Quantum Mechanical Path Integrals: from Transition Amplitudes to Worldline Formalism

Lecture notes from the "School on Spinning Particles in Quantum Field Theory: Worldline Formalism, Higher Spins and Conformal Geometry," held in Morelia (Michoacán) Mexico, 19-23 November 2012

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ABSTRACT: The following set of lectures cover introductory material on quantum-mechanical Feynman path integrals. We define and apply the path integrals to several particles models in flat space. We start considering the nonrelativistic bosonic particle in a potential for which we compute the exact path integrals for the free particle and for the harmonic oscillator and then consider perturbation theory for an arbitrary potential. We then move to relativistic particles, bosonic and fermionic (spinning) ones, and start considering them from the classical view point studying the symmetries of their actions. We then consider canonical quantization and path integrals and underline the role these models have in the study of space-time higher-spin fields.

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1 Introduction

The notion of *path integral* as integral over trajectories was first introduced by Wiener in the 1920's to solve problems related to the Brownian motion. Later, in 1940's, it was reintroduced by Feynman as an alternative to operatorial methods to compute transition amplitudes in quantum mechanics: Feynman path integrals use a lagrangian formulation instead of a hamiltonian one and can be seen as a quantum-mechanical generalization of the least-action principle (see e.g. [1]).

In electromagnetism the linearity of the Maxwell equations in vacuum allows to formulate the Huygens-Fresnel principle that in turn allows to write the wave scattered by a multiple slit as a sum of waves generated by each slit, where each single wave is characterized by the optical length $I(x_i, x)$ between the i-th slit and the field point x, and the final amplitude is thus given by $A = \sum_i e^{iI(x_i,x)}$, whose squared modulus describes patterns of interference between single waves. In quantum mechanics, a superposition principle can be formulated in strict analogy to electromagnetism and a linear equation of motion, the Schrödinger equation, can be correspondingly postulated. Therefore, the analogy can be carried on further replacing the electromagnetic wave amplitude by a transition amplitude between an initial point x' at time 0 and a final point x at time t, whereas the optical length is replaced by the classical action for going from x' and x in time t (divided by $\hbar$, that has dimensions of action). The full transition amplitude will thus be a "sum" over all paths connecting x' and x in time t:

$$K(x, x'; t) \sim \sum_{\{x(\tau)\}} e^{iS[x(\tau)]/\hbar} . \tag{1.1}$$

In the following we try to make sense of the previous expression. Let us now just try to justify the presence of the action in the previous expression by recalling that the action principle applied to the action function $S(x,t)$ (that corresponds to $S[x(\tau)]$ with only the initial point $x' = x(0)$ fixed) implies that the latter satisfy the classical Hamilton-Jacobi equation. Hence the Schrödinger equation imposed onto the amplitude $K(x, x'; t) \sim e^{iS(x,t)/\hbar}$ yields an equation that deviates from Hamilton-Jacobi equation by a linear term in $\hbar$. So that $\hbar$ measures the departure from classical mechanics, that corresponds to $\hbar \to 0$, and the classical action function determines the transition amplitude to leading order in $\hbar$. It will be often useful to parametrize an arbitrary path $x(\tau)$ between x' and x as $x(\tau) = x_{cl}(\tau) + q(\tau)$ with $x_{cl}(\tau)$ being the "classical path" i.e. the path that satisfies the equations of motion with the above boundary conditions, and $q(\tau)$ is an arbitrary deviation (see Figure 1 for graphical description.) With such parametrization we have

$$K(x, x'; t) \sim \sum_{\{q(\tau)\}} e^{iS[x_{cl}(\tau)+q(\tau)]/\hbar} . \tag{1.2}$$

and $x_{cl}(\tau)$ represents as a sort of "origin" in the space of all paths connecting x' and x in time t.
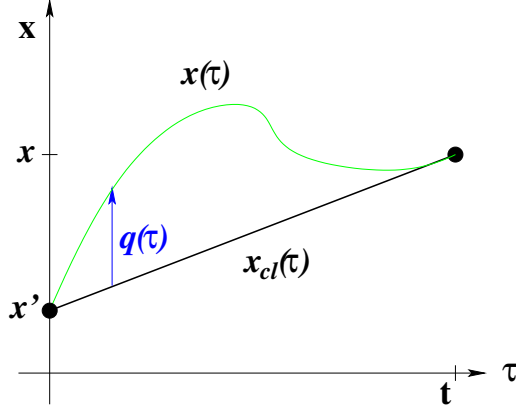
Figure 1. A useful parametrization of paths

