Lecture #1 GEOMETRY & FIELD THEORY

## Single particle mechanics

EX $\ddot{x} = f(x)$ $\quad (F = ma)$

Crucial tool ACTIONS

Step 1 Integrate the force $f(x)$ to a potential

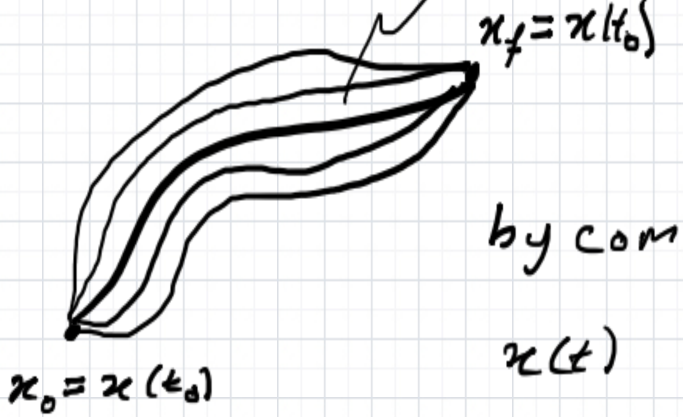
$$f(x) = -V'(x)$$

Step 2 Variational principle $L = T - V = \frac{1}{2}\dot{x}^2 - V(x)$

$$S[x] = \int_{t_0}^{t_F} \left(\frac{1}{2}\dot{x}^2 - V(x)\right)$$

functional

actual trajectory extremizes $S[x]$




Extremize

by comparing $S$ along paths $x(t)$

Varying $S[x+\delta x] - S[x]$

$$= \int\left[\tfrac{1}{2}(\dot{x}+\dot{\delta x})^2 - V(x+\delta x)\right] - \int\left[\tfrac{1}{2}\dot{x}^2 - V(x)\right]$$

Drop $O(\delta x^2)$

$$\approx \int\Big[\dot{x}\,\dot{\delta x} - \underbrace{\big(V(x+\delta x) - V(x)\big)}_{\delta x\, V'(x)}\Big]$$

integrate by parts to isolate $\delta x$

Assume $\delta x$ vanishes at $t_0$ & $t_f$

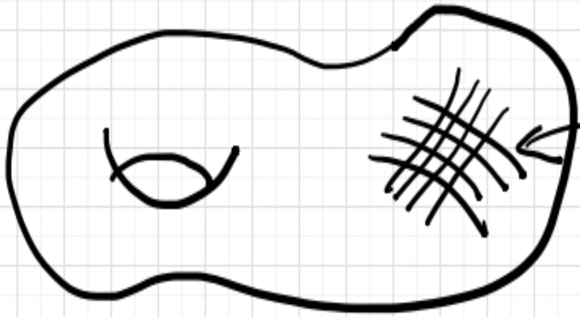
$$= -\int \delta x\left[\ddot{x} + V'(x)\right] =: \delta S \qquad \text{vanishes when } \ddot{x} = -V'(x) = f(x)$$

ARBITRARY

Remarks
- "Never" assume field equations before varying.
- Arbitrary variations encode EoMs, special choices of variations encode symmetries.

EXAMPLE Particle on a Riemannian manifold $(M, g)$

Manifold $M$



space that can be (locally) coordinatized, or better said mapped by gluing together $\mathbb{R}^n$'s.

Riemannian Means $M$ has a notion of lengths / distances encoded by a metric $g$. Given coordinates $(x^1, \ldots, x^n) = x^i$ write

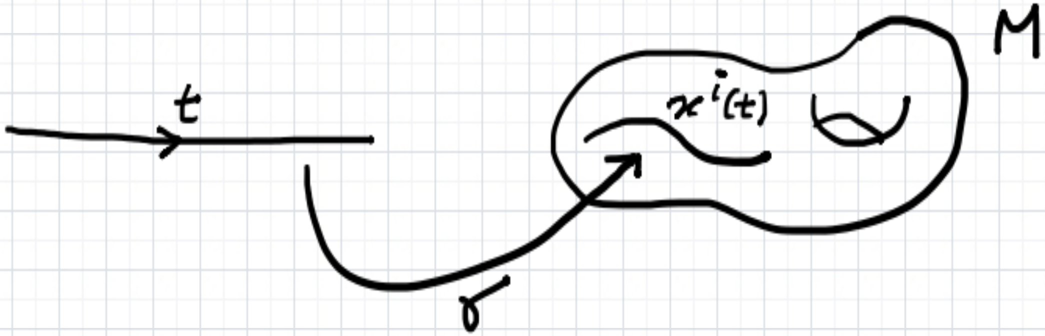
$$ds^2 = \sum_{i,j=1}^{n} dx^i \, g_{ij}(x) \, dx^j \qquad n = \dim M$$

