Thermalization of massive partons in anisotropic medium

W.Florkowski[1,2], R.Ryblewski[2]

[1] Jan Kochanowski University, Kielce, Poland
[2] Institute of Nuclear Physics, Krakow, Poland

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Motivation

- heavy-ion experimental data from RHIC and the LHC very well described by the 2nd order viscous hydrodynamics with early starting time, $\tau_0 < 1$ fm/c
- only at such early times the transverse distribution of matter in the colliding nuclei is known and may be used to model the initial energy/entropy density profile for hydrodynamic calculations
- viscous corrections combined with rapid longitudinal expansion induce a substantial pressure asymmetry in the created system
- at early times the microscopic models (string models, color glass condensate, pQCD kinetic calculations) predict also a large momentum anisotropy
- AdS/CFT correspondence predicts a large difference between $P_\perp$ and $P_\parallel$, which slowly decays with time (Heller, Janik, Witaszczyk)

Motivation

- viscous hydrodynamics is based on the linearization around an isotropic background $\longrightarrow$ large shear corrections (of the order of isotropic pressure) are present, this leads often to unphysical results
- new framework of anisotropic hydrodynamics (Florkowski, Ryblewski, Martinez, Strickland), it is based on the reorganization of the hydrodynamic expansion, anisotropy included in the leading order
- anisotropic hydrodynamics agrees with viscous hydrodynamics when the anisotropy is small, several problems are solved i.e. negative particle pressures, incorrectly reproduced the free-streaming limit

Motivation

- check of various hydrodynamic approximation methods

  first, we solve exactly the kinetic equation for transversely homogenous and boost-invariant system of massless particles in the relaxation time approximation

  subsequently, we compare two different second order viscous hydrodynamics approximations and anisotropic hydrodynamics to the exact solutions

- analyze impact of finite parton masses on the thermalization process

Kinetic equation
General setup

- Boltzmann equation (BE) in the relaxation-time approximation (RTA)

$$p^\mu \partial_\mu f(x,p) = C[f(x,p)] \qquad C[f] = p\cdot u \, \frac{f^{\rm eq} - f}{\tau_{\rm eq}}$$

background distribution

$$f^{\rm eq} = \frac{2}{(2\pi)^3} \exp(-p\cdot u/T)$$

- boost-invariant variables (Bialas, Czyz)

$$w = t p_L - z E \qquad v = tE - z p_L = \sqrt{w^2 + \left(m^2 + \vec{p}_T^{\,2}\right)\tau^2}$$

$$E = \frac{vt + wz}{\tau^2} \qquad p_L = \frac{wt + vz}{\tau^2}$$

- boost-invariant form of the kinetic equation

$$\frac{\partial f}{\partial \tau} = \frac{f^{\rm eq} - f}{\tau_{\rm eq}}$$

$$f^{\rm eq}(\tau, w, p_T) = \frac{2}{(2\pi)^3} \exp\left[- \frac{\sqrt{w^2 + \left(m^2 + \vec{p}_T^{\,2}\right)\tau^2}}{T(\tau)\tau} \right]$$

Kinetic equation
Moments

- zeroth moment (particle production $\leftrightarrow$ gluon emission)

$$\partial_\mu \int dP\, p^\mu f = \int dP\, C \qquad \frac{dn}{d\tau} + \frac{n}{\tau} = \frac{n^{\rm eq} - n}{\tau_{\rm eq}}$$

- first moment (energy-momentum conservation)

$$\partial_\mu \underbrace{\int dP\, p^\nu p^\mu f}_{} = \int dP\, p^\nu C = 0 \qquad \frac{d\mathcal{E}}{d\tau} = -\frac{\mathcal{E} + \mathcal{P}_L}{\tau}$$

$$T^{\mu\nu} = (\mathcal{E} + \mathcal{P}_T) u^\mu u^\nu - \mathcal{P}_T g^{\mu\nu} + (\mathcal{P}_L - \mathcal{P}_T) z^\mu z^\nu$$

$$u^\mu = \left(\frac{t}{\tau}, 0, 0, \frac{z}{\tau}\right) \qquad z^\mu = \left(\frac{z}{\tau}, 0, 0, \frac{t}{\tau}\right)$$

- Landau matching (effective temperature determination)

$$\mathcal{E}(\tau) = \mathcal{E}^{\rm eq}(\tau) \qquad \frac{g_0}{\tau^2} \int dP\, v^2\, f(\tau, w, p_T) = \mathcal{E}^{\rm eq}(T(\tau))$$

