Coarse Graining in Hydrodynamics and Effects of Fluctuations

T. Kodama and T. Koide

Abstract. We address the physical meaning of coarse graining scale and the hidden statistical ensemble in event by event analysis in hydrodynamic approach to heavy-ion collisions. It is shown that these features are appropriately understood in the frame work of variational formulation and pointed out that the local thermal equilibrium does not necessarily play a critical role in the description of the collective flow patterns. We further discuss that the effect of viscosity is also formlated in the form of the variational method including fluctuations.

Instituto de Física, Universidade Federal do Rio de Janeiro, C.P. 68528, 21941-972, Rio de Janeiro, Brazil

E-mail: `tkodama@if.ufrj.br, koide@if.ufrj.br`

1. Introduction

Hydrodynamic approach on event-by-event (EbE) basis has been applied successfully to describe the global and collective features of the data from relativistic heavy-ion collisions, particularly to the behavior of the Fourier components of flow pattern $\{v_n\}$ as a function of centrality and transverse momenta data [1], leading to a general expectation that we may determine the equation of state (EoS) and transport coefficients in addition to the information on initial collision dynamics. On the other hand, these successes brought us several new interesting questions and mysteries. The most crucial one is why at all the hydrodynamic approaches work so well for such violent and almost microscopic collisional processes. It is commonly believed that the fundamental hypothesis for the validity of hydrodynamics is the local thermal equilibrium (LTE). In this vision, we would conclude that the thermalization time and correlation length should be extremely small. For AA collisions this could still be acceptable due to the large number of partons involved, but surprisingly, the recent ALICE experiment reports a similar collective flow pattern in the pA data, too [2]. This casts a very serious question for the proper physical meaning of hydrodynamic description in pA collisions or, even in AA collisions [3]. In this report, we address this question and point out that hydrodynamic behavior for a finite set of observables does not necessarily mean the validity of LTE on real EbE basis, but works effectively in statistical basis for the ensemble of events specified by these observables.

2. Relativistic Hydrodynamics and Role of Coarse Graining

Let us denote the conserved four-current density by $n^\mu(x)$, satisfying the continuity equation, $\partial_\mu n^\mu(x) = 0$. The energy-momentum tensor $T^{\mu\nu}(x)$ of the system also conserves, $\partial_\mu T^{\mu\nu}(x) = 0$. Although these 4 equations are not enough to determine the time evolution of these quantities, in some special physical situations they become sufficient since the total number of variables drastically reduces. Suppose that, in the Landau frame (the energy flow rest frame), the

spatial part of $T^{\mu\nu}(x)$ becomes isotropic for any x. In addition, if there is no matter diffusion in this frame and there exists a functional relation among the local quantities, ε, P and n, as $P = P(\varepsilon, n)$, then we obtain a set of closed dynamical equations for $n^{\mu}(x)$ and $T^{\mu\nu}(x)$, constituting the ideal hydrodynamics. As was mentioned, these conditions can be realized simply by LTE so that usually the success of (almost) ideal hydrodynamic description of heavy-ion collisions means the realization of LTE. However, as we will discuss here, LTE is a sufficient but not a necessary condition for the hydrodynamic picture on EbE basis, especially for the description of a limited set of observables.

Although hydrodynamic equation has a form of local classical field theory, its variables such as n and ε are defined as averages in a certain finite volume which we refer to as *fluid element*. In this sense, the assumption that EoS $P = P(\varepsilon, n)$ should be satisfied strictly at a local point is physically meaningless in a finite-size system. For the hydrodynamic description of relativistic heavy-ion collisions, the locality means that size of the typical fluid element should be sufficiently small compared to that of the whole system (i.e., at least less than half fm), but at the same time it cannot be taken too much small, otherwise the number of degrees of freedom contained in the fluid element would not be enough to define thermodynamical quantities. Similarly, the time scale of the hydrodynamic motion should be much larger than the microscopic one so that LTE can be attained, but it should be much smaller than the time-scale of the expansion of the system, which is already very small due to the rapid expansion of the system. Therefore, for an allowed size of fluid element for the description of heavy-ion collisions, there exist large fluctuations and inhomogeneity in terms of the microscopic configurations, leading to a large deviation from the thermodynamic limit necessary to define the EoS.

