Relativistic distribution function for particles with spin at local thermodinamical equilibrium

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23 July 2013

F.Becattini, V.Chandra, L. Del Zanna, E.G. arXiv:1303.3431

Outline

- Motivation and introduction
- Global thermodynamical equilibrium with rotation
- Local thermodynamical equilibrium
- Polarization of spin 1/2 particles in Heavy Ion collision: Extension of Cooper Frye Formula
- Conclusion

Motivation

Cooper-Frye Formula

$$\varepsilon \frac{\mathrm{d}N}{\mathrm{d}^3 p} = \int_\Sigma \mathrm{d}\Sigma_\mu p^\mu f(x,p)$$

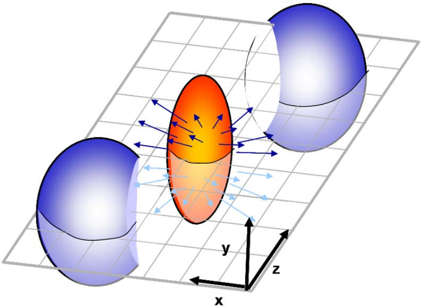
$$f(x,p) = \frac{1}{e^{\beta(x)\cdot p - \xi(x)} \pm 1} \qquad \beta(x) = \frac{1}{T_0} u^\mu \quad \xi = \frac{\mu_0}{T_0}$$

Usually the spin of the particles is taken into account only as a degeneracy factor $(2S+1)$, but in general, at the local thermodinamical equilibrium, different polarization states can be unevenly populated.

- Can this formula be extended to take into account the spin of the particles?
- Is it possible to predict the value of particle polarization at the freeze out?

Polarization in Heavy-Ion collision

In peripheral high energy heavy ion collisions the system has a large angular momentum and may manifest itself in the polarization of secondary produced particles



Z. T. Liang, X. N. Wang, Phys. Rev. Lett. 94 102301 (2005) and others
B. Betz, M. Gyulassy and G. Torrieri, Phys. Rev. C 76 044901 (2007)
F. Becattini, F. Piccinini and J. Rizzo, Phys. Rev. C 77 024906 (2008)

yet no definite formula connecting the polarization of hadrons to the hydrodynamical model
To understand this connection we must review some standard concepts.

Covariant Wigner Function for Dirac Field

The covariant Wigner function makes the connection between Kinetic Theory and Quantum Fields

$$W(x,k)_{AB} = -\frac{1}{(2\pi)^4}\int \mathrm{d}^4 y 2 e^{-ik\cdot y}\langle : \Psi_A(x+y/2)\bar{\Psi}_B(x-y/2) : \rangle$$

and for quasi-free theory, neglecting the interaction over scale shorter than the Compton wavelenght, k is almost on-shell

$$W^+(x,p) = \theta(k^0)W(k,x) = \int \frac{\mathrm{d}^3 p}{\varepsilon}\delta^4(p-k)\sum_{r,s}\bar{u}_r(p)f(x,p)_{rs}u_s(p)$$

$$W^-(x,p) = \theta(-k^0)W(k,x) = \int \frac{\mathrm{d}^3 p}{\varepsilon}\delta^4(p+k)\sum_{r,s}\bar{v}_s(p)\bar{f}(x,p)_{rs}v_r(p)$$

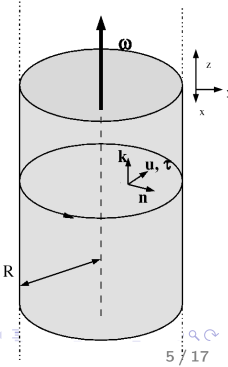
The $u(p)$ and $v(p)$ are the usually solution of the Dirac equation.
The distribution function for Dirac field is 2×2 matrix

Global thermodinamical equilibrium with rotation

Thermodinamical equilibrium with angular momentum

$$\widehat{\rho} = \frac{1}{Z}\exp\left[-\widehat{H}/T + \mu\widehat{Q}/T + \boldsymbol{\omega}\cdot\widehat{\mathbf{J}}/T\right] P_V$$

- Obtained by maximixing the entropy $\widehat{S} = -\mathrm{tr}\widehat{\rho}\log\widehat{\rho}$ with fixed mean total energy, momentum, charges and Angular Momentum
- $\boldsymbol{\omega}/T$ is the Lagrange multiplier of the angular momentum, with the physical meaning of angular velocity $\boldsymbol{v} = \boldsymbol{\omega}\times\boldsymbol{x}$
- P_V is the projector operator onto localized state witch is needed to avoid the singularity at $r = c/\omega$