2 Path integral representation of quantum mechanical transition amplitude: non relativistic bosonic particle

The quantum-mechanical transition amplitude for a time-independent hamiltonian operator is given by (here and henceforth we use natural units and thus set $\hbar = c = 1$; see Appendix A for a brief review on the argument) [1]

$$K(x, x'; t) = \langle x | e^{-it\mathbb{H}} | x' \rangle = \langle x, t | x', 0 \rangle \tag{2.1}$$

$$K(x, x'; 0) = \delta(x - x') \tag{2.2}$$

and describes the evolution of the wave function from time 0 to time t

$$\psi(x, t) = \int dx' \ K(x, x'; t) \psi(x', 0) \tag{2.3}$$

and satisfies the Schrödinger equation

$$i\partial_t K(x, x'; t) = H K(x, x'; t) \tag{2.4}$$

with H being the hamiltonian operator in coordinate representation; for a non-relativistic particle on a line we have $H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$. In particular, for a free particle $V = 0$, it is easy to solve the Schrödinger equation (2.4) with boundary conditions (2.2). One obtains

$$K_f(x, x'; t) = N_f(t) \ e^{iS_{cl}(x, x'; t)} \ , \tag{2.5}$$

with

$$N_f(t) = \sqrt{\frac{m}{i2\pi t}} \tag{2.6}$$

$$S_{cl}(x, x'; t) \equiv \frac{m(x - x')^2}{2t} \ . \tag{2.7}$$

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[1]A generalization to time-dependent hamiltonian $\mathbb{H}(t)$ can be obtained with the replacement $e^{-it\mathbb{H}} \to \mathcal{T} e^{-i \int_0^t d\tau \mathbb{H}(\tau)}$ and $\mathcal{T}e$ being a time-ordered exponential.

The latter as we will soon see is the action for the classical path of the free particle.

In order to introduce the path integral we "slice" the evolution operator in (2.1) by defining $\epsilon = t/N$ and insert $N-1$ decompositions of unity in terms of position eigenstates $\mathbb{1} = \int dx |x\rangle\langle x|$; namely

$$K(x, x'; t) = \langle x|e^{-i\epsilon\mathbb{H}} e^{-i\epsilon\mathbb{H}} \cdots e^{-i\epsilon\mathbb{H}}|x'\rangle \tag{2.8}$$

$$= \int \left(\prod_{i=1}^{N-1} dx_i\right) \langle x_N|e^{-i\epsilon\mathbb{H}}|x_{N-1}\rangle \prod_{j=1}^{N-1} \langle x_j|e^{-i\epsilon\mathbb{H}}|x_{j-1}\rangle \tag{2.9}$$

with $x_N \equiv x$ and $x_0 \equiv x'$. We now insert N spectral decomposition of unity in terms of momentum eigenstates $\mathbb{1} = \int \frac{dx}{2\pi}|p\rangle\langle p|$ and get

$$K(x, x'; t) = \int \left(\prod_{i=1}^{N-1} dx_i\right) \left(\prod_{k=1}^{N} \frac{dp_k}{2\pi}\right) \langle x_N|e^{-i\epsilon\mathbb{H}}|p_N\rangle\langle p_N|x_{N-1}\rangle$$
$$\times \prod_{j=1}^{N-1} \langle x_j|e^{-i\epsilon\mathbb{H}}|p_j\rangle\langle p_j|x_{j-1}\rangle. \tag{2.10}$$

For large N, assuming $\mathbb{H} = \mathbb{T} + \mathbb{V} = \frac{1}{2m}\mathbb{p}^2 + V(\mathbb{q})$, we can use the "Trotter formula" (see e.g. [2])

$$\left(e^{-i\epsilon\mathbb{H}}\right)^N = \left(e^{-i\epsilon\mathbb{V}}e^{-i\epsilon\mathbb{T}} + O(1/N^2)\right)^N \approx \left(e^{-i\epsilon\mathbb{V}}e^{-i\epsilon\mathbb{T}}\right)^N \tag{2.11}$$

that allows to replace $e^{-i\epsilon\mathbb{H}}$ with $e^{-i\epsilon\mathbb{V}}e^{-i\epsilon\mathbb{T}}$ in (2.10), so that one gets $\langle x_j|e^{-i\epsilon\mathbb{H}}|p_j\rangle \approx e^{i(x_j p_j - \epsilon H(x_j, p_j))}$, with $\langle x|p\rangle = e^{ixp}$. Hence,

$$K(x, x'; t) = \int \left(\prod_{i=1}^{N-1} dx_i\right) \left(\prod_{k=1}^{N} \frac{dp_k}{2\pi}\right) \exp\left[i\sum_{j=1}^{N} \epsilon\left(p_j \frac{x_j - x_{j-1}}{\epsilon} - H(x_j, p_j)\right)\right] \tag{2.12}$$

with

$$H(x_j, p_j) = \frac{1}{2m}p_j^2 + V(x_j) \tag{2.13}$$

the hamiltonian function. In the large N limit we can formally write the latter as

$$K(x, x'; t) = \int_{x(0)=x'}^{x(t)=x} Dx Dp \exp\left[i\int_0^t d\tau\left(p\dot{x} - H(x, p)\right)\right] \tag{2.14}$$

$$Dx \equiv \prod_{0<\tau<t} dx(\tau), \quad Dp \equiv \prod_{0<\tau<t} dp(\tau) \tag{2.15}$$

that is referred to as the "phase-space path integral." Alternatively, momenta can be integrated out in (2.10) as they are (analytic continuation of) gaussian integrals. Completing the square one gets

$$K(x, x'; t) = \int \left(\prod_{i=1}^{N-1} dx_i\right) \left(\frac{m}{2\pi i\epsilon}\right)^{N/2} \exp\left[i\sum_{j=1}^{N} \epsilon\left(\frac{m}{2}\left(\frac{x_j - x_{j-1}}{\epsilon}\right)^2 - V(x_j)\right)\right] \tag{2.16}$$

that can be formally written as

$$K(x, x'; t) = \int_{x(0)=x'}^{x(t)=x} Dx \ e^{iS[x(\tau)]} \tag{2.17}$$

with

$$S[x(\tau)] = \int_0^t d\tau \left(\frac{m}{2}\dot{x}^2 - V(x(\tau))\right) . \tag{2.18}$$

Expression (2.17) is referred to as "configuration space path integral" and is interpreted as a functional integral over trajectories with boundary condition $x(0) = x'$ and $x(t) = x$.

