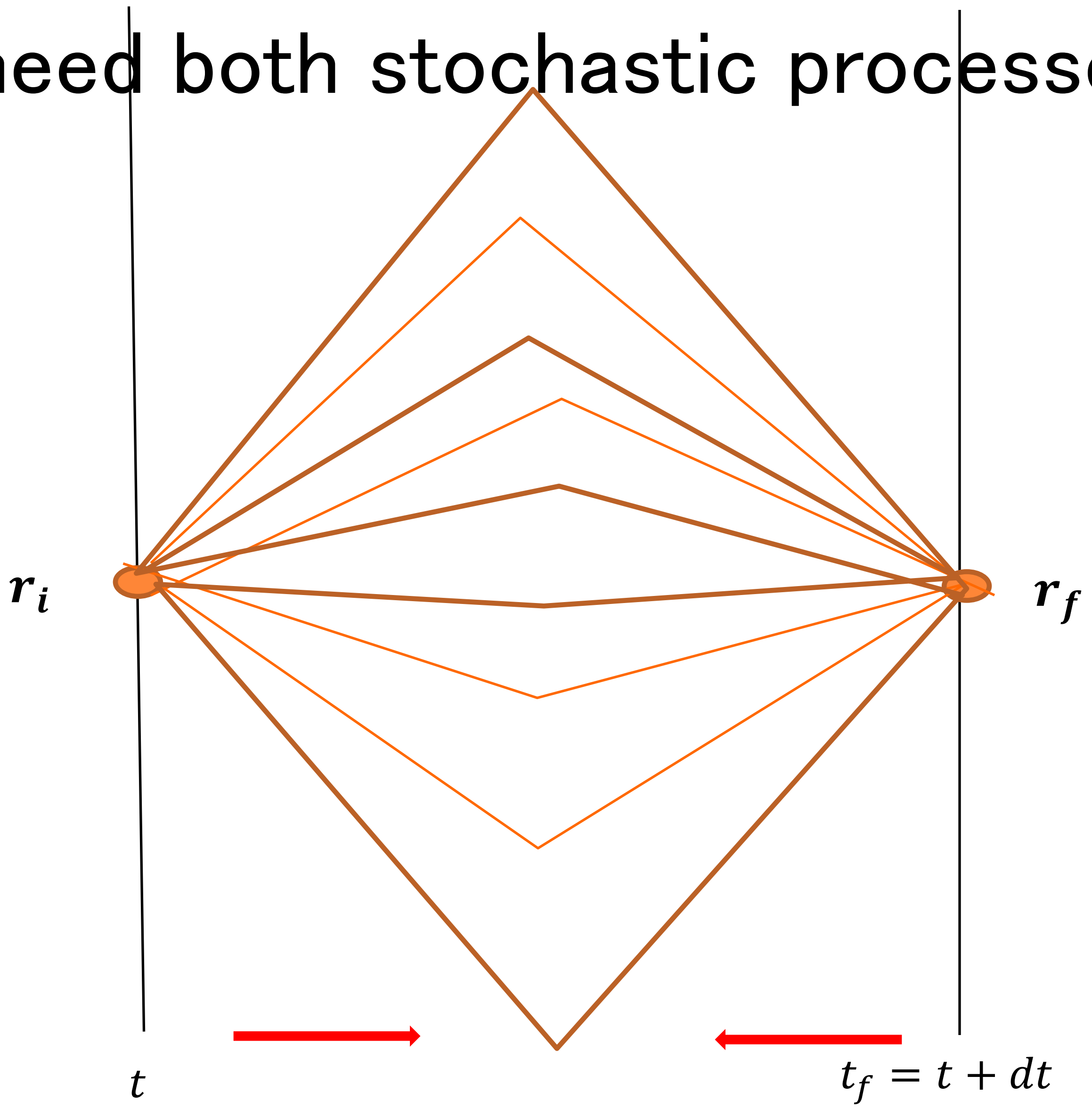
If we start from t, at $\boldsymbol{r_i}$, we don't arrive at $\boldsymbol{r_f}$