Global thermodynamical equilibrium with rotation

In Boltzman limit, using only statistical mechanics and group theory arguments, the single particle distribution function is

$$f(x,p)_{rs} = e^{\xi} e^{-\beta\cdot p} \frac{1}{2} \left(D^S([p]^{-1} R_{\hat{\boldsymbol{\omega}}}(i\omega/T)[p]) + D^S([p]^{\dagger} R_{\hat{\boldsymbol{\omega}}}(i\omega/T)[p]^{\dagger -1}) \right)_{rs}$$

F. Becattini, L. Tinti, Ann. Phys. 325, 1566 (2010)

- $\beta = \frac{1}{T}(1, \boldsymbol{\omega} \times \mathbf{x}) = \frac{1}{T_0}(\gamma, \gamma \mathbf{v})$
- $\mathrm{R}_{\hat{\boldsymbol{\omega}}}(i\omega/T)$ SL$(2, C)$ Matrix representation of a rotation around $\hat{\boldsymbol{\omega}}$ axis for a complex angle $i\omega/T$

The degeneration factor $2S+1$ is replaced with the character of the representation $\chi(\frac{i\omega}{T})$ for a complex angle

$$\sum_{r=-S}^{S} f(x,p)_{rr} = e^{\xi} e^{-\beta\cdot p} \mathrm{tr} \mathrm{R}_{\hat{\boldsymbol{\omega}}}(i\omega/T) = e^{\xi} e^{-\beta\cdot p} \chi(\frac{i\omega}{T})$$

S=1/2 case

For $S = 1/2$ particles we can rewrite this formula using the Dirac spinors $u(p)$ and $v(p)$

$$f(x,p)_{rs} = e^{\xi} e^{-\beta\cdot p} \frac{1}{2m} \left[\bar{u}_r(p) \exp[(\omega/T)\Sigma_z] u_s(p)\right]$$
$$\bar{f}(x,p)_{rs} = -e^{-\xi} e^{-\beta\cdot p} \frac{1}{2m} \left[\bar{v}_s(p) \exp[-(\omega/T)\Sigma_z]^T v_r(p)\right]$$

$\Sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu] \quad \Sigma_z = \Sigma_{1,2}$

In fully covariant form

$$f(x,p)_{rs} = e^{\xi} e^{-\beta\cdot p} \frac{1}{2m} \left[\bar{u}_r(p) \exp[\varpi_{\mu\nu}\Sigma^{\mu\nu}] u_s(p)\right]$$
$$\bar{f}(x,p)_{rs} = -e^{-\xi} e^{-\beta\cdot p} \frac{1}{2m} \left[\bar{v}_s(p) \exp[-\varpi_{\mu\nu}\Sigma^{\mu\nu}]^T v_r(p)\right]$$

$$\varpi_{\mu\nu} = \omega/T(\delta^1_\mu \delta^2_\nu - \delta^1_\nu \delta^2_\mu) = \sqrt{\beta^2}\Omega_{\mu\nu} = \sqrt{\beta^2}\frac{De^\mu_i}{d\tau} e^{i\nu}$$

$\Omega_{\mu\nu}$ acceleration tensor of the Frenet-Serret tetrad of the velocity field lines

Fermi-Dirac extension

Ansatz

$$f(x,p)_{rs} = \frac{1}{2m}\bar{u}_r(p)\Big(\exp[\beta\cdot p - \xi]\exp[\varpi^{\mu\nu}\Sigma_{\mu\nu}] + I\Big)^{-1} u_s(p)$$
$$\bar{f}(x,p)_{rs} = -\frac{1}{2m}\left[\bar{v}_s(p)\Big(\exp[\beta\cdot p + \xi]\exp[-\varpi^{\mu\nu}\Sigma_{\mu\nu}] + I\Big)^{-1T} v_r(p)\right]$$

It reduces to Fermi-Dirac distribution function for no rotating case

$$f(x,p)_{rs} = \frac{\delta_{rs}}{e^{\beta\cdot p - \xi} + 1}$$

and if we take the Boltzmann limit becomes

$$f(x,p)_{rs} = e^{\xi}e^{-\beta\cdot p}\frac{1}{2m}\left[\bar{u}(p)\exp[\varpi^{\mu\nu}\Sigma_{\mu\nu}](p)\right]_{rs}$$

Local equilibrium

The local equilibrium density matrix

$$\widehat{\rho}_{LE}(t) = \frac{1}{Z_{LE}} \exp\left[- \int \mathrm{d}^3 x \widehat{T}^{0\nu} \beta_\nu(x) - \widehat{j}^0 \xi(x) - \frac{1}{2} \widehat{S}^{0,\mu\nu} \varpi_{\mu\nu}(x) \right]$$

Obtained by maximizing the entropy $\widehat{S} = -\mathrm{tr}(\widehat{\rho} \log \widehat{\rho})$, with the constraints of fixing mean energy-momentum, angular momentum density

