Systematic Properties of the Tsallis Distribution: Energy Dependence of Parameters

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Outline

History
Boltzmann
Tsallis
Comparison Boltzmann vs. Tsallis
Transverse Momentum Distributions
Tsallis works well in p-p collisions
Tsallis does NOT describe Pb-Pb
... but works for p-Pb collisions
Summary of Results
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Transverse Momentum Distribution

**STAR** collaboration, B.I. Abelev at al.

**PHENIX** collaboration, A. Adare et al.

**ALICE** collaboration, K. Aamodt et al.

**CMS** collaboration, V. Khachatryan et al.

**ATLAS** collaboration, G. Aad et al.
Transverse Momentum Distribution

STAR, PHENIX, ALICE, CMS, ATLAS use:

\[
\frac{d^2N}{d\rho_T dy} = \rho_T \times \frac{dN}{dy} \frac{(n-1)(n-2)}{nT(nT + m_0(n-2))} \left(1 + \frac{m_T - m_0}{nT}\right)^{-n}
\]

What is the connection with the Tsallis distribution? Also, the physical significance of the parameters \(n\) and \(T\) has never been discussed by STAR, PHENIX, ALICE, ATLAS, CMS.
In the grand canonical ensemble the particle number, energy density and pressure are given by

\[
N = gV \int \frac{d^3p}{(2\pi)^3} \exp \left( -\frac{E - \mu}{T} \right),
\]

\[
\epsilon = g \int \frac{d^3p}{(2\pi)^3} E \exp \left( -\frac{E - \mu}{T} \right),
\]

\[
P = g \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3E} \exp \left( -\frac{E - \mu}{T} \right),
\]

where \( T \) and \( \mu \) are the temperature and the chemical potential, \( V \) is the volume and \( g \) is the degeneracy factor.
In particular, the particle number is:

\[
E \frac{d^3 N}{d^3 p} = \frac{gVE}{(2\pi)^3} e^{-\frac{E-\mu}{T}},
\]

\[
\frac{d^2 N}{m_T dm_T dy} = \frac{gVm_T \cosh y}{(2\pi)^2} e^{-\frac{m_T \cosh y-\mu}{T}},
\]

at mid-rapidity, \( y = 0 \) and zero chemical potential this becomes

\[
\left. \frac{d^2 N}{m_T dm_T dy} \right|_{y=0} = \frac{gVm_T}{(2\pi)^2} e^{-\frac{m_T}{T}}
\]

\( m_T \) scaling! Hopeless!
Possible generalization of Boltzmann-Gibbs statistics

Constantino Tsallis
Rio de Janeiro, TBPF

Citations: 1 389
Citations in HEP: 513
Entropy: Tsallis vs Boltzmann

The Boltzmann entropy is given by

\[ S^B = -g \sum_i [f_i \ln f_i - f_i] \] \tag{1}

The Tsallis entropy is given by

\[ S^T_T = -g \sum_i [f_i^q \ln_q f_i - f_i] \] \tag{2}

which uses

\[ \ln_q(x) \equiv \frac{x^{1-q} - 1}{1 - q} \] \tag{3}

often referred to as q-logarithm.

By maximizing the entropy one obtains expressions for particle density, energy density and pressure.
For high energy physics a consistent form of Tsallis statistics for the particle number, energy density and pressure is given by

\[
N = gV \int \frac{d^3p}{(2\pi)^3} \left[ 1 + (q - 1) \frac{E - \mu}{T} \right]^{\frac{q}{q-1}},
\]

\[
\epsilon = g \int \frac{d^3p}{(2\pi)^3} E \left[ 1 + (q - 1) \frac{E - \mu}{T} \right]^{\frac{q}{q-1}},
\]

\[
P = g \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3E} \left[ 1 + (q - 1) \frac{E - \mu}{T} \right]^{\frac{q}{q-1}}.
\]

where \( T \) and \( \mu \) are the temperature and the chemical potential, \( V \) is the volume and \( g \) is the degeneracy factor. The Tsallis distribution introduces a new parameter \( q \) which for transverse momentum spectra is always close to 1.
Thermodynamic consistency

\[dE = -pdV + TdS + \mu dN\]

Inserting \(E = \epsilon V, S = sV\) and \(N = nV\) leads to

\[d\epsilon = Tds + \mu dn\]

\[dP = nd\mu + sdT\]

In particular

\[n = \left. \frac{\partial P}{\partial \mu} \right|_T, \quad s = \left. \frac{\partial P}{\partial T} \right|_\mu, \quad T = \left. \frac{\partial \epsilon}{\partial s} \right|_n, \quad \mu = \left. \frac{\partial \epsilon}{\partial n} \right|_s.\]

are satisfied for the Tsallis distribution.
In the Tsallis distribution the total number of particles is given by:

\[ N = gV \int \frac{d^3p}{(2\pi)^3} \left[ 1 + (q - 1) \frac{E - \mu}{T} \right]^{-\frac{q}{q-1}}. \]