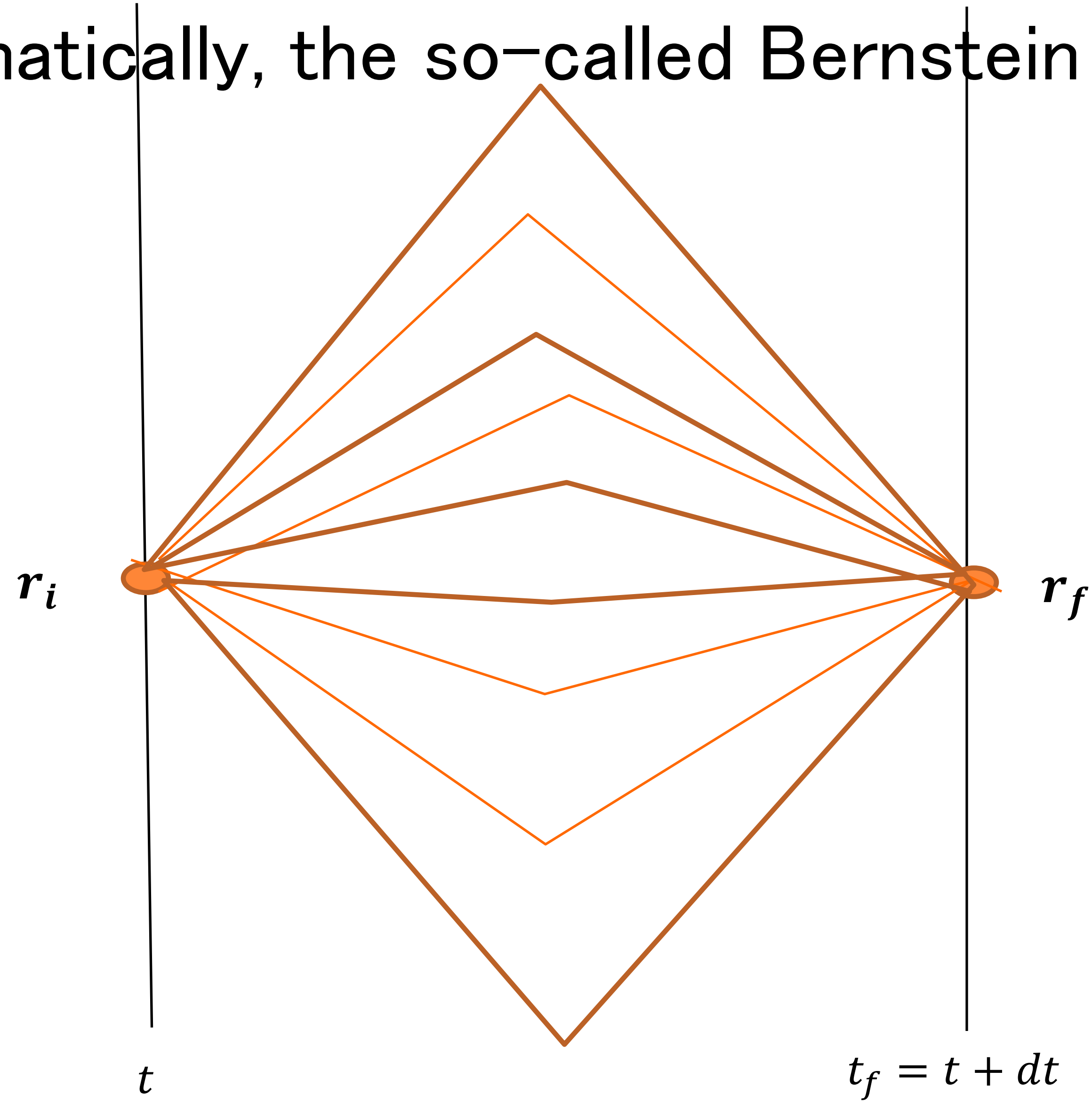
We need Stochastic Process to arrive at $\boldsymbol{r_f}$ but then we cannot start from $\boldsymbol{r_i}$



We need both stochastic processes… !



Mathematically, the so-called Bernstein Process



Variational Principle with Noises?