- 0th and 1st moments are fulfilled automatically for the exact solution of BE

Kinetic equation
Formal solution

- formal structure of the solutions (Gordon Baym)

$$f(\tau, w, p_T) \quad = \quad D(\tau,\tau_0) f_0(w,p_T) + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\rm eq}(\tau')} D(\tau,\tau')\, f^{\rm eq}(\tau', w, p_T)$$

$$D(\tau_2,\tau_1) = \exp\left[-\int_{\tau_1}^{\tau_2} \frac{d\tau''}{\tau_{\rm eq}(\tau'')} \right]$$

- equilibration time in RTA (Anderson and Witting)

$$\tau_{\rm eq}(\tau) = \frac{\varepsilon + P^{\rm eq}}{T P^{\rm eq}} \frac{\bar\eta}{\tilde\eta(\frac{m}{T})} \qquad \lim_{\frac{m}{T}\to 0} \tau_{\rm eq}(\tau) = \frac{5\bar\eta}{T(\tau)} \qquad \bar\eta = \eta/\mathcal{S} = \frac{1}{4\pi}, \frac{3}{4\pi}, \ldots, \frac{10}{4\pi}$$

- initial distribution $f_0(w,p_T) = f_{\rm RS}(\tau_0, w, p_T) \quad \Lambda(\tau_0) = \Lambda_0, \xi(\tau_0) = \xi_0$
- Romatschke-Strickland (RS) form

$$f_{\rm RS}(\tau, w, p_T) = \frac{1}{4\pi^3} \exp\left[-\frac{\sqrt{(1+\xi(\tau))w^2 + (m^2+p_T^2)\tau^2}}{\Lambda(\tau)\tau} \right]$$

Kinetic equation
Numerical method

$$\varepsilon^{\rm eq}(T(\tau)) = D(\tau,\tau_0)\frac{\Lambda_0^4}{2}\tilde{\mathcal{H}}_2\left(\frac{\tau_0 x_0^{-1/2}}{\tau},\frac{m}{\Lambda_0}\right) + \int_{\tau_0}^{\tau}\frac{d\tau'}{2\tau_{\rm eq}(\tau')}D(\tau,\tau')\,T^4(\tau')\tilde{\mathcal{H}}_2\left(\frac{\tau'}{\tau},\frac{m}{T}\right)$$

$$\tilde{\mathcal{H}}_2(y,z) = \int_0^\infty dr\, r^3 \mathcal{H}_2\left(y,\frac{z}{r}\right)\exp\left(-\sqrt{r^2+z^2}\right)$$

$$\mathcal{H}_2(y,z) = y\int_0^\pi d\phi\,\sin\phi\sqrt{y^2\cos^2\phi+\sin^2\phi+z^2}$$

- iterative method (Banerjee, Bhalerao, Ravishankar):
  1.) use trial function $T(\tau)$ on the RHS of the dynamic equation
  2.) the LHS of the dynamic equation determines the new $T(\tau)$
  3.) use the new $T(\tau)$ as the trial one
  4.) repeat steps 1-3 until the stable $T(\tau)$ is found
- particle density, transverse and longitudinal pressure

$$n(\tau) = \frac{g_0}{\tau}\int dP\, v\, f(\tau,w,p_T)$$

$$\mathcal{P}_L(\tau) = \frac{g_0}{\tau^2}\int dP\, w^2\, f(\tau,w,p_T), \quad \mathcal{P}_T(\tau) = \frac{g_0}{2}\int dP\, p_T^2\, f(\tau,w,p_T)$$

Massless particles

Anisotropic hydrodynamics

- assumption : distribution function is always well approximated by RS form
- the RS form is defined by transverse momentum scale $\Lambda(\tau)$ and anisotropy parameter $\xi(\tau)$
- energy density and pressures are given by simple formulas

$$\mathcal{E} = \frac{6 g_0 \Lambda^4}{\pi^2} \mathcal{R}(\xi) \qquad \mathcal{P}_T = \frac{3 g_0 \Lambda^4}{\pi^2} \mathcal{R}_T(\xi) \qquad \mathcal{P}_L = \frac{3 g_0 \Lambda^4}{\pi^2} \mathcal{R}_L(\xi) \qquad n = \frac{2 g_0 \Lambda^3}{\pi^2 \sqrt{1+\xi}}$$

