Lecture #3 Symmetries of Quantum Particle In Curved Space

Quantized $S[\Gamma] = \int [p_\mu \dot{x}^\mu - \frac{1}{2} p_\mu g^{\mu\nu} p_\nu]$ via covariance

$-2\hat{H} = \Delta$ scalar Laplacian

Further symmetries? Ward operators commuting with Δ.

Remark: We can view this as a QM problem or a geometry problem. Either perspective is fine.

Geometry solution

Ansatz Use the Lie derivative $\mathcal{L}_\xi$. On scalars, the Lie derivative along a vector field ξ^μ is simply

$$\mathcal{L}_\xi \psi = \xi^\mu \partial_\mu \psi$$

We must now compute $[\mathcal{L}_\xi, \Delta]$ on scalars.

$$\xi^\rho \partial_\rho \nabla^\mu \partial_\mu \psi - \nabla^\mu \partial_\mu (\xi^\rho \partial_\rho \psi)$$

(use $\partial_\mu (V_\nu W^\nu) = (\nabla_\mu V_\nu) W^\nu + V_\nu \nabla_\mu W^\nu$)
<u>Leibnitz</u>

$$= \mathcal{L}_\xi \Delta \psi - \nabla^\mu \left((\nabla_\mu \xi^\rho) \partial_\rho \psi + \xi^\rho \nabla_\mu \partial_\rho \psi \right)$$

$$= \cancel{\mathcal{L}_\xi \Delta \psi} \underbrace{- (\Delta \xi^\rho) \partial_\rho \psi}_{\substack{\nabla_\mu (\nabla^\mu \xi^\rho + \nabla^\rho \xi^\mu) \partial_\rho \psi \\ -(\nabla^\rho \nabla \cdot \xi + \cancel{R_\mu{}^{\rho\mu}{}_\sigma \xi^\sigma}) \partial_\rho \psi}} - 2 (\nabla^\mu \xi^\rho) \nabla_\mu \partial_\rho \psi \underbrace{- \xi^\rho \Delta \partial_\rho \psi}_{-\cancel{\mathcal{L}_\xi \Delta \psi} - \cancel{\xi^\rho R^\mu{}_{\rho\mu}{}^\sigma} \partial_\sigma \psi}$$

$$= -(\nabla^\mu \xi^\rho + \nabla^\rho \xi^\mu) \nabla_\mu \partial_\rho \psi - (\nabla_\mu [\nabla^\mu \xi^\rho + \nabla^\rho \xi^\mu]) \partial_\rho \psi - (\nabla^\rho \nabla \cdot \xi) \partial_\rho \psi$$

$$= - \nabla_\mu \left[(\nabla^\mu \xi^\rho + \nabla^\rho \xi^\mu) \partial_\rho \psi \right] - (\nabla^\rho \nabla \cdot \xi) \partial_\rho \psi$$

Thus if $\boxed{\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0}$ $\Rightarrow \nabla \cdot \xi = 0$ these cancel

and $\mathcal{L}_\xi$ is a <u>symmetry</u>. Vectors obeying this equation are called <u>Killing vectors</u> and correspond to <u>isometries</u>.

Isometries Under infinitesimal diffeomorphisms

$$\delta g_{\mu\nu} = \underbrace{\xi^\rho \partial_\rho g_{\mu\nu}}_{\text{transport term}} + \underbrace{\partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho}}_{\text{orbital term}}$$

$$= \mathcal{L}_\xi g_{\mu\nu} \qquad \text{This is the Lie derivative}$$

$$= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \qquad \text{Using christoffels.}$$

i.e. Isometries are diffeomorphisms leaving the metric inert: $\mathcal{L}_\xi g_{\mu\nu} = 0$

We will also be interested in conformal isometries

$$\mathcal{L}_\xi g_{\mu\nu} = \underbrace{\alpha}_{\text{some function}} g_{\mu\nu}$$

under which the metric only rescales.

