Lecture #4

GRAVITY FROM YANG–MILLS THEORY

Gauge group $SO(3,1) \ltimes \mathbb{R}^4$ = Poincaré

$$\begin{array}{ccc} & \begin{array}{cc} \omega & \end{array} & \\ \text{Lorentz rotation} \rightarrow & \left(\begin{array}{c|c} \Lambda & P \\ \hline 0 & 1 \end{array}\right) & \leftarrow \text{translation} \end{array}$$

Connection $\nabla = d + A$

$$= dx^\mu \nabla_\mu$$

$$A = \begin{array}{c} \begin{array}{cc} 4\times 4 & 4\times 1 \end{array} \\ \left(\begin{array}{c|c} \omega & e \\ \hline & \end{array}\right) \end{array}$$

Curvature $\nabla^2 = \frac{1}{2} dx^\mu \wedge dx^\nu F_{\mu\nu}$

$$= \left(\begin{array}{c|c} d+\omega & e \\ \hline & d \end{array}\right)^2 = \left(\begin{array}{c|c} (d+\omega)^2 & de + \omega e + ed \\ \hline 0 & 0 \end{array}\right)$$

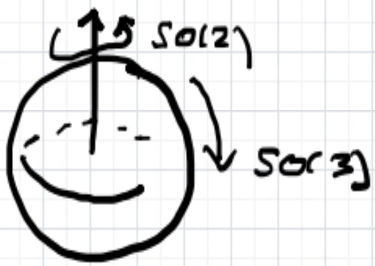
$$= \left(\begin{array}{c|c} (d\omega) + \omega^2 & (de) + \omega e \\ \hline 0 & 0 \end{array}\right)$$

In indices

$$F_{\mu\nu} = \left(\begin{array}{c|c} R_{\mu\nu} & T_{\mu\nu} \\ \hline 0 & 0 \end{array}\right) \Rightarrow \begin{cases} R_{\mu\nu}{}^{m}{}_{n} = \partial_\mu \omega_\nu{}^{m}{}_{n} - \partial_\nu \omega_\mu{}^{m}{}_{n} + 2\omega_\mu{}^{m}{}_{r}\, \omega_\nu{}^{r}{}_{n} \\ T_{\mu\nu}{}^{m} = \partial_\mu e_\nu{}^{m} - \partial_\nu e_\mu{}^{m} + 2\omega_\mu{}^{m}{}_{n}\, e_\nu{}^{n} \end{cases}$$

<u>FLAT STRUCTURE</u> $\quad$ Minkowski $= \dfrac{\text{Poincaré}}{\text{Lorentz}} = \dfrac{\text{Poincaré}}{\text{stab}\begin{pmatrix}0\\1\end{pmatrix}} \rightsquigarrow$ <u>Leave translations</u>

<u>EX</u> $\quad S^2 = \dfrac{SO(3)}{SO(2)} = \dfrac{SO(3)}{\text{stab}(\uparrow)}$



This suggests the <u>constraint</u> $\quad F\begin{pmatrix}0\\1\end{pmatrix} = 0 \Rightarrow T_{\mu\nu} = 0$

But $T_{\mu\nu}{}^{m} = 0$ is exactly the <u>torsion constraint</u> of Palatini formalism where

Metric $\longrightarrow$ $g_{\mu\nu} = e_\mu{}^{m} e_{\nu m} \Rightarrow R_{\mu\nu}{}^{m}{}_{n}$ = Riemann Tensor

and $\nabla_\mu = \partial_\mu + \omega_\mu{}^{\#}$ becomes the <u>Levi-Civita connexion</u>

CONFORMAL GEOMETRY

The flat model

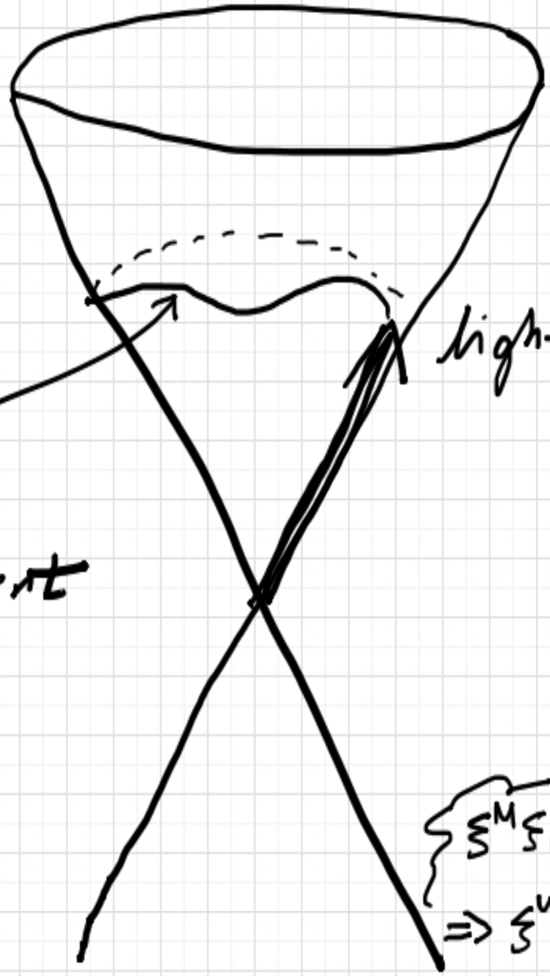
Conformal Group

$$\frac{SO(d,2)}{\mathbb{R}^* \ltimes SO(d-1,1) \ltimes \mathbb{R}^d} = \frac{SO(d,2)}{\text{stab(light like ray)}} \rightsquigarrow \text{leaves translations}$$