As an example, let us consider a classical microscopic system which contains a large number of quickly moving point-like particles. Then, the density n_0^* is a sum of the Dirac delta functions. However, we usually do not require a very precise resolution both in space and in time to describe the collective flow behaviors. Thus we introduce an averaged smooth density distribution $\tilde{n}^*(\vec{x}, t)$ from $n_0^*(x)$ as $\tilde{n}^*(\vec{x}, t) = \int dt' \int d^3\vec{X} \, n_0^*\left(\vec{X}\right) \, U_\tau(t' - t) \, W_h\left(\left|\vec{x} - \vec{r}\left(t'; \vec{X}\right)\right|\right)$ using a 4 dimensional smoothing kernel [4] $W(x) = U_\tau(t) \times W_h(\vec{x})$. Typically U and W are given by the Gaussian distributions with, respectively, width τ and h, which characterize the scales of the time and space resolutions. Similarly, the smoothed spatial current vector $\tilde{\mathbf{j}}(\vec{x}, t)$ can be defined, satisfying the continuity equation, $\partial \tilde{n}^*/\partial t + \nabla_x \cdot \tilde{\mathbf{j}} = 0$. Using these current and density, the four-current, $\tilde{j}^{\mu} = \left(\tilde{n}^*, \tilde{\mathbf{j}}\right)$ and the proper density, $\tilde{n} = \sqrt{\tilde{j}^{\nu}\tilde{j}_{\nu}}$ can be composed. The smoothed four-velocity field is then defined as $\tilde{u}^{\mu} = \tilde{j}^{\mu}/n$.

On the other hand, the smoothed energy-momentum tensor $\tilde{T}^{\mu\nu}$ can be introduced in analogous way as the convolution of the original $T^{\mu\nu}$ using the same smoothing kernel. Such an energy-momentum tensor again satisfies the continuity equation, $\partial_\mu \tilde{T}^{\mu\nu} = 0$. From this smoothed energy-momentum tensor, we can calculate the smoothed proper energy density as $\tilde{\varepsilon} \equiv \tilde{u}_\mu \tilde{u}_\nu \tilde{T}^{\mu\nu}$. The energy density defined in this way is an average of the energy density observed in the rest frame of the *matter* flow. The average is taken over all contributions within the range of the coarse-graining scale in space-time. In the following, we take these smoothed quantities as dynamical variables to represent this system [4].

When the above procedure is applied to one collision event which is characterized by a microscopic state, then we can construct the corresponding hydrodynamic description. However, it is obvious that there exist many different microscopic configurations which give the same hydrodynamic response. Let us prepare the set of collision events described by microscopic configurations which gives a specified four-current $\tilde{j}^{\mu}$ at the initial time t_0, and call this set Ω. If we calculate $\tilde{\varepsilon}$ at a space-time point x for each event in Ω, the value of $\tilde{\varepsilon}$ is not the same in general.

However, if the coarse-graining size is increased, the number of independent microscopic

configurations in Ω may become sufficiently large in a way that $\tilde{\varepsilon}$ and $\tilde{n}$ distribute sharply around their mean-values, $\tilde{\varepsilon}$ and $\tilde{\mathbf{n}}$, respectively, as a consequence of the central limit theorem. If this happens, since $\tilde{\varepsilon}$ and $\tilde{\mathbf{n}}$ are the averaged energy and matter densities belonging to the same fluid element, it is possible that they are strongly correlated so that $\tilde{\varepsilon}$ can be expressed as a function of $\tilde{\mathbf{n}}$, $\tilde{\varepsilon} = \tilde{\varepsilon}(\tilde{\mathbf{n}})$ (barotropic fluid).

Suppose that the fluctuations in $\tilde{\varepsilon}$ and $\tilde{j}^\mu$ are not important in the way that the system is characterized basically by the densities $\tilde{\mathbf{n}}$ and $\tilde{\varepsilon}$. In such a case, we expect that the most promising dynamics will be determined by the optimization of the model action,

$$I = -\int d^4x \, \tilde{\varepsilon} \left(\frac{1}{\gamma} \tilde{\mathbf{n}}^* \right), \tag{1}$$

where $\tilde{\mathbf{n}}^* \equiv \gamma \tilde{\mathbf{n}}$ denotes the mean matter density observed in the reference frame and γ is the Lorentz factor. This variational procedure leads to the ideal hydrodynamics. In other words, the hydrodynamic model can be considered as the optimized dynamics of the coarse-grained system (of course only the largeness of coarse-graining is not a sufficient condition to realize such situations). See Ref. [5] for details.

This is a mere single example to show that the collective behavior does not necessarily mean LTE in an single event. In realistic situations, we further extend the ensemble Ω in a way that Ω is a whole set of events which have the same final state observables, such as collective flow parameters. Such Ω might be a huge statistical ensemble of many different events. That is, we have to have in mind that the final state observables are still far from exclusive even for the collective behaviors of the system to conclude that the realization of LTE is attained in a real single event. For the effect of the fluctuations in $\tilde{\varepsilon}$ and $\tilde{j}^\mu$, see Sec. 4.

