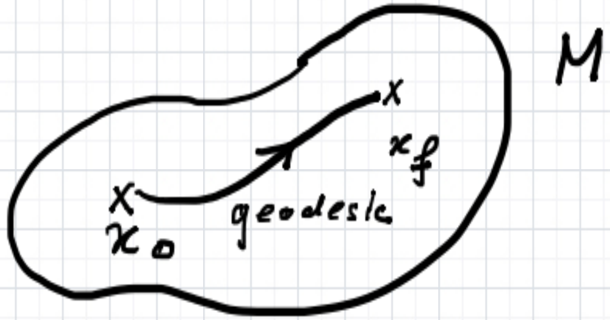
Lecture #2 Quantum Mechanics

Question



What is the probability, given we measure the particle at time t_0 at x_0, that we will observe it at x_f at time t_f?

Must "quantize" classical system $\Rightarrow$ use Hamiltonian approach.

$$S[\gamma] = \int \frac{1}{2}\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu \cong S[\Gamma] = \int \left[p_\mu \dot{x}^\mu - \frac{1}{2} p_\mu g^{\mu\nu}(x) p_\nu \right]$$

path in M — path in T^*M — symplectic current — Hamiltonian H

Observe

$$\frac{\delta S}{\delta p_\mu} = 0 \Rightarrow \dot{x}^\mu = \frac{\partial H}{\partial p_\mu} \Rightarrow \dot{x}^\mu = g^{\mu\nu} p_\nu$$

$$\frac{\delta S}{\delta x^\mu} = 0 \Rightarrow \dot{p}_\mu = -\frac{\partial H}{\partial x^\mu} \Rightarrow \dot{p}_\mu = -\frac{1}{2}\partial_\mu g^{\alpha\beta} p_\alpha p_\beta$$

"Hamilton's Equations"

$\Rightarrow$

$$p_\mu = g_{\mu\nu}\dot{x}^\nu$$

canonical momentum

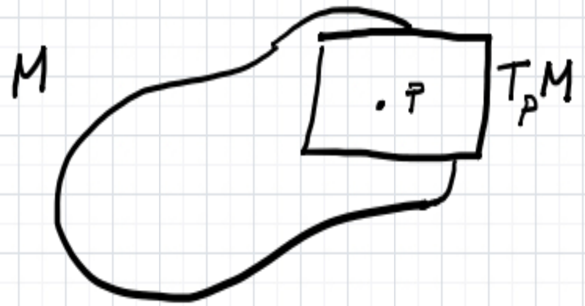
$$\frac{\nabla p_\mu}{dt} = \dot{p}_\mu - \Gamma^\rho_{\mu\nu}\dot{x}^\nu p_\rho = 0$$

geodesic equation

Symplectic Manifolds & Classical Mechanics

A symplectic manifold (Z, ω) consists of an even dimensional manifold (phase space) and a closed, non-denerate 2-form ω.

EX T^*M the cotangent bundle



space of tangent vectors at P, $v^\mu \frac{\partial}{\partial x^\mu}$ (column vectors v^μ)

T_p^*M = space of cotangent vectors at P, $p_\mu dx^\mu$ (row vectors ω_μ) (alias linear maps $T_pM \longrightarrow \mathbb{R}$)

Each point in T^*M is equipped with a tautological 1-form $\theta = p_\mu dx^\mu$.

$$\Rightarrow \underset{\text{symplectic form}}{\omega} = \underset{\text{symplectic current}}{d\theta} = dp_\mu \wedge dx^\mu$$

(has components $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ invertible, coordinates for T^*M are $Z^A = (p_\mu, x^\mu)$)

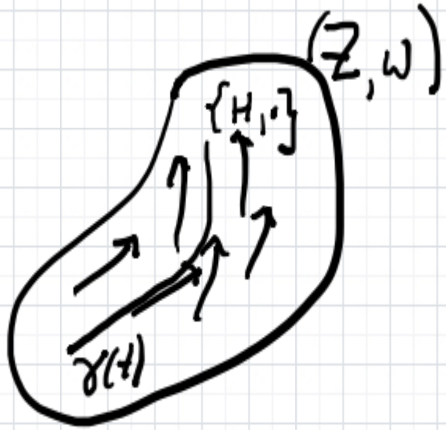
Poisson Brackets B/c $\omega = \omega_{AB}\, dZ^A \wedge dZ^B$ we can invert ω_{AB} to obtain the Poisson bi-vector Ω^{AB}. Then given functions $f(Z), g(Z)$

$$\{f, g\}_{PB} = \partial_A f\, \Omega^{AB}\, \partial_B g = -\{g, f\}_{PB}$$

Dynamics Given a special choice $H(Z)$ (perhaps based on an experiment!), the operator on functions

$$\{H, \cdot\} = \frac{\partial H}{\partial Z^A} \Omega^{AB} \frac{\partial}{\partial Z^B}$$

defines a vector field



Dynamics are integral flows $\gamma(t)$ along which

$$\{H, \cdot\} = \frac{d}{dt} \qquad \text{i.e.} \qquad \{H, \gamma\} = \dot{\gamma}$$