The corresponding momentum distribution is given by

\[ E \frac{dN}{d^3p} = gVE \frac{1}{(2\pi)^3} \left[ 1 + (q - 1) \frac{E - \mu}{T} \right]^{-\frac{q}{q-1}}, \]

which, in terms of the rapidity and transverse mass variables, \( E = m_T \cosh y \), becomes (at mid-rapidity for \( \mu = 0 \))

\[ \frac{d^2N}{dp_T \, dy} \bigg|_{y=0} = gV \frac{p_T m_T}{(2\pi)^2} \left[ 1 + (q - 1) \frac{m_T}{T} \right]^{-\frac{q}{q-1}}, \]

Comparison of the Tsallis form with the STAR, PHENIX, ALICE, ATLAS, CMS distributions

\[
\frac{d^2N}{dp_T dy} = gV \frac{p_T m_T}{(2\pi)^2} \left[ 1 + (q - 1) \frac{m_T}{T} \right]^{-q/(q-1)},
\]

\[
\frac{d^2N}{dp_T dy} = p_T \times \frac{dN}{dy} \frac{(n-1)(n-2)}{nT(nT + m_0(n-2))} \left[ 1 + \frac{m_T - m_0}{nT} \right]^{-n}
\]

For the comparison use the following substitution:

\[
n \rightarrow \frac{q}{q-1}
\]

\[
nT \rightarrow \frac{T + m_0(q-1)}{q-1}
\]
After this substitution one obtains

\[
\frac{d^2N}{dp_T\,dy} = \rho_T \frac{dN}{dy} \frac{(n-1)(n-2)}{nT(nT + m_0(n-2))}
\]

\[
\left[ \frac{T}{T + m_0(q-1)} \right]^{-q/(q-1)}
\]

\[
\left[ 1 + (q-1)\frac{m_T}{T} \right]^{-q/(q-1)}.
\]

To be compared with

\[
\frac{d^2N}{d\rho_T\,dy} = gV \frac{\rho_T m_T}{(2\pi)^2} \left[ 1 + (q-1)\frac{m_T}{T} \right]^{-q/(q-1)}.
\]

Apart from several constant factors, which can be absorbed in the volume \( V \), only a factor of \( m_T \) differs! However, \( m_0 \) shouldn’t appear as it destroys \( m_T \) scaling. The inclusion of the factor \( m_T \) leads to a more consistent interpretation of the variables \( q \) and \( T \).
Interpretation of Tsallis Parameter $q$


$$\left[ 1 + (q - 1) \frac{E - \mu}{T} \right]^{-1/(q-1)} = \int d \left( \frac{1}{T_B} \right) f \left( \frac{1}{T_B} \right) \exp \left( - \frac{E - \mu}{T_B} \right)$$
Interpretation of Tsallis Parameter $q$


\[ \left\langle \frac{1}{T_B} \right\rangle = \frac{1}{T} \]

and also

\[ \frac{\left\langle \left( \frac{1}{T_B} \right)^2 \right\rangle - \left( \frac{1}{T_B} \right)^2}{\left\langle \frac{1}{T_B} \right\rangle^2} = q - 1 \]
Tsallis works well in p-p collisions. However, it does not describe Pb-Pb collisions as well as for p-Pb collisions. Summary of Results: Tsallis distribution fits experimental data better for p-Pb collisions, especially at high transverse momenta. Conclusion: The ALICE collaboration's results confirm the superior performance of Tsallis distribution in p-Pb collisions compared to Pb-Pb collisions.
Tsallis Distribution p-p

Transverse momentum spectra of $K_s^0$ in pp collisions at $\sqrt{s} = 900$ GeV

M.D. Azmi
Tsallis Distribution p-p

Transverse momentum spectra of $\Lambda$ in pp collisions at $\sqrt{s} = 900$ GeV

M.D. Azmi
Tsallis Distribution p-p

Transverse momentum spectra of $\phi$ in pp collisions at $\sqrt{s} = 900$ GeV

M.D. Azmi
Tsallis Distribution p-p

Transverse momentum spectra of $\Xi^- + \Xi^+$ in pp collisions at $\sqrt{s} = 900$ GeV