3. Necessity of Real Event by Event Analysis

As shown above, the hydrodynamic description in heavy-ion collisions reduces to a coarse-grained dynamics obtained by the optimization of the model action (1) under the assumption of existence of an effective EoS, $\tilde{\varepsilon} = \tilde{\varepsilon}(\tilde{\mathbf{n}})$. Therefore, the success of the ideal hydrodynamic modeling of relativistic heavy-ion collisions depends on the consistent choice of this EoS and the model action. These two conditions will be satisfied for a broader range of microscopic configurations than those required by the real "local thermal equilibrium" for each event.

On the other hand, the size of Ω depends on the coarse-graining scale and final state observables. For larger Ω, the two conditions are more likely to be satisfied. We however loose the better resolution in the space-time recognition for larger Ω. In fact, we cannot observe inhomogeneities with smaller wavelength than the coarse-graining scale. This affects directly the class of observables that the model can describe. Even though some observables might be insensitive to inhomogeneities in each event. As an extreme example, we take the situation where the coarse-graining size is larger than the system size and the total time evolution. Then the ensemble Ω can be regarded as the statistical ensemble of the whole system itself, and the resultant system reduces simply to the so-called fire-ball. The thermal model for particle ratio can be considered in this category.

In general, the less the resolution is, the larger Ω becomes. Thus for some observables which do not require a precise space-time resolution, the effective hydrodynamic description for the statistical ensemble Ω will be sufficient for the understanding of the physics of these observables, without implying LTE on EbE basis. As a matter of fact, the experimental observables are usually averaged over collision events classified in terms of their initial configurations rather loosely defined, such as centrality, event plane, etc. In other words, the present collective flow data are still of inclusive nature. In order to claim that the true hydrodynamics with LTE is valid, we need to have observables that reflect the genuine hydrodynamic profile in EbE basis. For example, the remnant of a sharp shock wave propagation, if exists, would be a good

evidence and it also tells the possible limit size of coarse-graining for the collective flow. The shock thickness should not be larger than the coarse-graining scale of the collective flow.

The key point is that when we apply the hydrodynamic modeling, we do not know a priori the suitable coarse-graining scale in the real scenario. This puts a certain limitation in extracting the meaningful information of the initial condition from the hydrodynamic analysis. For this purpose, it is essential to find out the set of observables which carry the information on the inhomogeneities of the initial conditions on EbE basis. The flow parameters $\{v_n\}$, often called "event-by-event" analysis, in the sense that correlations among different observables measured for each event in coincidence, but there still exists a huge statistical ensemble which gives the same observed correlation as we have discussed. For example, the cumulant method to determine the flow parameters eliminates the information of event plane. In the recent paper, it is pointed out that event plane may differ in low and high p_T domain [4], according to the coarse-graining scale. If it can be experimentally measured, it would furnish some information on coarse graining scale in heavy-ion collisions.

4. Fluctuation of Fluid Variables and Stochastic Variational Method

Within the vision that hydrodynamic evolution is an effective dynamics for coarse-grained variables of the energy-momentum tensor, each real collision event is an element of the statistical ensemble Ω and does not obey a unique time evolution equation due to the difference in the microscopic degrees of freedom to which our macroscopic hydrodynamic variables are blind. When the fluctuation of events in Ω is large, they should be taken into account in the determination of dynamics of coarse-grained hydrodynamic variables. As was discussed so far, the variational approach is an optimized method for the formulation of the coarse-grained dynamics. Then the variation procedure in Eq. (1) should be modified so as to include the effect of the fluctuation which was ignored in Sec. 3. The stochastic variational method (SVM) is known as an appropriate approach for such situations [6].

In order to formulate the variational approach involving stochastic processes, we have to introduce two stochastic differential equations (SDE), one for the forward direction in time (FSDE), and the other, backward in time (BSDE) which describes the time reversed process of FSDE. These two SDEs are necessary to accommodate the fixed initial and final boundary conditions in the variational procedure, but they cannot be completely independent. For further discussion on this point, see [6]. Such stochastic processes are known as Bernstein process. For this process, the two Fokker-Planck equations for FSDE and BSDE should be equivalent. This leads to the consistency condition, $\mathbf{u} = \tilde{\mathbf{u}} + 2\nu\nabla\ln\rho$, where $\mathbf{u}$ and $\tilde{\mathbf{u}}$ are the velocity fields for FSDE and BSDE, respectively. Here ρ is the particle density which is given by the solution of the Fokker-Planck equation and ν is a parameter representing the intensity of the noise.