- the 0th and 1st moments of the BE in the RTA are evaluated

$$\frac{\partial_\tau \xi}{1+\xi} = \frac{2}{\tau} - \frac{4\mathcal{R}(\xi)}{\tau_{\rm eq}^{\rm AH}} \frac{\mathcal{R}^{3/4}(\xi)\sqrt{1+\xi} - 1}{2\mathcal{R}(\xi) + 3(1+\xi)\mathcal{R}'(\xi)}$$

$$\frac{1}{1+\xi} \frac{\partial_\tau \Lambda}{\Lambda} = \frac{\mathcal{R}'(\xi)}{\tau_{\rm eq}^{\rm AH}} \frac{\mathcal{R}^{3/4}(\xi)\sqrt{1+\xi} - 1}{2\mathcal{R}(\xi) + 3(1+\xi)\mathcal{R}'(\xi)}$$

- relaxation time

$$\tau_{\rm eq}^{\rm AH}(\tau) = \frac{5\bar{\eta}}{2\Lambda(\tau)}$$

First-order viscous hydrodynamics

- equations of the first-order viscous hydrodynamics

$$
\begin{aligned}
\partial_\tau \mathcal{E} &= -\frac{\mathcal{E} + \mathcal{P}_{\rm eq}}{\tau} + \frac{\Pi}{\tau} \qquad\qquad (\mathcal{E}_{\rm eq} = \mathcal{E}) \\
\Pi &= \frac{4\eta_{\rm eff}}{3\tau}
\end{aligned}
$$

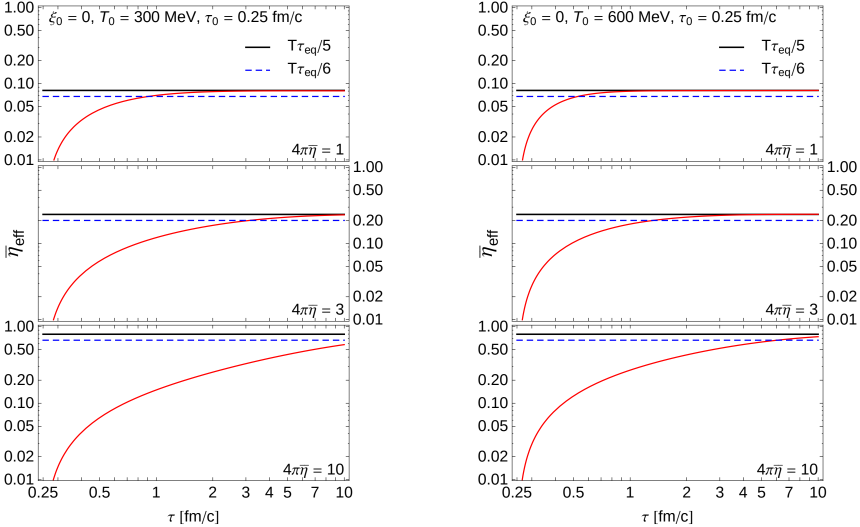
- using $T(\tau)$ from the exact solution of BE the following equivalent equation allows us to calculate effective $\bar{\eta}_{\rm eff} = \eta_{\rm eff}/\mathcal{S}_{\rm eq}$

$$
\frac{dT}{d\tau} + \frac{T}{3\tau} = \frac{4\bar{\eta}_{\rm eff}}{9\tau^2}
$$

- we may compare $\bar{\eta}_{\rm eff}$ with $\bar{\eta}$ used in kinetic equation

First-order viscous hydrodynamics

Extraction of shear viscosity



Second-order viscous hydrodynamics

- equations of the second-order viscous hydrodynamics

$$\partial_\tau \mathcal{E} = -\frac{\mathcal{E}+\mathcal{P}_{\rm eq}}{\tau} + \frac{\Pi}{\tau}$$
$$\partial_\tau \Pi = -\frac{\Pi}{\tau_\pi} + \frac{4}{3}\frac{\eta}{\tau_\pi \tau} - \beta\frac{\Pi}{\tau}$$