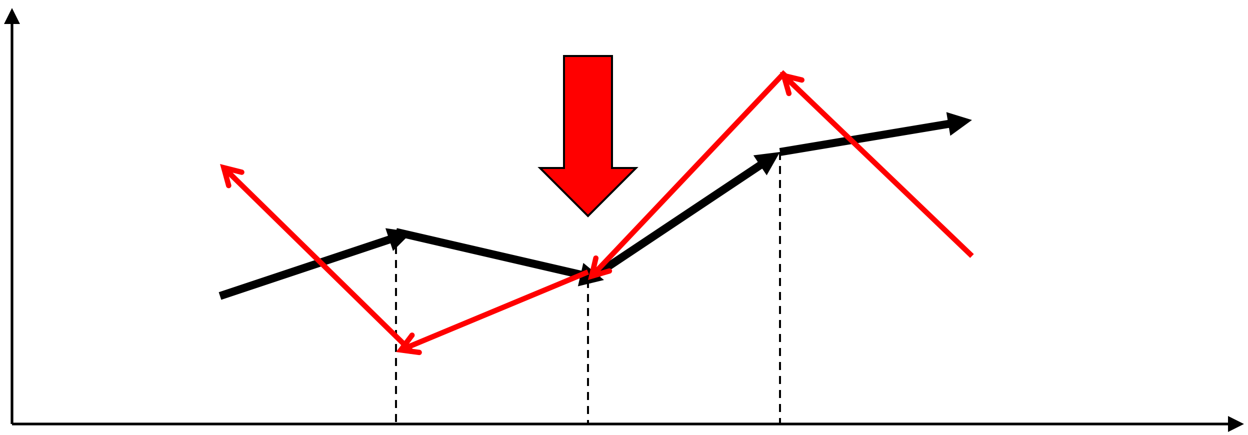
Generalize variables into the domain of stochastic variables

$$I = \left\langle \int_a^b dt L(X, DX) \right\rangle$$

We are talking necessarily the ensemble of trajectories ….

Yasue, J. Funct. Anal, 41, 327 (‘81), Guerra&Morato, Phys. Rev. D27, 1774 (‘83), Nelson, “Quantum Fluctuations” (‘85).

There are two velocities at a point



$$\vec{v} = \lim_{dt \to 0+} \frac{\vec{r}(\vec{R}, t+dt) - \vec{r}(\vec{R}, t)}{dt}$$

Forward SDE

$$\tilde{\vec{v}} = \lim_{dt \to 0+} \frac{\vec{r}(\vec{R}, t) - \vec{r}(\vec{R}, t-dt)}{dt}$$

Backward SDE

Fokker–Plank Equation for a given Stochastic Motion

We define the probability density function as

$$\rho(\vec{x},t) = \left\langle \delta(\vec{x} - \vec{x}(t)) \right\rangle$$

Average over all solutions SDE for a given initial condition.

One Solution of the SDE

Fokker–Plank Equation for a given Stochastic Motion

We define the probability density function as

$$\rho(\vec{x},t)=\left\langle \delta(\vec{x}-\vec{x}(t))\right\rangle$$

Average over all solutions SDE for a given initial condition.

One Solution of the SDE

$$\rho(\vec{x},t+dt)-\rho(\vec{x},t)=-\vec{u}(\vec{x}(t),t)\nabla\left\langle \delta(\vec{x}-\vec{x}(t))\right\rangle dt+\nu\Delta\left\langle \delta(\vec{x}-\vec{x}(t))\right\rangle dt$$

We get the Fokker–Plank Equation

$$\partial_t\rho(\vec{x},t)=-\nabla\left(\vec{u}(\vec{x},t)-\nu\nabla\right)\rho(\vec{x},t)$$

Consistency Condition for the Statistical Ensemble

Fokker–Plank equation (Forward)