The purpose of SVM is to determine these velocity fields $\mathbf{u}$ and $\tilde{\mathbf{u}}$ from an action through the variation principle introducing noises, starting from the classical (non-dissipative) Lagrangian. For a non-relativistic fluid, the most natural form of the stochastic Lagrangian density is given by [6] $\mathcal{L} = 1/2\,\rho^m\left[(1/2+\alpha_2)\left\{(1/2+\alpha_1)\mathbf{u}^2 + (1/2-\alpha_1)\tilde{\mathbf{u}}^2\right\} + (1/2-\alpha_2)\tilde{\mathbf{u}}\cdot\mathbf{u}\right] - \varepsilon$, here α_1 and α_2 is arbitrary constants and ρ^m is the mass density. The corresponding action is an average over the whole SDE solutions, and the variation is taken with respect to these stochastic trajectories to determine the unknown fields, $\mathbf{u}$ and $\tilde{\mathbf{u}}$. The dynamical equation obtained from the stochastic variation is found to be

$$\rho^m\left(\partial_t + \mathbf{u}_m\cdot\nabla\right)u_m^i - \sum_j \partial_j(\mu\rho_m e_{ij}^m) - 2\kappa\rho^m\sum_j \partial_i(\sqrt{\rho^m}^{-1}\partial_j^2\sqrt{\rho^m}) = -\nabla P,$$

where $\eta = \alpha_1(1+2\alpha_2)\nu\rho^m$, $\kappa = 2\alpha_2\nu^2$ and $e_{ij}^m = \partial_j u_m^i + \partial_i u_m^j$. The pressure P is defined by $(\rho_m)^2 d(\varepsilon/\rho_m)/d\rho_m$. One can see easily that the second term on the left hand side corresponds

to the viscous term (containing the gradient of velocity field) and this equation is reduced to the Navier-Stokes-Fourier equation when we set $\alpha_2 = 0$. That is, the fluctuation effects which were ignored in Sec. 3 induces the effects of viscosity in accordance with the fluctuation-dissipation theorem (more precisely, the variation of entropy should be taken into account to get the second coefficient of viscosity). For other results of different values of $\alpha's$, see Ref. [6].

5. Concluding remarks

In this work, we addressed the physical meaning of coarse-graining scale and the hidden statistical ensemble in EbE analysis in hydrodynamic approach to heavy-ion collisions. We introduced explicitly the coarse-graining procedure for the hydrodynamic modeling together with its variational formulation. In this picture, the collective flow patterns can be reproduced without requiring the LTE in a strict sense for event by event. That is, the hydrodynamic behavior observed in relativistic heavy-ion collisions does not necessarily imply the realization of LTE for a one collisional event. Furthermore, we call attention that the validity of EoS does not mean the thermal equilibrium as has already been pointed out before (see for example, Ref. [7] and also the recent work on the dynamical isotropization of pressure in the Color Glass Condensate scheme [8]). We further discussed possible signals for coarse-graining scale and genuine hydrodynamic behaviors on EbE basis. For example, the remnant of a sharp shock wave propagation would be a good observable which tells the possible coarse-graining size of the collective flow. Another example is to determine the event plane for different transverse momentum domain. Finally we showed that the coarse-graining is intimately related to the origin of viscosity and this effect can be formulated in the variational method extending dynamical variables to stochastic domain. In order to quantify the questions raised here, it will be useful to perform the analysis of coarse-graining described in this work for a certain microscopic model which gives complete dynamical evolution of the energy-momentum tensor, such as PHSD [9].

References

[1] Paiva S et al 1997 Phys. Rev. C **55** 1455
    Aguiar C E et al 2002 Nucl. Phys. A **698** 639
    Andrade R et al 2006 Phys. Rev. Lett. **97** 202302
    Andrade R et al 2007 Braz. J. Phys. **37** 717
    Takahashi J et al 2009 Phys. Rev. Lett. **103** 242301
    Wang D J et al 2012 arXiv:1205.4934
    Qiu Z et al 2012 Phys. Lett. B **707** 151
    Alver B and Roland G 2010 Phys. Rev. C **81** 054905
    Teaney D and Yan Li 2011 Phys. Rev. C **83** 064904
    Schenke B et al 2012 Phys. Rev. Lett. **108** 252301
    Dumitru A and Nara Y 2012 Phys. Rev. C **85** 034907
[2] ALICE Collaboration, arXiv:1307.3237
[3] Gyulassy M talk in 9th International Workshop on Relativistic Aspects of Nuclear Physics, Rio de Janeiro, October, 2013
[4] Mota Ph et al 2012 Eur. Phys. J. A **48** 165
[5] Elze H-T et al 1999 J. Phys. G **25** 1935
[6] See for example Koide T and Kodama T 2012 J. Phys. A**45** 255204 and references therein
[7] Kodama T et al 2007 Int. J. Mod. Phys. E**16** 763.
[8] Epelbaum T and Gelis F 2013 Phys. Rev. Lett. **111** 232301
[9] See for example, Bratkovskaya E L et al 2011 Nucl.Phys. A**856** 162 and references therein