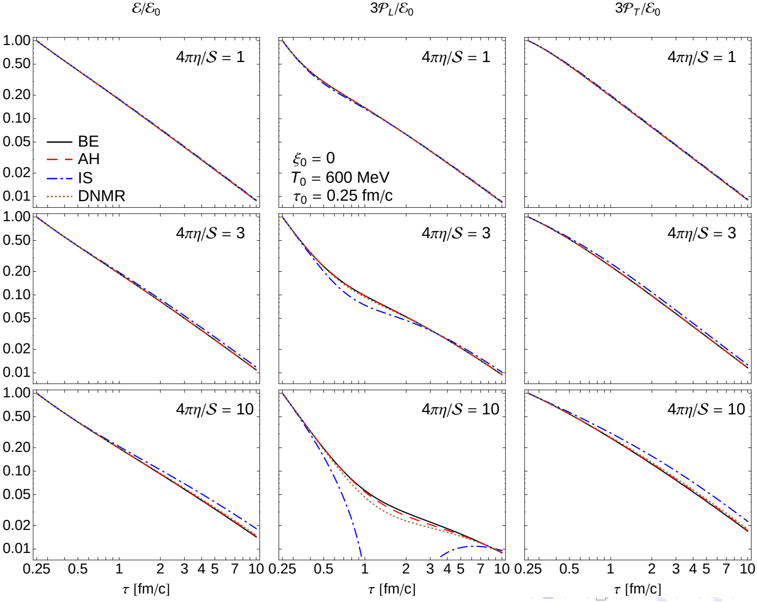
shear relaxation time

$$\tau_\pi(\tau) = \frac{5\bar{\eta}}{T(\tau)}$$

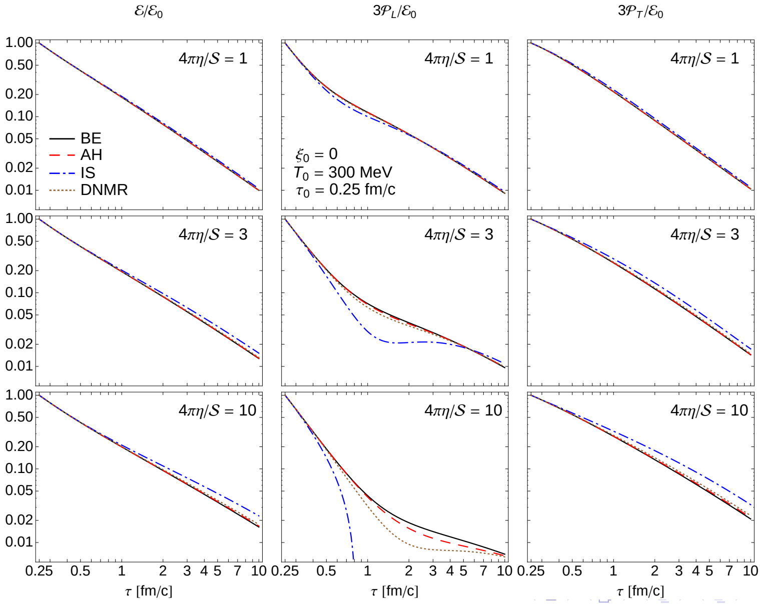
- $\beta = 4/3$ Israel-Stewart (IS)
- $\beta = 38/21$ Denicol, Niemi, Molnar, Rischke (DNMR)

- $\mathcal{P}_T = \mathcal{P}_{\rm eq} + \Pi/2 \qquad \mathcal{P}_L = \mathcal{P}_{\rm eq} - \Pi \qquad n \propto T^3 \quad (T \propto \mathcal{E}^{1/4})$