$$\partial_t \rho = -\nabla\left(\vec{u} - \nu\nabla\right)\rho$$

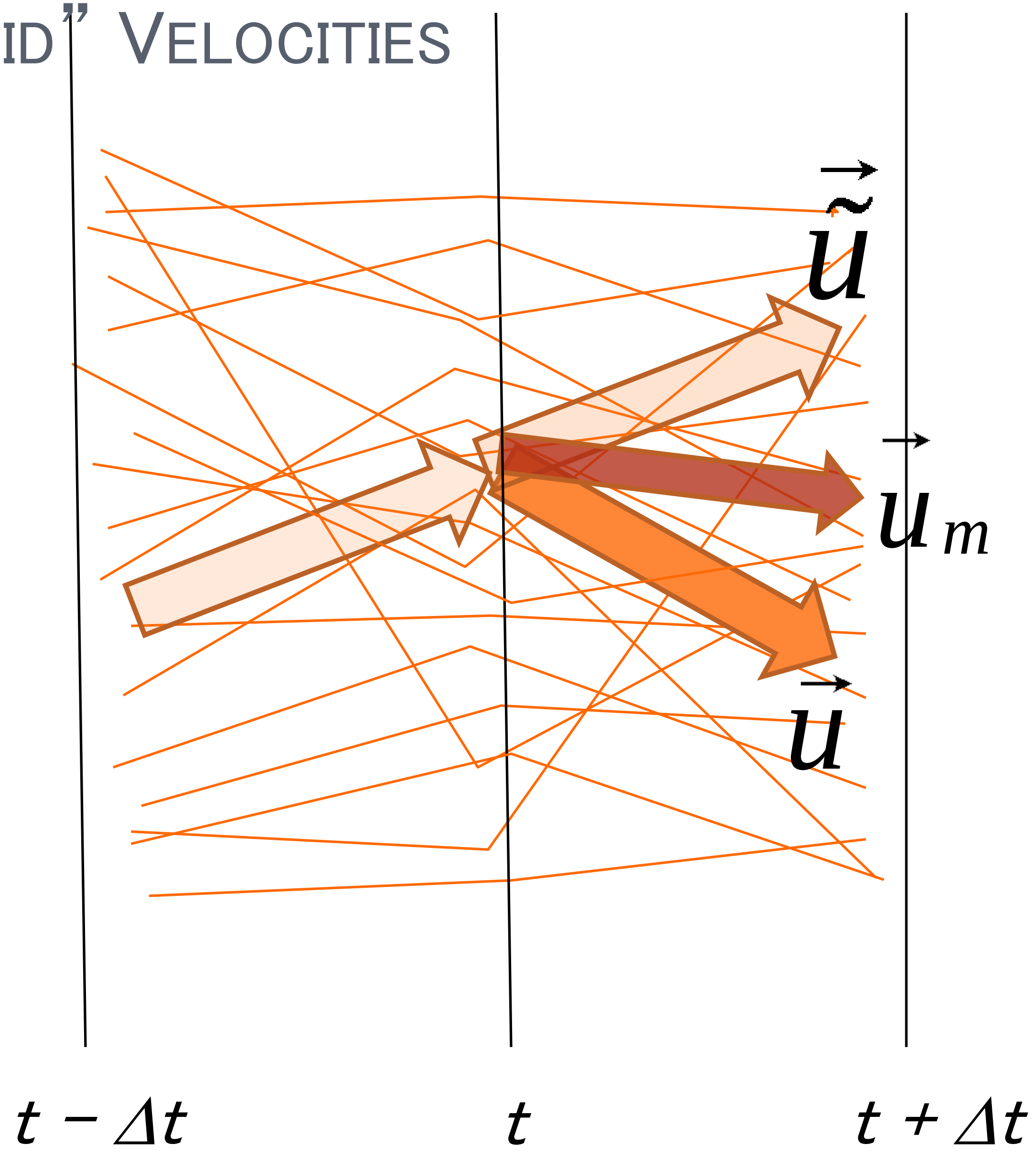
Fokker–Plank equation (Backward)

$$\partial_t \rho = -\nabla\left(\vec{\tilde{u}} + \nu\nabla\right)\rho$$

The two equation must be equivalent.

$$\Rightarrow \quad \vec{\tilde{u}} = \vec{u} + 2\nu\nabla \ln \rho$$

Two "Fluid" Velocities



What makes difference in Variational Method when variables are stochastic?

What makes difference in Variational Method when variables are stochastic? Partial Integration Formula !