dilations Lorentz Conformal boosts

A picture

$M^{d,2}$: $d+2$ dim. Minkowski



light like ray ξ^M , $\xi^M \xi_M = 0$, $\xi^M \sim \Omega \xi^M$

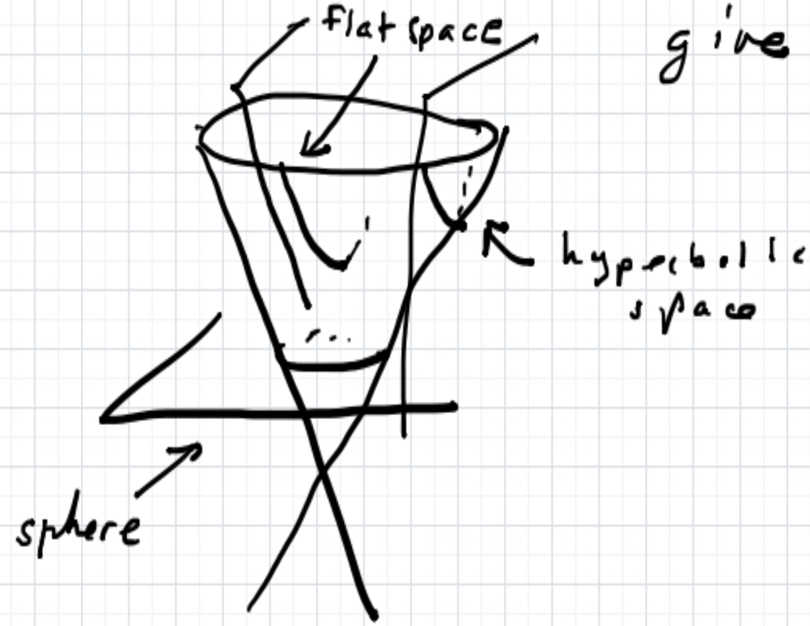
Any parameterization of light like rays inherits a conformally equivalent metric, so this generates

$$[g_{\mu\nu}] = [\Omega^2 g_{\mu\nu}]$$

$$\begin{aligned} ds^2\big| &= d\xi^M \, d\xi_M \\ &= \frac{\partial \xi^M}{\partial x^\mu} dx^\mu \frac{\partial \xi_M}{\partial x^\nu} dx^\nu = g_{\mu\nu} dx^\mu dx^\nu \\ &\sim \frac{\partial(\Omega \xi^M)}{\partial x^\mu} dx^\mu \frac{\partial(\Omega \xi_M)}{\partial x^\nu} dx^\nu \\ &= \Omega^2 g_{\mu\nu} dx^\mu dx^\nu \end{aligned}$$

$\{\xi^M \xi_M = 0 \Rightarrow \xi^M d\xi_M = 0\}$

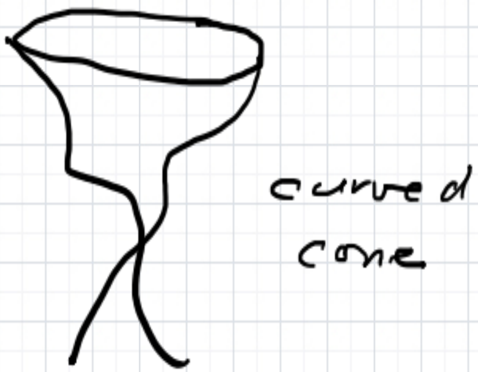
The special slices



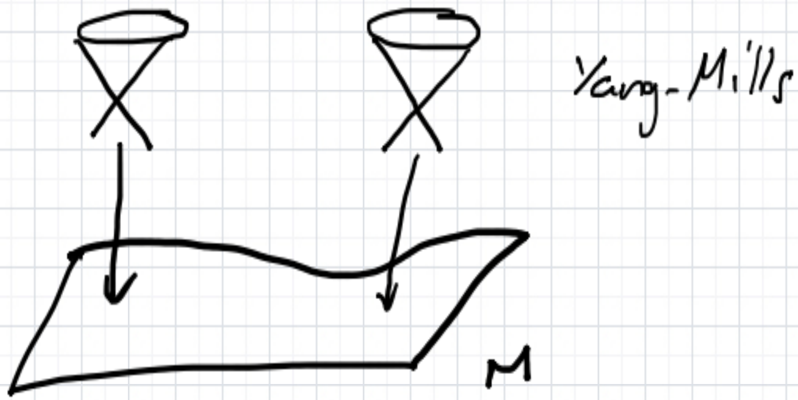
give which are all <u>conformally related</u>. In Lorentzian signature $[ds^2_{dS}] = [ds^2_{Mink}] = [dr^2_{AdS}]$