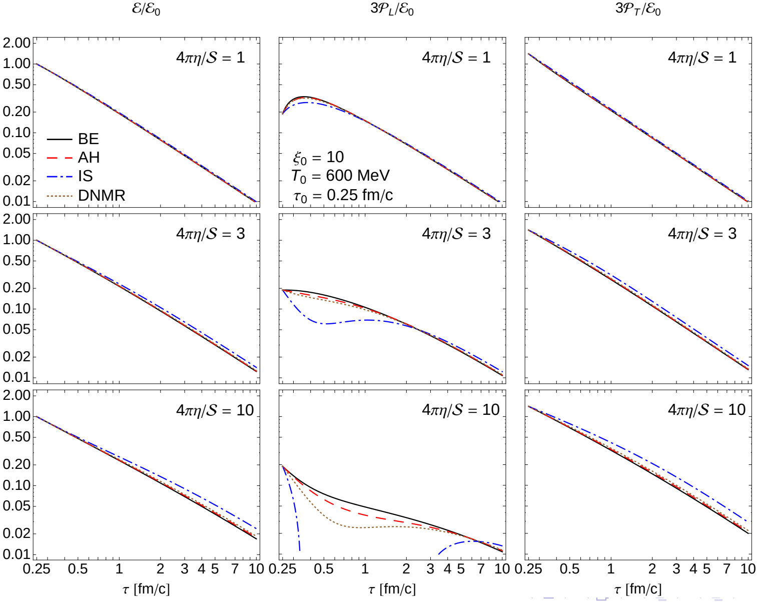
Comparison with exact solutions



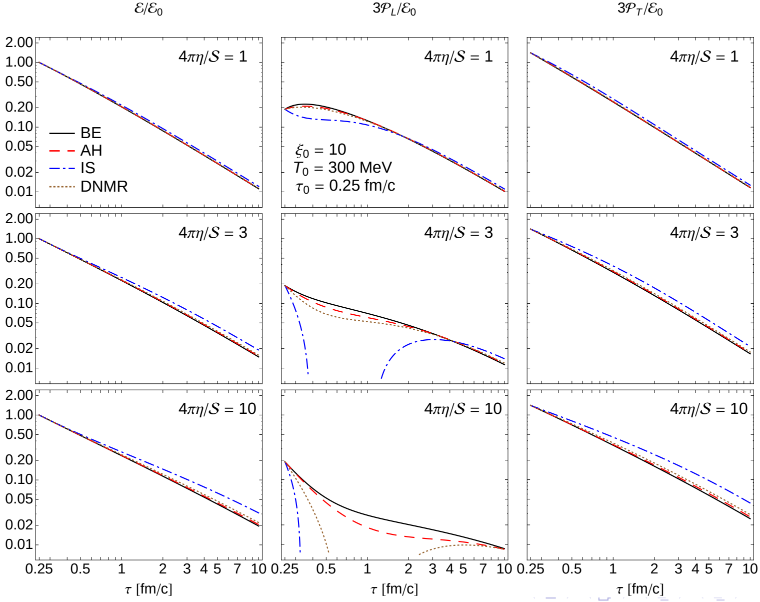
Comparison with exact solutions



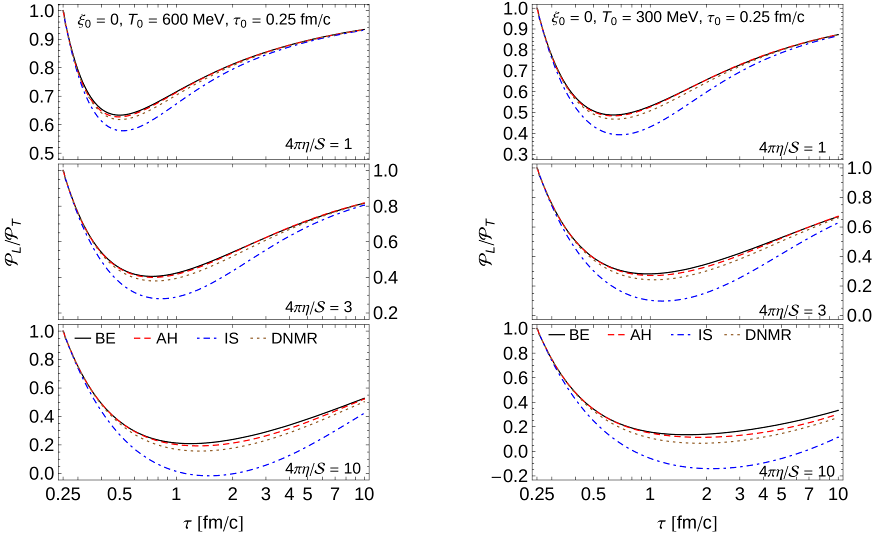
Comparison with exact solutions



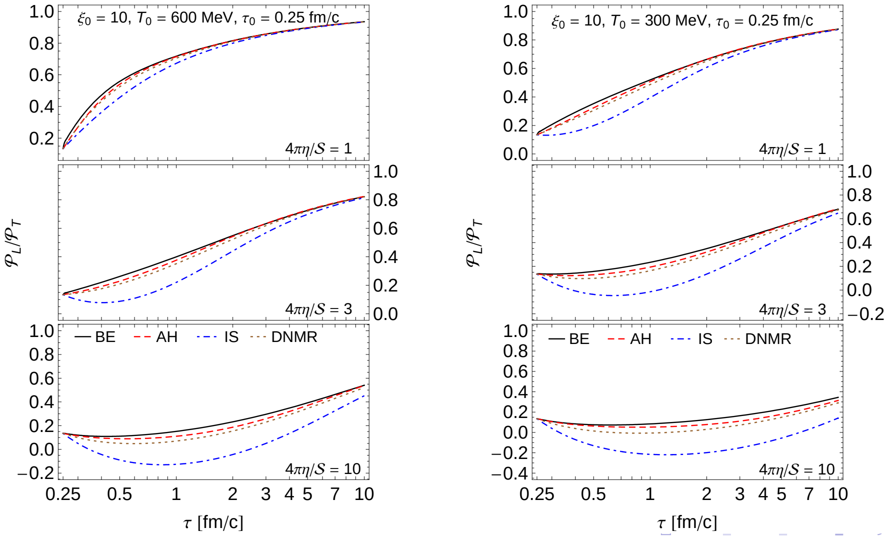
Comparison with exact solutions



Comparison with exact solutions



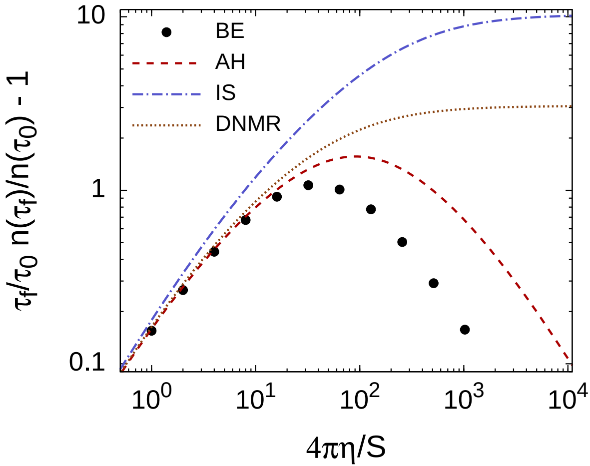
Comparison with exact solutions



Conclusions I

- in all considered cases the deviations of IS description from the exact solution are large
- for larger $\eta/\mathcal{S}$, smaller $\mathcal{E}$ or larger ξ the description is even worse
- usage of the DNMR approach highly improves the agreement
- in both IS and DNMR cases the problem with negative particle pressures remains