To date no good definition of path integral measure Dx is known and one has to rely on some regularization methods. For example one may expand paths on a suitable basis to turn the functional integral into a infinite-dimensional Riemann integral. One thus may regularize taking a large (though finite) number of vectors in the basis, perform the integrals and take the limit at the very end. This regularization procedure, as we will see, fixes the path integral up to an overall normalization constant that must be fixed using some consistency conditions. Nevertheless, ratios of path integrals are well-defined objects and turn out to be quite convenient tools in several areas of modern physics. Moreover one may fix –as we do in the next section– the above constant using the simplest possible model (the free particle) and compute other path integrals via their ratios with the free particle path integral.

2.1 Wick rotation to euclidean time: from quantum mechanics to statistical mechanics

As already mentioned path integrals were born in statistical physics. In fact we can easily obtain the particle partition function from (2.17) by, (a) "Wick rotating" time to imaginary time, namely $it \equiv \beta = 1/k\theta$ (where θ is the temperature) and (b) taking the trace

$$Z(\beta) = \text{tr } e^{-\beta\mathbb{H}} = \int dx \ \langle x|e^{-\beta\mathbb{H}}|x\rangle = \int dx \ K(x, x, : -i\beta)$$

$$= \int_{PBC} Dx \ e^{-S_E[x(\tau)]} \tag{2.19}$$

where the euclidean action

$$S_E[x(\tau)] = \int_0^\beta d\tau \left(\frac{m}{2}\dot{x}^2 + V(x(\tau))\right) \tag{2.20}$$

has been obtained by Wick rotating the worldline time $i\tau \to \tau$, and PBC stands for *periodic boundary conditions* and means that the path integral is taken over all closed paths.

2.2 The free particle path integral

We consider the path integral (2.17) for the special case of a free particle, i.e. $V = 0$. For simplicity we consider a particle confined on a line and rescale the worldline time $\tau \to t\tau$ in such a way that free action and boundary conditions turn into

$$S_f[x(\tau)] = \frac{m}{2t} \int_0^1 d\tau \ \dot{x}^2 , \quad x(0) = x', \ x(1) = x \tag{2.21}$$

where x and x' are points on the line. The free equation of motion is obviously $\ddot{x} = 0$ so that the aforementioned parametrization yields

$$x(\tau) = x_{cl}(\tau) + q(\tau) \tag{2.22}$$

$$x_{cl}(\tau) = x' + (x - x')\tau = x + (x' - x)(1 - \tau), \quad q(0) = q(1) = 0 \tag{2.23}$$

and the above action reads

$$S_f[x_{cl} + q] = \frac{m}{2t} \int_0^1 d\tau \; (\dot{x}_{cl}^2 + \dot{q}^2 + 2\dot{x}_{cl}\dot{q}) = \frac{m}{2t}(x - x')^2 + \frac{m}{2t} \int_0^1 d\tau \; \dot{q}^2 \tag{2.24}$$

$$= S_f[x_{cl}] + S_f[q]$$

and the mixed term identically vanishes due to equation of motion and boundary conditions. The path integral can thus be written as

$$K_f(x, x'; t) = e^{iS_f[x_{cl}]} \int_{q(0)=0}^{q(1)=0} Dq \; e^{iS_f[q(\tau)]} = e^{i\frac{m}{2t}(x-x')^2} \int_{q(0)=0}^{q(1)=0} Dq \; e^{i\frac{m}{2t}\int_0^1 \dot{q}^2} \tag{2.25}$$

so that by comparison with (2.5,2.6,2.7) one can infer that, for the free particle,

$$\int_{q(0)=0}^{q(1)=0} Dq \; e^{i\frac{m}{2t}\int_0^1 \dot{q}^2} = \sqrt{\frac{m}{i2\pi t}} \; . \tag{2.26}$$

Alternatively, one can expand $q(\tau)$ on a basis of function with Dirichlet boundary condition on the line

$$q(\tau) = \sum_{n=1}^{\infty} q_n \sin(n\pi\tau) \tag{2.27}$$

with q_n arbitrary real coefficients. We can thus define the measure as

$$Dq \equiv A \prod_{n=1}^{\infty} dq_n \sqrt{\frac{mn^2\pi}{4t}} \tag{2.28}$$

with A an unknown coefficient that we fix shortly

$$\int_{q(0)=0}^{q(1)=0} Dq \; e^{i\frac{m}{2t}\int_0^1 \dot{q}^2} = A \int \prod_{n=1}^{\infty} dq_n \sqrt{\frac{mn^2\pi}{4it}} e^{i\frac{m}{4t}\sum_n (\pi n q_n)^2} = A \tag{2.29}$$

from which $A = \sqrt{\frac{m}{i2\pi t}}$ follows. The latter results easily generalize to d space dimensions where $A = \int_{q(0)=0}^{q(1)=0} Dq \; e^{i\frac{m}{2t}\int_0^1 \dot{q}^2} = \left(\frac{m}{i2\pi t}\right)^{d/2}$.