M.D. Azmi
Tsallis Distribution p-p

Transverse momentum spectra of $\pi^-$ in pp collisions at $\sqrt{s} = 7$ TeV

CMS, pp @ $\sqrt{s} = 7$ TeV
$|y| < 1$

M. Danish Azmi
Tsallis Distribution p-p

Transverse momentum spectra of proton from pp collisions at $\sqrt{s} = 7$ TeV

$1/N_{ev} \frac{d^2N}{dy dp_T}$

CMS, pp @ $\sqrt{s} = 7$ TeV
$|y| < 1$

M. Danish Azmi
Tsallis Distribution p-p

Transverse momentum spectra of $\pi^-$ in pp collisions at $\sqrt{s} = 7$ TeV

CMS, pp @ $\sqrt{s} = 7$ TeV
$|y| < 1$

M. Danish Azmi
Tsallis Distribution $p-p$

Transverse momentum spectra of $K^-$ in $pp$ collisions at $\sqrt{s} = 7$ TeV

$1/N_{ev} (d^2N)/(dy dp_T)$ (GeV/c)$^{-1}$

CMS, $pp @ \sqrt{s} = 7$ TeV

$|y| < 1$

M. Danish Azmi
Tsallis works well in p-p collisions but does not describe Pb-Pb collisions. However, it works for p-Pb collisions.

Summary of Results

Conclusion
Tsallis Distribution does not describe Pb-Pb

M. Danish Azmi
... but works for p-Pb

Transverse momentum distribution of charged particles in NSD p-Pb collisions

q = 1.140  M. Danish Azmi
... but works for p-Pb at all rapidities

Transverse momentum distribution of charged particles in NSD p-Pb collisions

$\frac{1}{N_{\text{evt}}} \frac{1}{(2\pi p_T)} (d^2N_{\text{ch}})(dp_T d\eta)$

$\eta < 0.8, p_T > 0.3$ ALICE, p-Pb $\sqrt{s} = 5.02$ TeV

$q = 1.139$ M. Danish Azmi M. Danish Azmi
... but works for p-Pb at all rapidities.

Transverse momentum distribution of charged particles in NSD p-Pb collisions

\[ \frac{1}{N_{\text{evt}}} \frac{1}{(2\pi)p_T} \left( \frac{d^2N_{ch}}{dp_T d\eta} \right) \text{ (GeV/c)}^2 \]

ALICE, p-Pb
\[ \sqrt{s} = 5.02 \text{ TeV} \]

\[ q = 1.139 \]

M. Danish Azmi
Tsallis works well in p-p collisions but does NOT describe Pb-Pb collisions. It does well for p-Pb collisions.

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<th>q, ALICE</th>
<th>q, UA1</th>
<th>q, ATLAS</th>
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<tr>
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</tbody>
</table>

The Tsallis parameter q varies with energy for different experiments. The comparison between these parameter values is shown in the graph.
History Boltzmann Tsallis Comparison Boltzmann vs. Tsallis Transverse Momentum Distributions Tsallis works well in p-p collisions Tsallis does NOT describe Pb-Pb collisions but works for p-Pb collisions Summary of Results Conclusion

![Graph showing Tsallis Temperature (MeV) vs. 
\( \sqrt{s} \) (TeV) for CMS, ALICE, ATLAS, and UA1.](image)
Open symbols effective temperature $T$ obtained using ALICE distribution. Closed symbols use Tsallis distribution.

History Boltzmann Tsallis Comparison Boltzmann vs. Tsallis Transverse Momentum Distributions Tsallis works well in p-p collisions Tsallis does NOT describe Pb-Pb collisions but works for p-Pb collisions

Summary of Results Conclusion

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Graph showing the comparison between Boltzmann and Tsallis distributions for transverse momentum in p-Pb collisions.}
\end{figure}
Tsallis works well in p-p collisions but does NOT describe Pb-Pb collisions, whereas it works for p-Pb collisions.

J. C., G. Lykasov, A. Parvan, O. Teryaev, A. Sorin, D.S. Worku
PLB (2013)
Comparison Boltzmann vs. Tsallis

Transverse Momentum Distributions

Tsallis works well in p-p collisions
Tsallis does NOT describe Pb-Pb collisions
but works for p-Pb collisions

Summary of Results

Conclusion

Charged Hadrons
UA1 and ATLAS

\[ \frac{dN}{2\pi p_T dp_T dy} \]

\[ p_T \text{(GeV)} \]

\[ \begin{array}{c}
14 \text{ TeV} \\
7 \text{ TeV} \\
2.36 \text{ TeV} \\
0.9 \text{ TeV} \\
0.54 \text{ TeV}
\end{array} \]

J.C., G.I. Lykasov, A.S. Sorin, O.V. Teryaev, A.S. Pravan, D. Worku
Conclusion:

Use

\[
\frac{d^2N}{dp_T dy} = gV \frac{p_T m_T}{(2\pi)^2} \left[ 1 + (q - 1) \frac{m_T}{T} \right]^{-\frac{q}{q-1}}, \tag{4}
\]

instead of

\[
\frac{d^2N}{dp_T dy} = p_T \times \frac{dN}{dy} \frac{(n-1)(n-2)}{nT(nT + m_0(n-2))} \left[ 1 + \frac{m_T - m_0}{nT} \right]^{-n} \tag{5}
\]