We would like to "curve this situation up". Two (equivalent) approaches

i)



ii)



Yang-Mills Theory for Conformal Group

$SO(d,2)$ connexion

$$\nabla = d + A \qquad A = \left(\begin{array}{c|c|c} b & -e & 0 \\ \hline f & \omega & e \\ \hline 0 & f & -b \end{array}\right) \qquad \eta^{MN} = \left(\begin{array}{c|c|c} & & 1 \\ \hline & \eta^{mn} & \\ \hline 1 & & \end{array}\right)$$

$so(d,2)$-valued

Lightlike ray $X^M = \left(\begin{array}{c} 0 \\ \hline 0 \\ \hline 1 \end{array}\right)$ again require $F_{\mu\nu}{}^M{}_N X^N = 0$

Also only want diffeos, local Lorentz but <u>not</u> local conformal boosts, so use these to gauge away (Weyl's) dilation gauge field b_μ.

In addition, want to eliminate gauge field $f_\mu{}^m$. For that, the constraint

$$e^\mu{}_m F_{\mu\nu}{}^m{}_n = 0$$

is consistent with the remaining YM symmetries.

Calling

$$R_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma} + (g_{\mu\rho} P_{\nu\sigma} \pm 3\,\text{more})$$

Riemann

Weyl Tensor

Schouten tensor (trace-adjusted Ricci)

Then constraints are solved via

$$\mathcal{F}_\mu{}^m = P_\mu{}^m$$

Solutions for A & F are

Levi-Civita

Weyl tensor

$$\tilde{\nabla} = \nabla_\omega + \begin{pmatrix} 0 & -e & 0 \\ P & 0 & e \\ 0 & -P & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 & 0 \\ C & W & 0 \\ 0 & -C & 0 \end{pmatrix}$$

Schouten

Cotton Tensor $C = \nabla_\omega P$

Only independent field is $e_\mu{}^m \longmapsto \Omega(x)\, e_\mu{}^m$ under local <u>dilations</u>.

* Weyl tensor is Weyl invariant $W(g) = W(\Omega^2 g)$ <u>obstructs</u> conformal flatness

vanishes in $d=3$

* Cotton tensor $\longmapsto$ Cotton + Weyl Tensor $\Rightarrow$ Cotton obstructs $d=3$ <u>c-flatness</u>

TRACTORS

What about Yang-Mills matter?

What are gauge transformations?

Diffeos
Local Lorentz } Easy

Weyl

so(d,2) matrix

Weyl

$$\nabla^T \longmapsto U\,\nabla^T U^{-1} \qquad \log U = \begin{pmatrix} \omega & & \\ k & & \\ & -k & -\omega \end{pmatrix}$$

local c-boosts

But $b_\mu \longmapsto b_\mu - k_\mu + \partial_\mu \omega$ but we gauged $b_\mu = 0 \Rightarrow k_\mu = \partial_\mu \omega \equiv \Upsilon_\mu$

$$\Rightarrow U = \exp\begin{pmatrix} \omega & & \\ \Upsilon & & \\ & -\Upsilon & -\omega \end{pmatrix} = \begin{pmatrix} \overset{\sim e^{\omega}}{\Omega} & 0 & \\ \Upsilon & \mathbb{1} & \\ -\frac{1}{2}\frac{\Upsilon^2}{\Omega} & -\frac{\Upsilon}{\Omega} & \frac{1}{\Omega} \end{pmatrix} \longleftarrow \text{Tractor Gauge transformation}$$

TRACTOR VECTOR

weight w

$$V^M = \begin{pmatrix} V^+ \\ V^m \\ V^- \end{pmatrix} \longmapsto \Omega^w U^M{}_N V^N = \Omega^w \begin{pmatrix} \Omega V^+ \\ V^m + \Upsilon^m V^+ \\ \frac{1}{\Omega}\left(V^- - \Upsilon^m V_m - \frac{1}{2}\Upsilon^2 V^+\right) \end{pmatrix}$$

EX $X^M = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a weight 1 tractor b/c

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \Omega^1 \begin{pmatrix} \Omega & 0 & 0 \\ \Upsilon & 1 & 0 \\ \frac{1}{2\Omega}\Upsilon^2 & \frac{\Upsilon}{\Omega} & \frac{1}{\Omega} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Claim: Any problem with a vector will have a pair of scalars that form a tractor multiplet!

The tractor connexion ∇^T gives a covariant derivative w.r.t. tractor gauge transformations. To have a calculus for tractors need to build a multiplet of operators

$$D^M = \begin{pmatrix} ? \\ \nabla^T \\ ? \end{pmatrix}$$

The Thomas D-operator

$$D^M = \begin{pmatrix} w(d+2w-2) \\ (d+2w-2)\nabla^T \\ -(\Delta^T + wJ) \end{pmatrix}$$

is a weight -1 tractor vector operator.

Remarkably D^M & X^M are both null $D^M D_M = 0 = X^M X_M$

RAISE & LOWER WITH η^{MN} ← invariant

Next ~ GRAVITY & TRACTORS!