EX $H = \frac{1}{2} p_\mu g^{\mu\nu} p_\nu$, $Z^A = (p_\mu, x^\nu)$, $\omega = d(p_\mu dx^\mu) \Rightarrow \{p_\mu, x^\nu\} = \delta^\nu_\mu$

As an EXERCISE check

$$\begin{cases} \dot{x}^\mu = \{H, x^\mu\} \\ \dot{p}_\mu = \{H, p_\mu\} \end{cases}$$

gives the Hamiltonian version of the geodesic equation.

Quantization Z^A become linear operators $\hat{Z}^A$ on a **Hilbert space**.

$$\{Z^A, Z^B\}_{PB} = \Omega^{AB}(Z) \longmapsto [\hat{Z}^A, \hat{Z}^B] = \frac{1}{i\hbar}\,\Omega^{AB}(\hat{Z})$$

NB Potential ordering ambiguities in $\Omega^{AB}(\hat{Z})$ can be avoided thanks to the Darboux theorem — symplectic geometry have **no local invariants** à la the Riemann tensor so can always choose coordinates where $\Omega^{AB} = \left(\begin{array}{c|c} & \mathbb{1} \\ \hline -\mathbb{1} & \end{array}\right)$ (splitting into "positions & momenta" or "**choice of polarization**").

Then $\{H, \cdot\} = \frac{d}{dt} \longmapsto [H, \cdot] = \frac{1}{i\hbar}\frac{d}{dt}$ Schrödinger Equation.

If $H = H^\dagger$, $\exp iHt$ is unitary & generates time evolution. The **probability** of observing $|\psi\rangle$ having prepared a state $|\varphi\rangle$ a time t in the past is $\left|\langle\psi| e^{iHt} |\varphi\rangle\right|^2$.

Example $\quad H = \frac{1}{2} p_\mu g^{\mu\nu} p_\nu \qquad$ For Hilbert space $\mathcal{H}$ take wave functions $\psi(x)$

Then $\quad Z_A = (p_\mu, x^\nu) \longmapsto \left(\frac{1}{i} \frac{\partial}{\partial x^\mu}, x^\nu \right).$

Now we do have an ordering problem b/c $g^{\mu\nu} = g^{\mu\nu}(x)$.

Ultimately these must be resolved by experiments, but a good tool for theorists is to use symmetries. Here we have

$$H \longmapsto \hat{H} = -\frac{1}{2} \frac{\partial}{\partial x^\mu} g^{\mu\nu}(x) \frac{\partial}{\partial x^\nu} + \text{reordering terms}$$

this is (almost) the Laplace operator so require covariance:

EXERCISE $\quad$ Verify

$$\underbrace{\nabla^\mu \partial_\mu}_{\Delta} \psi(x) = g^{\mu\nu} (\partial_\mu \partial_\nu - \Gamma^\rho_{\mu\nu} \partial_\rho) \psi(x)$$

$$= \frac{1}{\sqrt{g}} \partial_\mu g^{\mu\nu} \sqrt{g} \partial_\nu \psi(x)$$

THIS IMPLIES THE ORDERING $\quad -\frac{1}{\sqrt{g(\hat{x})}} p_\mu g^{\mu\nu}(\hat{x}) \sqrt{g(\hat{x})} p_\nu$

Further symmetries? We have $-2\hat{H} = \Delta$ scalar Laplacian

Ward operators commuting with Δ.

Remark: We can view this as a QM problem or a geometry problem.
Either perspective is fine.