The expression (2.28) allows to use the so-called *mode regularization* to compute more generic particle path integrals where interaction terms may introduce computational ambiguities. Namely: whenever an ambiguity appears one can always rely on the mode expansion truncated at a finite mode M and then take the large limit at the very end. Other regularization schemes that have been adopted to such purpose are: *time slicing* that rely on the well-defined expression for the path integral as multiple slices (cfr. eq. (2.16)) and *dimensional regularization* that regulates ambiguities by dimensionally extending the worldline (see [3] for a review on such issues). However dimensional regularization is a regularization that only works in the perturbative approach to the path integral, by regulating single Feynman diagrams. Perturbation theory within the free particle path integral is a subject that will be discussed below.

2.2.1 The free particle partition function

The partition function for a free particle in d-dimensional space can be obtained as

$$Z_f(\beta) = \int d^d x \ K(x, x, : -i\beta) = \left(\frac{m}{2\pi\beta}\right)^{d/2} \int d^d x = \mathcal{V}\left(\frac{m}{2\pi\beta}\right)^{d/2} \tag{2.30}$$

$\mathcal{V}$ being the spatial volume. It is easy to recall that this is the correct result as

$$Z_f(\beta) = \sum_p e^{-\beta p^2/2m} = \frac{\mathcal{V}}{(2\pi)^d} \int d^d p \ e^{-\beta p^2/2m} = \mathcal{V}\left(\frac{m}{2\pi\beta}\right)^{d/2} \tag{2.31}$$

with $\frac{\mathcal{V}}{(2\pi)^d}$ being the "density of states".

2.2.2 Perturbation theory about free particle solution: Feynman diagrams

In the presence of an arbitrary potential the path integral for the transition amplitude is not exactly solvable. However if the potential is "small" compared to the kinetic term one can rely on perturbation theory about the free particle solution. The significance of being "small" will be clarified a posteriori.

Let us then obtain a perturbative expansion for the transition amplitude (2.17) with action (2.18). As done above we split the arbitrary path in terms of the classical path (with respect to the free action) and deviation $q(\tau)$ and again making use of the rescaled time we can rewrite the amplitude as

$$K(x, x'; t) = N_f(t) \ e^{i\frac{m}{2t}(x-x')^2} \ \frac{\int_{q(0)=0}^{q(1)=0} Dq \ e^{i\int_0^1 (\frac{m}{2t}\dot{q}^2 - tV(x_{cl}+q))}}{\int_{q(0)=0}^{q(1)=0} Dq \ e^{i\frac{m}{2t}\int_0^1 \dot{q}^2}} \ . \tag{2.32}$$

We then Taylor expand the potential in the exponent about the classical free solution: this gives rise to a infinite set of interaction terms (τ dependence in x_{cl} and q is left implied)

$$S_{int} = -t \int_0^1 d\tau \left(V(x_{cl}) + V'(x_{cl})q + \frac{1}{2!}V^{(2)}(x_{cl})q^2 + \frac{1}{3!}V^{(2)}(x_{cl})q^3 + \cdots \right)$$

$\tag{2.33}$

Next we expand the exponent $e^{iS_{int}}$ so that we only have polynomials in q to integrate with the path integral weight; in other words we only need to compute expressions like

$$\frac{\int_{q(0)=0}^{q(1)=0} Dq \ e^{i\frac{m}{2t}\int_0^1 \dot{q}^2} q(\tau_1)q(\tau_2)\cdots q(\tau_n)}{\int_{q(0)=0}^{q(1)=0} Dq \ e^{i\frac{m}{2t}\int_0^1 \dot{q}^2}} \equiv \Big\langle q(\tau_1)q(\tau_2)\cdots q(\tau_n)\Big\rangle \tag{2.34}$$

and the full (perturbative) path integral can be compactly written as

$$K(x, x'; t) = N_f(t) \ e^{i\frac{m}{2t}(x-x')^2} \Big\langle e^{-it\int_0^1 V(x_{cl}+q)} \Big\rangle \tag{2.35}$$

and the expressions $\left\langle f(q) \right\rangle$ are referred to as "correlation functions". In order to compute the above correlations functions we define and compute the so-called "generating functional"

$$\mathcal{Z}[j] \equiv \int_{q(0)=0}^{q(1)=0} Dq \ e^{i\frac{m}{2t}\int_0^1 \dot{q}^2 + i\int_0^1 qj} = N_f(t)\left\langle e^{i\int_0^1 qj} \right\rangle \tag{2.36}$$

in terms of which

$$\left\langle q(\tau_1)q(\tau_2)\cdots q(\tau_n) \right\rangle = (-i)^n \frac{1}{\mathcal{Z}[0]} \frac{\delta^n}{\delta j(\tau_1)\delta j(\tau_2)\cdots\delta j(\tau_n)} \mathcal{Z}[j]\Big|_{j=0} \ . \tag{2.37}$$