Because of the two definitions of velocities,
we introduce two different time derivative operators

Mean forward derivative $D\vec{r} = \vec{u}$

Mean backward derivative $\tilde{D}\vec{r} = \vec{\tilde{u}}$

stochastic partial integration formula

$$\int_a^b dt \left\langle (DX) \cdot Y \right\rangle$$

$$= \left\langle X(b)Y(b) - X(a)Y(a) \right\rangle - \int_a^b dt \left\langle X \cdot (\tilde{D}Y) \right\rangle$$

Exmample: Single particle action

Classical Action $$I_{cla} = \int_a^b dt \left(\frac{m}{2} \left(\frac{d\vec{r}(t)}{dt} \right)^2 - V(\vec{r}(t)) \right)$$

$$\left(\frac{d\vec{r}}{dt} \right)^2 \Rightarrow \begin{cases} 1) \quad D\vec{r} \cdot D\vec{r} \\ 2) \quad \tilde{D}\vec{r} \cdot \tilde{D}\vec{r} \\ 3) \quad \dfrac{D\vec{r} \cdot D\vec{r} + \tilde{D}\vec{r} \cdot \tilde{D}\vec{r}}{2} \end{cases}$$

Take the case 3 (time reversal symmetry)

$$\Rightarrow \quad I_{sto} = \int_a^b dt \left\langle \frac{m}{2} \frac{(D\vec{r})^2 + (\tilde{D}\vec{r})^2}{2} - V(\vec{r}) \right\rangle$$

VARIATIONAL PRCEDURE

$$r \rightarrow r + \delta r$$

$$\delta \int_a^b dt \frac{m}{2} \left\langle (D\vec{r}) \cdot (D\vec{r}) \right\rangle = m \int_a^b dt \left\langle (D\vec{r}) \cdot (D\delta\vec{r}) \right\rangle$$

$$= m \int_a^b dt \left\langle \vec{u} \cdot (D\delta\vec{r}) \right\rangle$$

$$= -m \int_a^b dt \left\langle \tilde{D}\vec{u} \cdot \delta\vec{r} \right\rangle$$

From Ito formula, $\tilde{D}\vec{u} = \left(\partial_t + \vec{\tilde{u}} \cdot \nabla - \nu \Delta \right) \vec{u}$

Single particle case

$\delta I = 0$ leads to

$$\left(\partial_t + \vec{\tilde{u}} \cdot \nabla - \nu\Delta\right)\vec{u} + \left(\partial_t + \vec{u} \cdot \nabla + \nu\Delta\right)\vec{\tilde{u}} = -\frac{2}{m}\nabla V$$

Note that when $\nu = 0$ (no noise), we have $\vec{u} = \vec{\tilde{u}}$

and $\partial_t + \vec{u} \cdot \nabla = d/dt$ ⟹ $\dfrac{d\vec{u}}{dt} = -\dfrac{1}{m}\nabla V$

Instead of two velocities, use $\vec{u}_m = (\vec{u} + \vec{\tilde{u}})/2$ and

$\partial_t \rho + \nabla \cdot \left(\rho \vec{u}_m\right) = 0,$ we get Euler – like equation

$$\left(\partial_t + \vec{u}_m \cdot \nabla\right)\vec{u}_m - 2\nu^2 \nabla\left(\rho^{-1/2}\nabla^2\sqrt{\rho}\right) = -\frac{1}{m}\nabla V$$

Single particle case

Equation for a trajectory

Equation for a distribution!

$\delta I = 0$ leads to

$$\left(\partial_t + \vec{\tilde{u}} \cdot \nabla - \nu\Delta\right)\vec{u} + \left(\partial_t + \vec{u} \cdot \nabla + \nu\Delta\right)\vec{\tilde{u}} = -\frac{2}{m}\nabla V$$

Note that when $\nu = 0$ (no noise), we have $\vec{u} = \vec{\tilde{u}}$

and $\partial_t + \vec{u} \cdot \nabla = d/dt$ → $\dfrac{d\vec{u}}{dt} = -\dfrac{1}{m}\nabla V$

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A closed set of equations

$$\partial_t \rho + \nabla \cdot \left(\rho \vec{u}_m \right) = 0,$$

$$\left(\partial_t + \vec{u}_m \cdot \nabla \right) \vec{u}_m - 2\nu^2 \nabla \left(\rho^{-1/2} \nabla^2 \sqrt{\rho} \right) = -\frac{1}{m} \nabla V$$

An interesting representation:
Suppose the velocity field is irrotational. Then we can introduce a scalar function ϑ such that

$\nabla \vartheta = \vec{u}_m / (2\nu)$ (Velocoty potential)

$$\longrightarrow \nabla \left[\partial_t \vartheta + \nu \left(\nabla \vartheta \right)^2 - \nu \left(\rho^{-1/2} \nabla^2 \sqrt{\rho} \right) + \frac{1}{m} \nabla V \right] = 0$$