- AH gives the better description that DNMR and IS, the problem with negative pressures is solved

Comparison with exact solutions



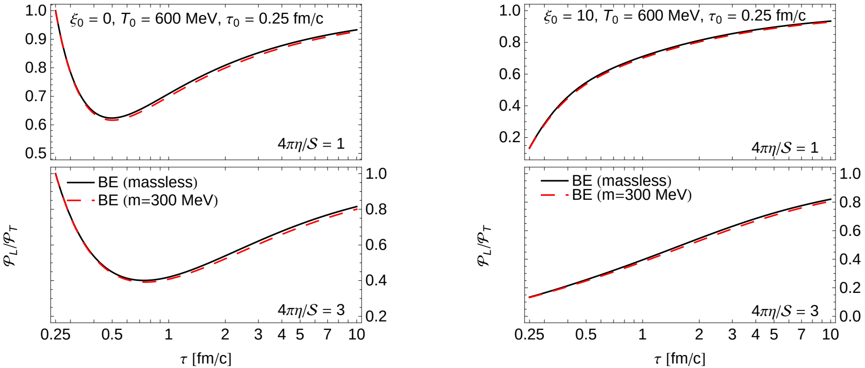
Conclusions II

- particle production measure $\Delta_n = \tau_f/\tau_0 n(\tau_f)/n(\tau_0) - 1$ is expected to vanish in the ideal hydrodynamical limit ($\eta/\mathcal{S} \to 0$) and the free-streaming limit ($\eta/\mathcal{S} \to \infty$)
- IS and DNMR predict that the Δ_n is a monotonically increasing function of the $\eta/\mathcal{S}$
- only AH framework qualitatively reproduces the ratio also at the limit $\eta/\mathcal{S} \to \infty$

Massive particles

Exact solutions of kinetic equation

Impact of finite parton masses on thermalization process



inclusion of finite parton masses weakly affects the thermalization of the system