By partially integrating the kinetic term and completing the square we get

$$\mathcal{Z}[j] = e^{\frac{i}{2}\iint jD^{-1}j} \int_{q(0)=0}^{q(1)=0} Dq \ e^{i\frac{m}{2t}\int_0^1 \dot{q}^2} \tag{2.38}$$

where $D^{-1}(\tau, \tau')$, "the propagator", is the inverse kinetic operator, with $D \equiv \frac{m}{t}\partial_\tau^2$, such that

$$DD^{-1}(\tau, \tau') = \delta(\tau, \tau') \tag{2.39}$$

in the basis of function with Dirichlet boundary conditions, and $\tilde{q}(\tau) \equiv q(\tau) - \int_0^1 j(\tau')D^{-1}(\tau', \tau)$. By defining $D^{-1}(\tau, \tau') = \frac{t}{m}\Delta(\tau, \tau')$ we get

$$^{\bullet\bullet}\Delta(\tau, \tau') = \delta(\tau, \tau') \tag{2.40}$$

$$\Rightarrow \quad \Delta(\tau, \tau') = \theta(\tau - \tau')(\tau - 1)\tau' + \theta(\tau' - \tau)(\tau' - 1)\tau \tag{2.41}$$

where "dot" on the left (right) means derivative with respect to τ (τ'). The propagator satisfies the following properties

$$\Delta(\tau, \tau') = \Delta(\tau', \tau) \tag{2.42}$$

$$\Delta(\tau, 0) = \Delta(\tau, 1) = 0 \tag{2.43}$$

from which $\tilde{q}(0) = \tilde{q}(1) = 0$. Therefore we can shift the integration variable in (2.38) from q to $\tilde{q}$ and get

$$\mathcal{Z}[j] = N_f(t) \ e^{\frac{it}{2m}\iint j\Delta j} \tag{2.44}$$

and finally obtain

$$\left\langle q(\tau_1)q(\tau_2)\cdots q(\tau_n) \right\rangle = (-i)^n \frac{\delta^n}{\delta j(\tau_1)\delta j(\tau_2)\cdots\delta j(\tau_n)} e^{\frac{it}{2m}\iint j\Delta j}\Big|_{j=0} \ . \tag{2.45}$$

In particular:

1. correlation functions of an odd number of "fields" vanish;

2. the 2-point function is nothing but the propagator

$$\left\langle q(\tau_1)q(\tau_2) \right\rangle = -i\frac{t}{m}\Delta(\tau_1, \tau_2) = \overset{\tau_1 \qquad\qquad \tau_2}{\rule{3cm}{0.4pt}}$$

3. correlation functions of an even number of fields are obtained by all possible contractions of pairs of fields. For example, for $n = 4$ we have $\langle q_1 q_2 q_3 q_4 \rangle = (-i\frac{t}{m})^2 (\Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23})$ where an obvious shortcut notation has been used. The latter statement is known as the "Wick theorem". Diagrammatically

$$\left\langle q_1 q_2 q_3 q_4 \right\rangle = \quad \overset{\tau_1 \quad \tau_2}{\underset{\tau_3 \quad \tau_4}{\rule{2cm}{0.4pt}}} \quad + \quad \Big|_{\tau_3}^{\tau_1} \Big|_{\tau_4}^{\tau_2} \quad + \quad \overset{\tau_1 \quad \tau_2}{\underset{\tau_3 \quad \tau_4}{\times}}$$

Noting that each vertex and each propagator carry a power of t/m we can write the perturbative expansion as a short-time expansion (or inverse-mass expansion). It is thus not difficult to convince oneself that the expansion reorganizes as

$$K(x, x'; t) = N_f(t) \, e^{i\frac{m}{2t}(x-x')^2} \exp\left\{\text{connected diagrams}\right\}$$

$$= N_f(t) \, e^{i\frac{m}{2t}(x-x')^2} \exp\left\{ \bullet + \bullet\!\!\!\bigcirc + \left[\begin{array}{c} \overset{\bullet\text{---}\bullet}{+} \\ \underset{\bullet}{\infty} \end{array} \right] + \cdots \right\} \tag{2.46}$$

where the diagrammatic expansion in the exponent (that is ordered by increasing powers of t/m) only involves connected diagrams, i.e. diagrams whose vertices are connected by at least one propagator. We recall that $N_f(t) = \sqrt{\frac{m}{i2\pi t}}$, and we can also give yet another representation for the transition amplitude, the so-called "heat-kernel expansion"

$$K(x, x'; t) = \sqrt{\frac{m}{i2\pi t}} \, e^{i\frac{m}{2t}(x-x')^2} \sum_{n=0}^{\infty} a_n(x, x') t^n \tag{2.47}$$

where the terms $a_n(x, x')$ are known as Seeley-DeWitt coefficients. We can thus express such coefficients in terms of Feynman diagrams and get

$$a_0(x, x') = 1 \tag{2.48}$$

$$a_1(x, x') = \bullet = -i \int_0^1 d\tau \, V(x_{cl}(\tau)) \tag{2.49}$$

$$a_2(x, x') = \frac{1}{2!} \langle \bullet \rangle^2 + \bullet\!\!\!\bigcirc$$

$$= \frac{1}{2!}\left(-i \int_0^1 d\tau \, V(x_{cl}(\tau))\right)^2 - \frac{1}{2!\,m}\int_0^1 d\tau \, V^{(2)}(x_{cl}(\tau))\tau(\tau - 1) \tag{2.50}$$

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where in (2.50) we have used that $\langle q(\tau)q(\tau) \rangle = -i\frac{t}{m}\Delta(\tau, \tau) = -i\frac{t}{m}\tau(\tau - 1)$. Let us now comment on the validity of the expansion: each propagator inserts a power of t/m.