The Fokker–Planck equation

$$\partial_t \rho + \nabla\cdot\left(\rho \vec{u}_m\right) = 0,$$

$$\left(\partial_t + \vec{u}_m\cdot\nabla\right)\vec{u}_m - 2\nu^2\nabla\left(\rho^{-1/2}\nabla^2\sqrt{\rho}\right) = -\frac{1}{m}\nabla V$$

$$\nabla\vartheta = \vec{u}_m/(2\nu)$$

$$\partial_t \rho + 2\nu\nabla\cdot\left(\rho\nabla\vartheta\right) = 0,$$

$$\nabla\left[\partial_t\vartheta + \nu\left(\nabla\vartheta\right)^2 - \nu\left(\rho^{-1/2}\nabla^2\sqrt{\rho}\right) + \frac{1}{m}\nabla V\right] = 0$$

These two equations are equivalent to a complex equation,

$$i\partial_t\varphi = \left[-\nu\nabla^2 + \frac{1}{2\nu m}V\right]\varphi,$$

with $\varphi \equiv \sqrt{\rho}e^{i\vartheta}$,

The Fokker–Planck equation

$$\partial_t \rho + \nabla \cdot \left(\rho \vec{u}_m \right) = 0,$$

$$\left(\partial_t + \vec{u}_m \cdot \nabla \right) \vec{u}_m - 2\nu^2 \nabla \left(\rho^{-1/2} \nabla^2 \sqrt{\rho} \right) = -\frac{1}{m} \nabla V$$

$$\Big\downarrow \quad \nabla \vartheta = \vec{u}_m / (2\nu)$$

$$\partial_t \rho + 2\nu \nabla \cdot \left(\rho \nabla \vartheta \right) = 0,$$

$$\nabla \left[\partial_t \vartheta + \nu \left(\nabla \vartheta \right)^2 - \nu \left(\rho^{-1/2} \nabla^2 \sqrt{\rho} \right) + \frac{1}{m} \nabla V \right] = 0$$

That is, this is equivalent to Schrödinger Equation

$$i\hbar \partial_t \varphi = \left[-\frac{\hbar^2}{2m} \nabla^2 + V \right] \varphi, \qquad \varphi \equiv \sqrt{\rho} e^{i\vartheta}, \nu = \hbar / 2m.$$

In resume,

Classical Action

$$I_{cla} = \int_a^b dt \left(\frac{m}{2} \left(\frac{d\vec{r}(t)}{dt} \right)^2 - V(\vec{r}(t)) \right)$$

Stochastic Action

$$I_{sto} = \int_a^b dt \left\langle \frac{m}{2} \frac{(D\vec{r})^2 + (\tilde{D}\vec{r})^2}{2} - V(\vec{r}) \right\rangle$$

The corresponding Fokker-Planck equation

$$\partial_t \rho + \nabla \cdot \left(\rho \vec{u}_m \right) = 0,$$
$$\left(\partial_t + \vec{u}_m \cdot \nabla \right) \vec{u}_m - 2\nu^2 \nabla \left(\rho^{-1/2} \nabla^2 \sqrt{\rho} \right) = -\frac{1}{m} \nabla V$$

Schrödinger Equation

$$i\hbar \partial_t \varphi = \left[-\frac{\hbar^2}{2m} \nabla^2 + V \right] \varphi,$$

$$\varphi \equiv \sqrt{\rho} e^{i\vartheta},$$
$$\vec{u} = 2\nu \nabla \vartheta,$$
$$\nu = \hbar / 2m.$$

Navier–Stokes Equation

$$I_{Traditional} \to I_{Stochastic} = \left\langle \int_a^b dt \int d^3R \left(\frac{\rho_0}{2} D\vec{r} \bullet D\vec{r} - \frac{\rho_0}{\rho} \varepsilon(\rho, S) \right) \right\rangle$$