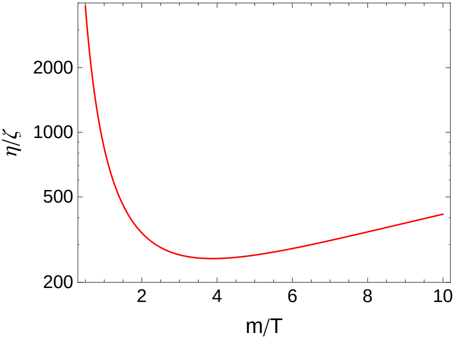
Viscous hydrodynamics

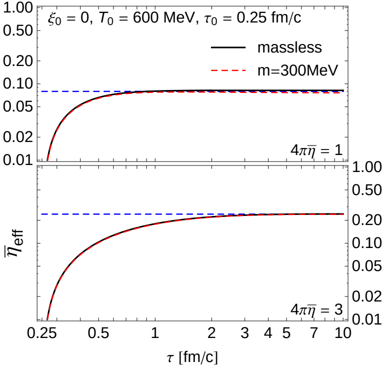
Extraction of shear η viscosity

- equations of the first-order viscous hydrodynamics in the case of massive particles

$$\partial_\tau \mathcal{E} = -\frac{\mathcal{E}+\mathcal{P}_{\rm eq}}{\tau} + \frac{\Pi - \cancel{\Pi_\zeta}}{\tau} \qquad (\mathcal{E}_{\rm eq} = \mathcal{E})$$

$$\Pi = \frac{4\eta}{3\tau} \qquad \Pi_\zeta = -\frac{\zeta}{\tau}$$



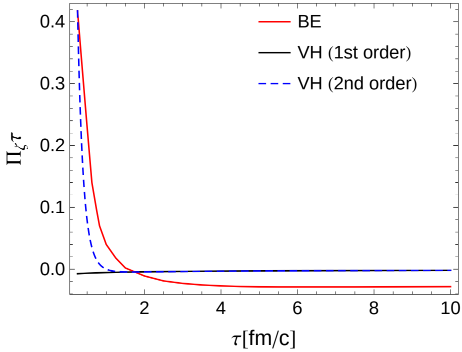


Viscous hydrodynamics
Extraction of bulk ζ viscosity

- second-order viscous hydrodynamics equation for bulk viscous pressure

$$\frac{\partial \Pi}{\partial \tau} = -\frac{\Pi}{\tau_\Pi} - \frac{1}{2}\frac{1}{\beta_0}\Pi\left[\beta_0\frac{1}{\tau} + T\frac{\partial}{\partial \tau}\left(\frac{\beta_0}{T}\right)\right] - \frac{1}{\beta_0}\frac{1}{\tau} \qquad +\lambda_{\Pi\pi}\eta \ \ (?)$$

$$\beta_0 = \tau_\Pi/\zeta \qquad \tau_\Pi = \tau_{\rm eq}$$



Conclusions III

- inclusion of finite parton massess does not affect the thermalization of the system
- bulk viscosity not reproduced by the 2nd order viscous hydrodynamics
- other kinetic coefficients (i.e. $\lambda_{\Pi\pi}$) may play an important role in the description of bulk viscosity of the quark-gluon plasma

Summary

- an exactly solvable case presented and the accuracy of different hydrodynamical approximation schemes tested
- DNMR scheme works much better than IS $\longrightarrow$ main lesson for more and more numerous hydro practioners

$$\sout{\beta = 4/3} \qquad \beta = 38/21$$

- AH scheme works even better than DNMR provided the relaxation times in BE and AH are properly matched
- the correct relationship between the shear viscosity and the relaxation time established

$$\sout{\eta = \frac{2}{3} P_{\rm eq} \tau_{\rm eq}} \qquad \eta = \frac{4}{5} P_{\rm eq} \tau_{\rm eq}$$

- the inclusion of finite parton masses does not affect the thermalization, the value of bulk viscosity in the close-to-equilibrium limit is not reproduced correctly

Thank you for your attention!