Isometries of Minkowski space

$g_{\mu\nu} = \eta_{\mu\nu}$, $ds^2 = -dt^2 + d\vec{x}^2$

(nb denoting killing vectors by $\xi^\mu \partial_\mu$)

Rotations $x_i \partial_j - x_j \partial_i$

Boosts $t\partial_i + x_i \frac{\partial}{\partial t}$

$$\left.\begin{matrix} x_i \partial_j - x_j \partial_i \\ t\partial_i + x_i \frac{\partial}{\partial t} \end{matrix}\right\} x_\mu \partial_\nu - x_\nu \partial_\mu = M_{\mu\nu}$$

Translations ∂_i

Evolution $\frac{\partial}{\partial t}$

$$\left.\begin{matrix} \partial_i \\ \frac{\partial}{\partial t} \end{matrix}\right\} \frac{\partial}{\partial x^\mu} = P_\mu$$

$$\underbrace{[M_{\mu\nu}, M_{\rho\sigma}]}_{\text{Lie Bracket}} = M_{\mu\sigma}\eta_{\nu\rho} \pm 3\text{more}$$

$$[M_{\mu\nu}, P_\rho] = -\eta_{\mu\rho} P_\nu + \eta_{\nu\rho} P_\mu$$

Lie algebra of the Poincaré group

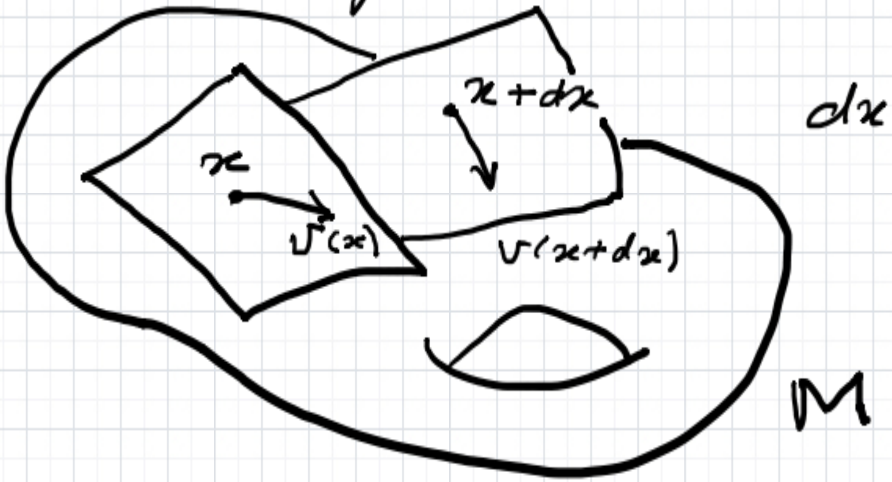
$$\underbrace{SO(d-1,1)}_{\text{Lorentz}} \ltimes \underbrace{\mathbb{R}^d}_{\text{Translations}}$$

Including dilations $x^\mu \partial_\mu$ & conformal boosts $(d + 2x\cdot\partial - 2)\partial_\mu - x_\mu \Delta$

this algebra becomes $SO(d,2)$ the <u>conformal group</u>

Connexions

The job of the Levi-Civita connection was to check whether tangent vectors were held parallel



$$dx^\mu \nabla_\mu v^\nu = 0 \Rightarrow v \text{ is parallel}$$

But we consider transporting other mathematical structures from point to point in M.

Connexions are often analyzed in terms of <u>gauge invariance</u>

<u>EX</u> Electromagnetism

$$\partial_\mu + ieA_\mu = \nabla_\mu \longmapsto \underbrace{e^{ie\alpha}}_{U(1)} \nabla_\mu e^{-ie\alpha} \qquad (\underline{ie}\ A_\mu \mapsto A_\mu + \partial_\mu \alpha)$$

Yang Mills

$$\partial_\mu + \underbrace{T_a}_{\substack{\text{generators} \\ \text{of } \mathfrak{g}\ (\text{e.g. } su(N))}} A^a_\mu = \nabla_\mu \longmapsto \underbrace{U}_{G\text{-valued}} \nabla_\mu U^{-1}$$

This example underlies all cases.