T. Koide and T. K, .J. PhysA: 45(25):255204

Navier–Stokes Equation

$$I_{Traditional} \to I_{Stochastic} = \left\langle \int_a^b dt \int d^3R \left(\frac{\rho_0}{2} D\vec{r} \bullet D\vec{r} - \frac{\rho_0}{\rho} \varepsilon(\rho, S) \right) \right\rangle$$

$$\delta I = -\int_a^b dt \int d^3R \rho_0 \left\langle \left(\left(\partial_t + \tilde{\vec{u}} \cdot \nabla - \nu\Delta \right) \vec{u} + \frac{1}{\rho} \nabla P \right) \delta\vec{r} + \frac{T}{\rho} \delta S \right\rangle$$

from kinetic term

from potential term

T. Koide and T. K, .J. PhysA: 45(25):255204

Navier–Stokes Equation

$$\rho\left(\partial_t + \vec{v}_m \cdot \nabla\right)\vec{v}_m + \sum_j \partial_j \left[\left(P - \varsigma \nabla \cdot \vec{v}_m\right)\delta_{ij} - \eta e^m_{ij}\right]$$

$$-\sum_j \partial_j \left(\eta \partial_j \frac{\eta}{\rho} \nabla \ln \rho\right) = 0,$$

$$e^m_{ij} = \partial_j v^i_m + \partial_i v^j_m - \frac{2}{3}(\nabla \cdot \vec{v}_m)\delta_{ij}$$

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Navier–Stokes Equation

$$\rho\left(\partial_t + \vec{v}_m \cdot \nabla\right)\vec{v}_m + \sum_j \partial_j \left[\left(P - \varsigma \nabla \cdot \vec{v}_m\right)\delta_{ij} - \eta e_{ij}^m\right]$$

$$-\sum_j \partial_j \left(\eta \partial_j \frac{\eta}{\rho} \nabla \ln \rho\right) = 0,$$

With a surface tension correction

$$e_{ij}^m = \partial_j v_m^i + \partial_i v_m^j - \frac{2}{3}(\nabla \cdot \vec{v}_m)\delta_{ij}$$

T. Koide and T. K, .J. PhysA: 45(25):255204

Gross–Pitaevskii Equation

$$I_{Stochastic} = \left\langle \int_a^b dt \int d^3R \left(\frac{\rho_0}{2} \frac{D\vec{r} \bullet D\vec{r} + \tilde{D}\vec{r} \bullet \tilde{D}\vec{r}}{2} - \frac{\rho_0}{\rho} \varepsilon(\rho) \right) \right\rangle$$

with $\varepsilon = \frac{1}{m} V(r) \rho + \frac{1}{2m^2} U_0 \rho^2,$

$$\psi \equiv \sqrt{\rho} e^{i\vartheta},$$

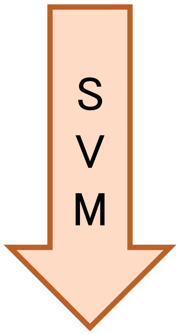
$$\vec{u} = 2\nu \nabla \vartheta,$$

$$\nu = \hbar / 2m.$$

$$i\hbar \partial_t \psi = \left[-\frac{\hbar^2}{m} \nabla^2 \psi + V + U_0 |\psi|^2 \right] \psi,$$

ANOTHER INTERESTING EAMPLE

Classical damped motion



$$I_{Classic} = \int_a^b dt\left[\frac{m}{2}\left(\frac{dx}{dt}\right)^2 - V(x)\right]e^{\lambda t}.$$

$$i\hbar\partial_t\psi = \left[-\frac{\hbar^2}{m}\nabla^2\psi + V + i\frac{\hbar}{2}\gamma\left(\delta\ln\frac{\psi^*}{\psi}\right)\right]\psi,$$

$$\delta\ln\frac{\psi^*}{\psi} \equiv \ln\frac{\psi^*}{\psi} - \left\langle \ln\frac{\psi^*}{\psi}\right\rangle$$

Optical-potential-like equation, known as Kostin Equation

Summary

- It is important to know what is the "Thermalization" scale realized in heavy ion collisions. Depends on what we observe.
- Transport coefficients, or even effective EoS may depend on this scale.
- Some observables are not sensitive to this scale. If we can see only these, we would think really the hydro works well …

Outlook

- Can the difference of identified particle flow pattern see this ?
- Variational approach with noises for Relativistic fluid.

- Use of transport code (PHSD*, UrQMD) and construct Hydro introducing coarse graining and see the effects…

* Elena Bratkovskaya's talk.

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