Backup slides

Late time behavior
Anisotropic hydrodynamics

- since $\xi \to 0 (\tau \to \infty)$ we linearize AH equations in ξ

$$\partial_\tau \xi = \frac{2}{\tau} + \left(\frac{2}{\tau} - \frac{\Gamma}{2}\right)\xi - \frac{17}{63}\Gamma\xi^2 + \mathcal{O}(\xi^3)$$

$$\partial_\tau \Lambda = -\frac{1}{12}\Gamma\Lambda\xi + \frac{187}{3780}\Gamma\Lambda\xi^2 + \mathcal{O}(\xi^3)$$

- one finds

$$\lim_{\tau\to\infty} \xi(\tau) = \frac{4}{\Gamma\tau} + \frac{968}{63(\Gamma\tau)^2} + \mathcal{O}\left(\frac{1}{\tau^3}\right)$$

$$\lim_{\tau\to\infty} \Lambda(\tau) = \frac{C}{\tau^{1/3}}\left(1 + \frac{22}{45}\frac{1}{\Gamma\tau} + \mathcal{O}\left(\tau^{-2}\right)\right)$$

- having determined ξ and Λ we may find $\mathcal{E} = \mathcal{R}(\xi)\mathcal{E}_{\rm eq}(\Lambda)$ in the form

$$\lim_{\tau\to\infty} \mathcal{E}(\tau) = \frac{D}{\tau^{4/3}}\left(1 - \frac{32}{45}\frac{1}{\Gamma\tau} + \mathcal{O}\left(\tau^{-2}\right)\right)$$

Late time behavior
Kinetic equation

- RTA integral equation

$$\mathcal{E}(\tau) = \underbrace{D(\tau,\tau_0)\mathcal{E}_0 \frac{\mathcal{H}\left(\frac{\tau_0}{\tau} x_o^{-1/2}\right)}{\mathcal{H}\left(x_o^{-1/2}\right)}}_{\to 0(\tau\to\infty)} + \int_{\tau_0}^{\tau} \frac{d\tau'}{2\tau_{\rm eq}(\tau')} D(\tau,\tau')\mathcal{E}(\tau')\mathcal{H}\left(\frac{\tau'}{\tau}\right)$$

- large-τ asymptotic expansion

$$\lim_{\tau\to\infty} \mathcal{E}(\tau) = A\left(\frac{\tau_{\rm eq}}{\tau}\right)^{4/3}\left(1 + B\frac{\tau_{\rm eq}}{\tau} + \mathcal{O}\left(\tau^{-2}\right)\right)$$

$$\lim_{\tau'\to\tau} \mathcal{H}\left(\frac{\tau'}{\tau}\right) = 2 + \frac{8(\tau'-\tau)}{3\tau} + \frac{4(\tau'-\tau)^2}{5\tau^2} + \mathcal{O}\left((\tau'-\tau)^3\right)$$

requiring equivalence we get $B = -16/45$

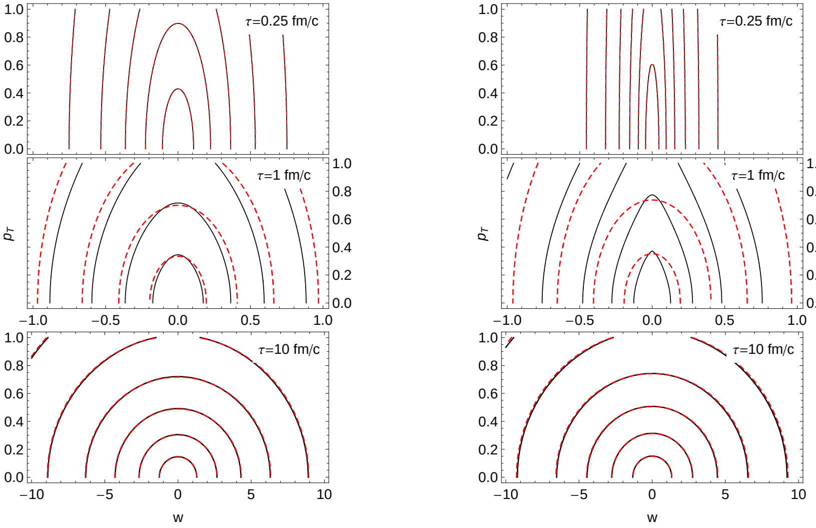
- comparing the two solutions we get

$$\tau_{\rm eq}^{\rm AH} = \frac{\tau_{\rm eq}}{2}, \quad |\xi| \ll 1$$

Comparison with exact solutions

$T_0 = 600$ MeV $\quad \xi_0 = 0 \quad \bar{\eta} = 3/4\pi$

$T_0 = 600$ MeV $\quad \xi_0 = 10 \quad \bar{\eta} = 3/4\pi$



Extra Conclusions

- the AH solution deviates from exact (BE) solution at early times
- the shape of the contours of the exact solution indicates two contributions to the solution, one part which is related to the initial free-streaming (the sharp one) and the other related to the late-time dynamics