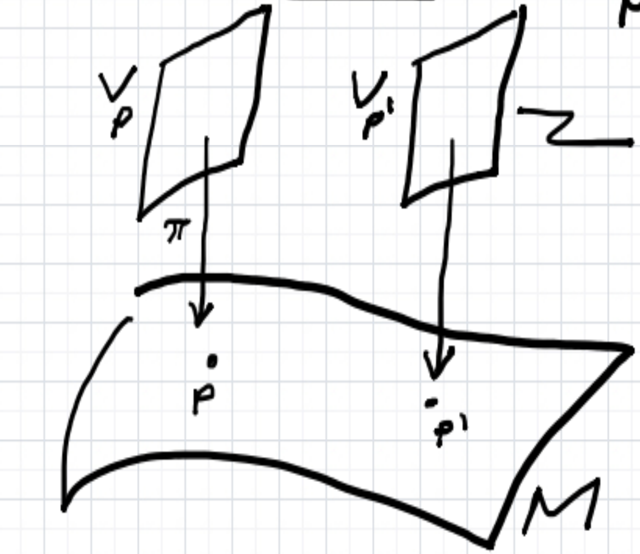
Christoffels

$$\partial_\mu + \underbrace{\Gamma^{\#}_\mu}_{\substack{\#\text{ denotes action} \\ \text{of indices } {}^\rho_\nu}} = \nabla_\mu \longrightarrow \underbrace{G}_{\substack{GL(d)\text{ matrix} \\ (\text{diffeomorphism})}} \nabla_\mu G^{-1}$$

(Recall Γ's are needed so that $\partial + \Gamma$ is a tensor)

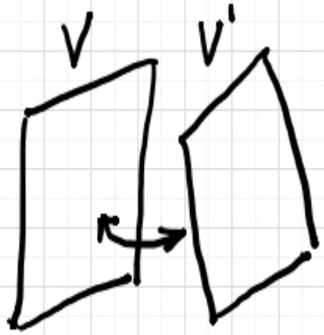
Vector Bundles

$$\begin{array}{ccc} F & \to & V \\ & & \downarrow \\ & & M \end{array}$$



some vector space V / or your favorite math-structure...

Want to compare vectors at p, p'

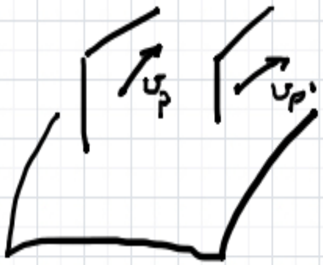


glued together modulo gauge transformations encoding how you coordinatize V

structure group

F

A choice of $v \in V_p$ for each p is called a <u>section</u>,



in physics these are just field.

Locally, just think of functions
$$\begin{array}{ccc} M & \longrightarrow & V \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x^\mu & \longmapsto & V^\mu(x) \end{array}$$

In the previous examples, sections of an (associated) vector bundle could be electron field $\psi(x)$, quark field $\chi^\alpha(x)$, tangent vectors $v^\mu(x)$

$\underline{\text{EX}}$ Conformal densities $\mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}[w] = \mathcal{E}M[w]$

$$\begin{array}{c} \mathbb{R}[w] \\ \downarrow \\ M \end{array}$$

These are dimensionful real scalar fields, sections defined up ~~to~~ gauge invariance

$$\varphi \longmapsto \Omega(x)^w \varphi \qquad \text{often write } [\varphi] = [\Omega^w \varphi] \quad \text{or} \quad \varphi \sim \Omega^w \varphi \Bigg\} \text{equivalence classes}$$

* A single conformal density carries little information b/c $\varphi \sim [\varphi(x)^{-\frac{1}{w}}]^w \varphi = 1$

$\underline{\text{ACHTUNG}}$ Assuming φ is nowhere vanishing!

This is just the statement that measurements are meaningless w/o specifying unit system.

* Need one field $\sigma \in \mathcal{E}M[1]$ that sets your unit systems, called the scale. Typical choice

GRAVITY FROM YANG-MILLS THEORY

Gauge group $SO(3,1) \ltimes \mathbb{R}^4$ = Poincaré

$$\begin{array}{c} \omega \\ \left(\begin{array}{c|c} \Lambda & P \\ \hline 0 & 1 \end{array}\right) \end{array}$$

Lorentz rotation $\rightarrow \Lambda$ translation $\rightarrow P$

Connection $\nabla = d + A$

$$= dx^\mu \nabla_\mu$$

$$A = \begin{array}{c} \begin{array}{cc} 4\times4 & 4\times1 \end{array} \\ \left(\begin{array}{c|c} \omega & e \\ \hline & \end{array}\right) \end{array}$$

Curvature $\nabla^2 = \frac{1}{2} dx^\mu \wedge dx^\nu F_{\mu\nu}$

$$= \left(\begin{array}{c|c} d+\omega & e \\ \hline & d \end{array}\right)^2 = \left(\begin{array}{c|c} (d+\omega)^2 & de + \omega e + ed \\ \hline 0 & 0 \end{array}\right)$$

$$= \left(\begin{array}{c|c} (d\omega) + \omega^2 & (de) + \omega e \\ \hline 0 & 0 \end{array}\right)$$