Lower order approximations

The lowest order single particle distribution function can be obtained replacing

$$\beta_\nu \to \beta_\nu(x) \quad \xi \to \xi(x) \quad \varpi_{\mu\nu} \to \varpi_{\mu\nu}(x)$$

Local equilibrium

$$f(x,p)_{rs} = \frac{1}{2m}\bar{u}_r(p)\Big(\exp[\beta(x)\cdot p - \xi(x)]\exp[\varpi_{\mu\nu}(x)\Sigma^{\mu\nu}] + I\Big)^{-1} u_s(p)$$
$$\bar{f}(x,p)_{rs} = \frac{-1}{2m}\left[\bar{v}_s(p)\Big(\exp[\beta(x)\cdot p + \xi(x)]\exp[-\varpi(x)_{\mu\nu}\Sigma^{\mu\nu}] + I\Big)^{-1T} v_r(p)\right]$$

For small value of $\varpi_{\mu\nu}$ the first correction to the charge density and to the stress-energy tensor is $\mathcal{O}(\varpi^2_{\mu\nu})$

$$j^0(x) = 2\int \mathrm{d}^3p(n_F - \bar{n}_F)$$
$$+\varpi_{\mu\nu}\varpi^{\mu\nu}\frac{1}{4}\int \mathrm{d}^3p[n_F(1-n_F)(1-2n_F) - \bar{n}_F(1-\bar{n}_F)(1-2\bar{n}_F)]$$

$n_F = 1/[\exp(\beta(x)\cdot p - \xi(x)) + 1]$
Very small correction $\hbar\omega/KT \ll 1$

What's the meaning of ϖ out of equilibrium?

The meaning of $\beta(x)$ and $\xi(x)$ are inferred from equilibrium limit.

Global equilibrium

Frenet-Serret tensor

$$\varpi_{\mu\nu} = \sqrt{\beta^2}\Omega_{\mu\nu}$$
$$\Omega_{\mu\nu} = \frac{De_i^\mu}{\mathrm{d}\tau}e^{i\nu}$$

Thermal vorticity

$$\varpi_{\mu\nu} = -(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu)/2$$
$$\beta^\mu = \frac{1}{T}(1, \boldsymbol{\omega} \times \mathbf{x}) = \frac{1}{T_0}(\gamma, \gamma\mathbf{v})$$

In non equilibrium situation what's the right choice? If the system isn't too far from equilibrium ϖ should differ from equilibrium value only for 2nd order in gradients.

$$\varpi_{\mu\nu} = -(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu)/2 + \mathcal{O}(\partial^2\beta)$$

Polarization vector

$$\Pi_\mu = -\frac{1}{2}\epsilon_{\mu\rho\sigma\tau} S^{\rho\sigma} \frac{p^\tau}{m}$$

$S^{\rho\sigma}$ is total angular momentum of single particle, then the mean polarization vector of a particle with momentum p around the space-time point x

$$\langle \Pi_\mu(x,p) \rangle = -\frac{1}{2}\frac{1}{\mathrm{tr} f}\epsilon_{\mu\rho\sigma\tau} \frac{\mathrm{d}\mathcal{J}^{0,\rho\sigma}(x,p)}{\mathrm{d}^3 p} \frac{p^\tau}{m}$$

with the total angular momentum density

$$\frac{\mathrm{d}\mathcal{J}^{0,\rho\sigma}(x,p)}{\mathrm{d}^3 p} = (p^\rho x^\sigma - p^\sigma x^\rho)\mathrm{tr} f(x,p) + \frac{\mathrm{d}\mathcal{S}^{0,\rho\sigma}(x,p)}{\mathrm{d}^3 p}$$

The Levi-Civita tensor makes irrelevant the orbital part of the angular momentum density.

Spin-tensor

The average of the *canonical* spin tensor using the distribution function

$$\mathcal{S}^{\lambda,\mu\nu} = \frac{1}{2}\langle : \Psi\{\gamma^\lambda, \Sigma^{\mu\nu}\}\Psi :\rangle$$
$$= \frac{1}{2}\int \frac{\mathrm{d}^3 p}{2\varepsilon}\sum_{rs} f(x,p)_{rs}\bar{u}(p)_s\{\gamma^\lambda, \Sigma^{\mu\nu}\}u_r(p) - \bar{f}(x,p)_{rs}\bar{v}(p)_r\{\gamma^\lambda, \Sigma^{\mu\nu}\}v(p)_s$$

After some algebra..