metric tensor

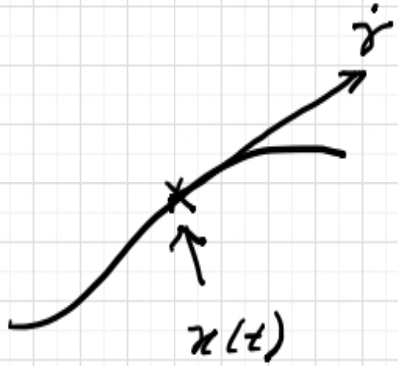
The infinitesimal arc length $ds$ measures distance b/w points with coordinates $x^i$ & $x^i + dx^i$.

**Path** $\gamma$ is a map $\gamma : \mathbb{R} \longrightarrow M$

$\uparrow$ time




The tangent vector to $\gamma$ is $\dot{\gamma} = \dot{x}^i(t)$ (at time $t$)




The arc length$^2$ $ds^2 = \frac{ds^2}{dt^2} dt^2 = \frac{dx^i}{dt} g_{ij}(t) \frac{dx^j}{dt}$.

Lets determine particle motions extremizing $ds^2$:

$$S[\gamma] = \frac{1}{2} \int_\gamma \frac{dx^i}{dt} g_{ij} \frac{dx^j}{dt}$$ "Energy integral"

functional of paths

Remark: It might seem preferable to extremize $ds$. The difference is unparameterized versus parameterized "geodesics".

Varying $S$

$$\delta S = \frac{1}{2}\delta\int \dot{x}^i g_{ij}\dot{x}^j = \int\left(\delta\dot{x}^i g_{ij}\dot{x}^j + \frac{1}{2}\dot{x}^i\delta g_{ij}\dot{x}^j\right)$$

$$= \int \delta x^i \underbrace{\left[-\frac{d}{dt}\left(g_{ij}\dot{x}^j\right) + \frac{1}{2}\dot{x}^j\frac{\partial}{\partial x^i}g_{jk}\dot{x}^k\right]}_{\text{Vanishes for } \delta S = 0}$$

This is the geodesic equation $\frac{d}{dt}\left(g_{ij}\dot{x}^j\right) = +\frac{1}{2}\dot{x}^j\dot{x}^k\partial_i g_{jk}$

FARNARKLING: $g_{ij}\ddot{x}^j + \dot{x}^k\partial_k g_{ij}\dot{x}^j = +\frac{1}{2}\dot{x}^j\dot{x}^k\partial_i g_{jk}$

Multiply by inverse metric $g^{li}g_{ij} = \delta^l_j$ (Kronecker $\delta$) which we use to raise indices

$$\ddot{x}^l + \dot{x}^k\dot{x}^j \underbrace{\frac{1}{2}g^{li}\left(\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk}\right)}_{=:\ \Gamma^l_{kj}\ \text{"Christoffel symbols"}} = 0$$

$$\Rightarrow \boxed{\frac{d}{dt}\dot{x}^l + \Gamma^l_{kj}\dot{x}^k\dot{x}^j = 0}$$

Define the <u>covariant derivative</u> $\frac{\nabla}{dt}$ along $\gamma$

any vector

$$\boxed{\frac{\nabla v^l}{dt} := \dot{v}^l + \Gamma^l_{jk}\,\dot{x}^j v^k}$$

velocity

<u>Then</u> our geodesic equation reads $\frac{\nabla \dot{x}^l}{dt} = 0$

$\Rightarrow$ covariantly constant velocity.