Therefore for a fixed potential V, the larger the mass, the larger the time for which the expansion is accurate. In other words for a very massive particle it results quite costy to move away from the classical path.

2.3 The harmonic oscillator path integral

If the particle is subject to a harmonic potential $V(x) = \frac{1}{2}m\omega^2 x^2$ the path integral is again exactly solvable. In the rescaled time adopted above the path integral can be formally written as in (2.17) with action

$$S_h[x(\tau)] = \frac{m}{2t} \int_0^1 d\tau \left[\dot{x}^2 - (\omega t)^2 x^2 \right] . \tag{2.51}$$

Again we parametrize $x(\tau) = x_{cl}(\tau) + q(\tau)$ with

$$\left.\begin{array}{l} \ddot{x}_{cl} + (\omega t)^2 x_{cl} = 0 \\ x_{cl}(0) = x', \ x_{cl}(1) = x \\ (q(0) = q(1) = 0) \end{array}\right\} \quad \Rightarrow \quad x_{cl}(\tau) = x' \cos(\omega t\tau) + \frac{x - x' \cos(\omega t)}{\sin(\omega t)} \sin(\omega t\tau) \tag{2.52}$$

and, since the action is quadratic, similarly to the free particle case there is no mixed term between $x_{cl}(\tau)$ and $q(\tau)$, i.e. $S_h[x(\tau)] = S_h[x_{cl}(\tau)] + S_h[q(\tau)]$ and the path integral reads

$$K_h(x, x'; t) = N_h(t) \, e^{iS_h[x_{cl}]} \tag{2.53}$$

with

$$S_h[x_{cl}] = \frac{m\omega}{2\sin(\omega t)} \left((x^2 + x'^2) \cos(\omega t) - 2xx' \right) \tag{2.54}$$

$$N_h(t) = \int_{q(0)=0}^{q(1)=0} Dq \, e^{i\frac{m}{2t} \int_0^1 d\tau \left(\dot{q}^2 - (\omega t)^2 q^2 \right)} . \tag{2.55}$$

We now use the above mode expansion to compute the latter:

$$\begin{aligned} N_h(t) = N_f(t) \frac{N_h(t)}{N_f(t)} &= \sqrt{\frac{m}{i2\pi t}} \frac{\displaystyle\int_{q(0)=0}^{q(1)=0} Dq \, e^{i\frac{m}{2t} \int_0^1 (\dot{q}^2 - (\omega t)^2 q^2)}}{\displaystyle\int_{q(0)=0}^{q(1)=0} Dq \, e^{i\frac{m}{2t} \int_0^1 \dot{q}^2}} \\ &= \sqrt{\frac{m}{i2\pi t}} \prod_{n=1}^{\infty} \frac{\int dq_n \, e^{i\frac{imt}{4} \sum_n (\omega_n^2 - \omega^2)}}{\int dq_n \, e^{i\frac{imt}{4} \sum_n \omega_n^2}} = \sqrt{\frac{m}{i2\pi t}} \prod_{n=1}^{\infty} \left(1 - \frac{\omega^2}{\omega_n^2} \right)^{-1/2} \\ &= \sqrt{\frac{m}{i2\pi t}} \left(\frac{\omega t}{\sin(\omega t)} \right)^{1/2} . \end{aligned} \tag{2.56}$$

Above $\omega_n \equiv \frac{\pi n}{t}$. It is not difficult to check that, with the previous expression for $N_h(t)$, path integral (2.53) satisfies the Schrödinger equation with hamiltonian $H = -\frac{1}{2m}\frac{d^2}{dx^2} +$

$\frac{1}{2}m\omega^2 x^2$ and boundary condition $K(x, x'; 0) = \delta(x - x')$. The propagator can also be easily generalized

$$\Delta_\omega(\tau, \tau') = \frac{1}{\omega t \sin(\omega t)} \Big\{ \theta(\tau - \tau') \sin(\omega t(\tau - 1)) \sin(\omega t \tau')$$
$$+ \theta(\tau' - \tau) \sin(\omega t(\tau' - 1)) \sin(\omega t \tau) \Big\} \tag{2.57}$$

and can be used in the perturbative approach.

In the limit $t \to 0$ (or $\omega \to 0$), all previous expressions reduce to their free particle counterparts.

2.3.1 The harmonic oscillator partition function

Similarly to the free particle case we can switch to statistical mechanics by Wick rotating the time, $it = \beta$ and get

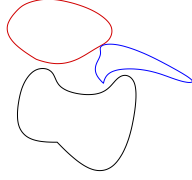$$K(x, x'; -i\beta) = e^{-S_h[x_{cl}]} \int_{q(0)=0}^{q(1)=0} Dq \, e^{-S_h[q]} \tag{2.58}$$

where now

$$S_h[x] = \frac{m}{2\beta} \int_0^1 d\tau \left(\dot{x}^2 + (\omega\beta)^2 x^2 \right) \tag{2.59}$$

is the euclidean action and

$$S_h[x_{cl}] = \frac{m\omega}{2\sinh(\beta\omega)} \Big[(x^2 + x'^2) \cosh(\beta\omega) - 2xx' \Big] \tag{2.60}$$