$$\mathcal{S}^{\lambda,\mu\nu} = \varpi^{\mu\nu}\frac{1}{2}\int \frac{\mathrm{d}^3 p}{\varepsilon} p^\lambda n_F(p)(1 - n_F(p)) + \text{rotations of indices}$$

and

$$\frac{\mathrm{d}\mathcal{S}^{\lambda,\mu\nu}(x,p)}{\mathrm{d}^3 p} = \varpi^{\mu\nu}\frac{1}{2\varepsilon}p^\lambda n_F(p)(1 - n_F(p)) + \text{rotations of indices}$$

It's a $\mathcal{O}(\varpi)$ term

Mean Polarization vector

$$\langle \Pi_\mu(x,p) \rangle = -\frac{1}{8}\epsilon_{\mu\rho\sigma\tau}(1-n_F(p))\varpi^{\rho\sigma}(x)\frac{p^\tau}{m} = \frac{1}{8}\epsilon_{\mu\rho\sigma\tau}(1-n_F(p))\partial^\rho\beta^\sigma(x)\frac{p^\tau}{m}$$

$$\Pi = (\Pi_0, \mathbf{\Pi}) = \frac{1-n_F}{8m}((\nabla \times \boldsymbol{\beta}) \cdot \hat{\mathbf{p}}, \varepsilon(\nabla \times \boldsymbol{\beta}) - \frac{\partial \boldsymbol{\beta}}{\partial t} \times \mathbf{p} - \nabla\beta^0 \times \mathbf{p})$$

Quasi-free particles get transversely polarized also in steady temperature gradient without velocity flow ($v = 0$ and $\nabla\beta^0 \neq 0$)

$$\Pi = (1-n_F)\frac{\hbar p}{8mKT^2}(0, \nabla T \times \mathbf{p})$$

Very tiny term but in some situation could be relevant!

For antiparticles just replace $n_F \to \bar{n}_F$

Particles and antiparticles have the same orientation of the polarization vector

Polarization in Heavy Ion Collision

In Heavy Ion Collision, one can now calculate the space-integral of the mean polarization vector for the 3-dimensional freezeout hypersurface Σ.

$$\langle \Pi_\mu(p) \rangle = \frac{\int \mathrm{d}\Sigma_\lambda \frac{p^\lambda}{\varepsilon}(-1/2)\epsilon_{\mu\rho\sigma\tau}\frac{\mathrm{d}\mathcal{S}^{0,\rho\sigma}}{\mathrm{d}^3 p}\frac{p^\tau}{m}}{\int \mathrm{d}\Sigma_\lambda \frac{p^\lambda}{\varepsilon}\mathrm{tr} f}$$

that for $\varpi \ll 1$ becomes

$$\langle \Pi_\mu(p) \rangle = \frac{1}{8}\epsilon_{\mu\rho\sigma\tau}\frac{p^\tau}{m}\frac{\int \mathrm{d}\Sigma_\lambda p^\lambda n_F(1-n_F)\partial^\rho \beta^\sigma}{\int \mathrm{d}\Sigma_\lambda p^\lambda n_F}$$

Cooper-Frye for polarization

Polarization of Λ

The Λ polarization is determined by measuring the angular distribution of the decay protons

$$\frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}\Omega^*} = \frac{1}{4\pi}(1 + \alpha\mathbf{\Pi}_0 \cdot \hat{\mathbf{p}}^*)$$

$$\mathbf{\Pi}(p) = \frac{\varepsilon}{8m}\frac{\int \mathrm{d}\Sigma_\lambda p^\lambda n_F (\nabla \times \boldsymbol{\beta})}{\int \mathrm{d}\Sigma_\lambda p^\lambda n_F} + \frac{\boldsymbol{p}}{8m} \times \frac{\int \mathrm{d}\Sigma_\lambda p^\lambda n_F (\partial_t \boldsymbol{\beta} + \nabla \beta^0)}{\int \mathrm{d}\Sigma_\lambda p^\lambda n_F}$$

Polarization 3-vector in the rest-frame of Λ

$$\mathbf{\Pi}_0 = \mathbf{\Pi} - \frac{\boldsymbol{p}}{\varepsilon(\varepsilon + m)}\mathbf{\Pi} \cdot \mathbf{p}$$

See L. Csernai's talk!

Conclusions and Outlook

- We have determinated the relativistic distribution function for spin $1/2$ particles at local equilibrium
- At leading order in the gradients of β the polarization of the particles is proportional to thermal vorticity $\partial_\mu \beta_\nu - \partial_\nu \beta_\mu$.
- We have obtained an extension of the Cooper-Frye formula to calculate spin 1/2 particle polarization in Heavy Ion collision at the freeze out.
- The detection of this polarization (see L. Csernai's talk) would be a striking confirmation of the achievement of local thermodynamical equilibrium, also for the spin degrees of freedom.