Geometry solution

Ansatz Use the Lie derivative $\mathcal{L}_\xi$. On scalars, the Lie derivative along a vector field ξ^μ is simply

$$\mathcal{L}_\xi \psi = \xi^\mu \partial_\mu \psi$$

We must now compute $[\mathcal{L}_\xi, \Delta]$ on scalars.

$$\xi^\rho \partial_\rho \nabla^\mu \partial_\mu \psi - \nabla^\mu \partial_\mu (\xi^\rho \partial_\rho \psi)$$

(use $\partial_\mu (V_\nu W^\nu) = (\nabla_\mu V_\nu) W^\nu + V_\nu \nabla_\mu W^\nu$)
<u>Leibnitz</u>

$$= \mathcal{L}_\xi \Delta \psi - \nabla^\mu \big((\nabla_\mu \xi^\rho) \partial_\rho \psi + \xi^\rho \nabla_\mu \partial_\rho \psi \big)$$

$$= \cancel{\mathcal{L}_\xi \Delta \psi} \underbrace{- (\Delta \xi^\rho) \partial_\rho \psi}_{\substack{\nabla_\mu (\nabla^\mu \xi^\rho + \nabla^\rho \xi^\mu) \partial_\rho \psi \\ - (\nabla^\rho \nabla \cdot \xi + \cancel{R_\mu{}^{\rho\mu}{}_\sigma \xi^\sigma}) \partial_\rho \psi}} - 2 (\nabla^\mu \xi^\rho) \nabla_\mu \partial_\rho \psi \underbrace{- \xi^\rho \Delta \partial_\rho \psi}_{- \cancel{\mathcal{L}_\xi \Delta \psi} - \cancel{\xi^\rho R^\mu{}_{\rho\mu}{}^\sigma \partial_\sigma \psi}}$$

$$= -(\nabla^\mu \xi^\rho + \nabla^\rho \xi^\mu) \nabla_\mu \partial_\rho \psi - (\nabla_\mu [\nabla^\mu \xi^\rho + \nabla^\rho \xi^\mu]) \partial_\rho \psi - (\nabla^\rho \nabla \cdot \xi) \partial_\rho \psi$$

$$= - \nabla_\mu \big[(\nabla^\mu \xi^\rho + \nabla^\rho \xi^\mu) \partial_\rho \psi \big] - (\nabla^\rho \nabla \cdot \xi) \partial_\rho \psi$$

Thus if $\boxed{\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0}$ $\Rightarrow \nabla \cdot \xi = 0$ these cancel

and $\mathcal{L}_\xi$ is a <u>symmetry</u>. Vectors obeying this equation are called <u>Killing vectors</u> and correspond to <u>isometries</u>.

Isometries Under infinitesimal diffeomorphisms

$$\delta g_{\mu\nu} = \underbrace{\xi^\rho \partial_\rho g_{\mu\nu}}_{\text{transport term}} + \underbrace{\partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho}}_{\text{orbital term}}$$

$$= \mathcal{L}_\xi g_{\mu\nu} \qquad \text{This is the Lie derivative}$$

$$= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \qquad \text{Using christoffels.}$$

I.e. Isometries are diffeomorphisms leaving the metric inert: $\mathcal{L}_\xi g_{\mu\nu} = 0$

We will also be interested in conformal isometries

$$\mathcal{L}_\xi g_{\mu\nu} = \underset{\text{some function}}{\alpha} g_{\mu\nu}$$

under which the metric only rescales.

Isometries of Minkowski space

$g_{\mu\nu} = \eta_{\mu\nu}$, $ds^2 = -dt^2 + d\vec{x}^2$

(nb denoting Killing vectors by $\xi^\mu \partial_\mu$)

$$\left.\begin{array}{ll} \underline{\text{Rotations}} & x_i \partial_j - x_j \partial_i \\ \underline{\text{Boosts}} & t\,\partial_i + x_i \dfrac{\partial}{\partial t} \end{array}\right\} x_\mu \partial_\nu - x_\nu \partial_\mu = M_{\mu\nu}$$

$$\left.\begin{array}{ll} \underline{\text{Translations}} & \partial_i \\ \underline{\text{Evolution}} & \dfrac{\partial}{\partial t} \end{array}\right\} \frac{\partial}{\partial x^\mu} = P_\mu$$

$$\left.\begin{array}{l} [M_{\mu\nu}, M_{\rho\sigma}] = M_{\mu\sigma}\eta_{\nu\rho} \pm 3\text{ more} \\ \\ [M_{\mu\nu}, P_\rho] = -\eta_{\mu\rho} P_\nu + \eta_{\nu\rho} P_\mu \end{array}\right\}$$

Lie Bracket

Lie algebra of the Poincaré group

$\underbrace{SO(d-1,1)}_{\text{Lorentz}} \ltimes \underbrace{\mathbb{R}^d}_{\text{Translations}}$

Including dilations $x^\mu \partial_\mu$ & conformal boosts $(d+2x\cdot\partial-2)\partial_\mu - x_\mu \Delta$

this algebra becomes $SO(d,2)$ the <u>conformal group</u>