<u>Remarks</u>

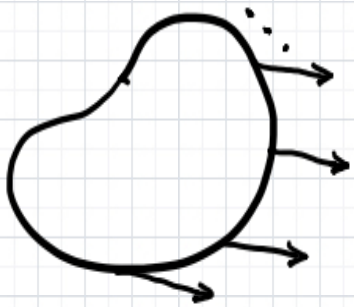
- This equation determines how particles move in <u>curved</u> spaces.
- It also handles changes of coordinates. As an <u>EXERCISE</u>, take polar coordinates in the plane so $ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$ and extremize $S = \frac{1}{2}\int(\dot{r}^2 + r^2\dot{\theta}^2)$.
- We can also define a covariant gradient operator

$$\nabla_i v^j = \partial_i v^j + \Gamma^j_{ik} v^k \Rightarrow \frac{\nabla v^j}{dt} = \dot{x}^i \nabla_i v^j =: \nabla_{\dot{\gamma}} v^j$$

Curvature: How do we decide if a manifold is curved?

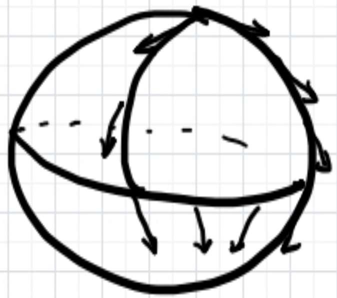
Answer Parallel transport vectors around Loops.

Plane



vs

Sphere



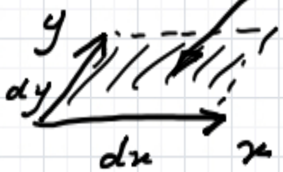
Need to measure how vectors rotate when you parallel transport them around infinitesimal Loops.

* Parallel transport measured by connexions $\nabla$.

Ex $\nabla_i = \partial_i + \Gamma_i^\#$ where $\Gamma_i^\#$ stands for $Gl(n)$ action of Christoffels.

$$\frac{\nabla \dot{x}^i}{dt} = 0$$ says the tangent vector $\dot{x}^i$ is held parallel.

* <u>Infinitesimal loops</u>: use parallelograms




in vector calculus area measured by length of cross product of vectors along sides.
This notion is generalized by 2-forms

$$dx \wedge dy = - dy \wedge dx$$

* <u>Rotations</u>: Infinitesimal rotations generated by anti-symmetric matrices b/c

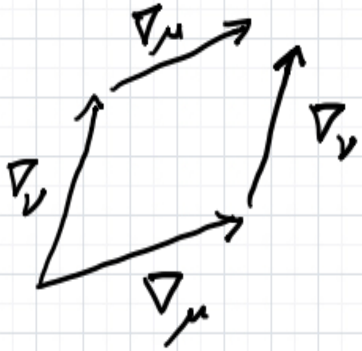
$$\underset{\text{orthogonal matrix}}{O^T O} = \mathbb{1} \quad \& \quad O = e^{\omega}$$

$$\Rightarrow \quad \mathbb{1} + \omega^T + \omega + \mathcal{O}(\omega^2) = \mathbb{1} \Rightarrow \omega = -\omega^T$$

So curvature must be a rotation-valued 2-form:

$$\underbrace{dx^\mu \wedge dy^\nu R_{\mu\nu}}_{\text{loops}} \underbrace{{}^{m}{}_{n}}_{\text{rotations}} \qquad \text{where } R_{\mu\nu mn} = - R_{\mu\nu nm}$$

To compute curvature compare parallel transports




Torsion (vanishes if parallelograms close)

rotation part

$$\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu = [\nabla_\mu, \nabla_\nu] = T_{\mu\nu}{}^{\rho} \nabla_\rho + R_{\mu\nu}{}^{\#}$$

curvature

<u>EXERCISE</u> Using $\nabla_\mu v^\nu = \partial_\mu V^\nu + \Gamma_\mu{}^\nu{}_\rho v^\rho$ & $2\Gamma_{\mu\nu\rho} = \partial_\mu g_{\nu\rho} + \partial_\rho g_{\nu\mu} - \partial_\nu g_{\mu\rho}$

compute

$$[\nabla_\mu, \nabla_\nu] v^\rho .$$

Express your answer as $R_{\mu\nu}{}^{\rho}{}_{\sigma} v^\sigma$ and verify that

$$R_{\mu\nu}{}^{\rho}{}_{\sigma} = \partial_\mu \Gamma_\nu{}^\rho{}_\sigma - \partial_\nu \Gamma_\mu{}^\rho{}_\sigma + \Gamma_\mu{}^\rho{}_\alpha \Gamma_\nu{}^\alpha{}_\sigma - \Gamma_\nu{}^\rho{}_\alpha \Gamma_\mu{}^\alpha{}_\sigma$$