$$\int_{q(0)=0}^{q(1)=0} Dq \, e^{-S_h[q]} = \left(\frac{m\omega}{2\pi \sinh(\beta\omega)} \right)^{1/2} \tag{2.61}$$

that is always regular. Taking the trace of the amplitude (heat kernel) one gets the partition function

$$Z_h(\beta) = \int dx \, K_h(x, x; -i\beta) = \int_{PBC} Dx \, e^{-S_h[x]} = \sum \, \text{[figure]}$$
$$= \frac{1}{\sqrt{2}} \frac{1}{(\cosh(\beta\omega) - 1)^{1/2}} = \frac{e^{-\frac{\beta\omega}{2}}}{1 - e^{-\beta\omega}} \tag{2.62}$$

where PBC stands for "periodic boundary conditions" $x(0) = x(1)$ and implies that the path integral is a "sum" over all closed trajectories. Above we used the fact that the partition function for the free particle can be obtained either using a transition amplitude computed with Dirichlet boundary conditions $x(0) = x(1) = x$ and integrating the over the initial=final point x, or with periodic boundary conditions $x(0) = x(1)$, integrating over the "center of mass" of the path $x_0 \equiv \int_0^1 d\tau x(\tau)$.

It is easy to check that the latter matches the geometric series $\sum\limits_{n=0}^{\infty} e^{-\beta\omega(n+\frac{1}{2})}$. In particular, taking the zero-temperature limit ($\beta \to \infty$), the latter singles out the vacuum energy $Z_h(\beta) \approx e^{-\beta\frac{\omega}{2}}$, so that for generic particle models, the computation of the above path integral can be seen as a method to obtain an estimate of the vacuum energy. For example, in the case of an anharmonic oscillator, if the deviation from harmonicity is small, perturbation theory can be used to compute the correction to the vacuum energy.

2.3.2 Perturbation theory about the harmonic oscillator partition function solution

Perturbation theory about the harmonic oscillator partition function solution goes essentially the same way as we did for the free particle transition amplitude, except that now we may use periodic boundary conditions for the quantum fields rather than Dirichlet boundary conditions. Of course one can keep using DBC, factor out the classical solution, with $x_{cl}(0) = x_{cl}(1) = x$ and integrate over x. However for completeness let us choose the former parametrization and let us focus on the case where the interacting action is polynomial $S_{int}[x] = \beta \int_0^1 d\tau \sum_{n>2} \frac{g_n}{n!} x^n$; we can define the generating functional

$$\mathcal{Z}[j] = \int_{PBC} Dx\ e^{-S_h[x]+\int_0^1 jx} \tag{2.63}$$

that similarly to the free particle case yields

$$\mathcal{Z}[j] = Z_h(\beta)\ e^{\frac{1}{2}\iint jD_h^{-1}j} \tag{2.64}$$

and the propagator results

$$\left\langle x(\tau)x(\tau')\right\rangle = D_h^{-1}(\tau - \tau') \equiv G_\omega(\tau - \tau')$$
$$\frac{m}{\beta}(-\partial_\tau^2 + (\omega\beta)^2)G_\omega(\tau - \tau') = \delta(\tau - \tau') \tag{2.65}$$

in the space of functions with periodicity $\tau \cong \tau + 1$. On the infinite line the latter equation can be easily inverted using the Fourier transformation that yields

$$G_\omega^l(u) = \frac{1}{2m\omega}e^{-\beta\omega|u|}\ , \quad u \equiv \tau - \tau' \ . \tag{2.66}$$

In order to get to Green's function on the circle (i.e. with periodicity $\tau \cong \tau + 1$) we need to render (2.66) periodic [4]. Using Fourier analysis in (2.65) we get

$$G_\omega(u) = \frac{\beta}{m}\sum_{k\in\mathbb{Z}}\frac{1}{(\beta\omega)^2 + (2\pi k)^2}e^{i2\pi ku} = \frac{\beta}{m}\int_{-\infty}^{\infty}d\lambda\sum_{k\in\mathbb{Z}}\delta(\lambda + k)\frac{1}{(\beta\omega)^2 + (2\pi\lambda)^2}e^{i2\pi\lambda u} \tag{2.67}$$

where in the second passage we inserted an auxiliary integral. Now we can use the Poisson resummation formula $\sum_k f(k) = \sum_n \hat{f}(n)$ where $\hat{f}(\nu)$ is the Fourier transform of $f(x)$,
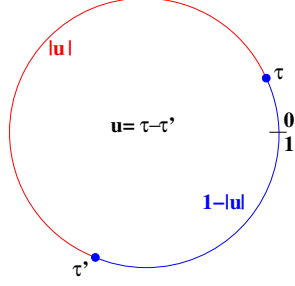
Figure 2. Shortest distance on the circle between τ and τ'

with ν being the frequency, i.e. $\hat{f}(\nu) = \int dx f(x) e^{-i2\pi\nu x}$. In the above case $\hat{\delta}(n) = e^{i2\pi\lambda n}$. Hence the leftover integral over λ yields

$$G_\omega(u) = \sum_{n \in \mathbb{Z}} G^l_\omega(u + n) \tag{2.68}$$

that is explicitly periodic. The latter sum involves simple geometric series that can be summed to give

$$G_\omega(u) = \frac{1}{2m\omega} \frac{\cosh\left(\omega\beta\left(\frac{1}{2} - |u|\right)\right)}{\sinh\left(\frac{\omega\beta}{2}\right)} \ . \tag{2.69}$$

This is the Green's function for the harmonic oscillator with periodic boundary conditions (on the circle). Notice that in the large β limit one gets an expression that is slightly different from the Green's function on the line (cfr. eq.(2.66)), namely:

$$G^\infty_\omega(u) = \frac{1}{2m\omega} \begin{cases} e^{-\beta\omega|u|} \ , & |u| < \frac{1}{2} \\ e^{-\beta\omega(1-|u|)} \ , & \frac{1}{2} < |u| < 1 \ . \end{cases} \tag{2.70}$$

Basically the Green's function is the exponential of the shortest distance (on the circle) between τ and τ', see Figure 2. The partition function for the anharmonic oscillator can be formally written as

$$Z_{ah}(\beta) = Z_h(\beta) \ e^{-S_{int}[\delta/\delta j]} \ e^{\frac{1}{2} \iint j G_\omega j}\Big|_{j=0}$$
$$= Z_h(\beta) \ e^{\{\text{connected diagrams}\}} \ . \tag{2.71}$$

As an example let us consider the case

$$S_{int}[x] = \beta \int_0^1 d\tau \left(\frac{g}{3!}x^3 + \frac{\lambda}{4!}x^4\right) = \beta \left(\text{⬤←⦚} \quad + \quad \text{⬤⦙⦙} \right) \tag{2.72}$$

so that, to lowest order

$$Z_{ah}(\beta) = Z_h(\beta) \exp\left\{ \beta \left(-3 \ \text{⬀⬁} \ + \ \beta/2! \left[\text{⑨ ◯⬤- -⬤◯} \ + 6 \ \text{⬤⤙⤚⬤} \right] + \cdots \right) \right\} \tag{2.73}$$

and the single diagrams read

$$\text{(diagram)} = \frac{\lambda}{4!} \int_0^1 d\tau \ \left(G_\omega(0)\right)^2 \overset{\beta\to\infty}{\longrightarrow} = \frac{\lambda}{4 \cdot 4!(m\omega)^2} \qquad (2.74)$$

$$\text{(diagram)} = \left(\frac{g}{3!}\right)^2 \int_0^1\!\!\int_0^1 \ \left(G_\omega(0)\right)^2 G_\omega(\tau - \tau') \overset{\beta\to\infty}{\longrightarrow} = \frac{g^2}{4(3!)^2 \beta m^3 \omega^4} \qquad (2.75)$$

$$\text{(diagram)} = \left(\frac{g}{3!}\right)^2 \int_0^1\!\!\int_0^1 \ \left(G_\omega(\tau - \tau')\right)^3 \overset{\beta\to\infty}{\longrightarrow} = \frac{g^2}{12(3!)^2 \beta m^3 \omega^4} \ . \qquad (2.76)$$

Then in the zero-temperature limit

$$Z_{ah}(\beta) \approx e^{-\beta E_0'} \qquad (2.77)$$

$$E_0' = \frac{\omega}{2} \left(1 + \frac{\lambda}{16m^2\omega^3} - \frac{11g^2}{144m^3\omega^5}\right) \qquad (2.78)$$

gives the sought estimate for the vacuum energy. However, the above expression (2.73) with diagrams (2.74,2.75,2.76) computed with the finite temperature Green's function (2.69) yields (the perturbative expansion for) the finite temperature partition function of the anharmonic oscillator described by (2.72).

2.4 Problems for Section 2

(1) Show that the classical action for the free particle on the line is $S_f[x_{cl}] = \frac{m(x-x')^2}{2t}$.

(2) Compute the 6-point correlation function.

(3) Compute the Seeley-DeWitt coefficient $a_3(x, x')$ both diagrammatically and in terms of vertex functions.

(4) Compute the classical action (2.54) for the harmonic oscillator on the line.

(5) Show that the transition amplitude (2.53) satisfies the Schrödinger equation for the harmonic oscillator.

(6) Show that the propagator (2.57) satisfies the Green equation $(\partial_\tau^2 + (\omega t)^2)\Delta_\omega(\tau, \tau') = \delta(\tau, \tau')$.

(7) Show that the propagator (2.66) satisfies eq. (2.65).

(8) Using Fourier transformation, derive (2.66) from (2.65).

(9) Using the geometric series obtain (2.69) from (2.68) .

(10) Check that, in the large β limit up to exponentially decreasing terms, expressions (2.74,2.75,2.76) give the same results, both with the Green's function (2.66) and with (2.70).

(11) Compute the next-order correction to the vacuum energy (2.78).

(12) Compute the finite temperature partition function for the anharmonic oscillator given by (2.72) to leading order in perturbation theory, i.e. only consider the eight-shaped diagram.

3 Path integral representation of quantum mechanical transition amplitude: fermionic degrees of freedom

We employ the coherent state approach to generalize the path integral to transition amplitude of models with fermionic degrees of freedom. The simplest fermionic system is a two-dimensional Hilbert space, representation of the anticommutators algebra

$$\{a, a^\dagger\} = 1 \,, \quad a^2 = (a^\dagger)^2 = 0 \,. \tag{3.1}$$

The spin basis for such algebra is given by $(|-\rangle, |+\rangle)$ where

$$a|-\rangle = 0 \,, \quad |+\rangle = a^\dagger|-\rangle \,, \quad |-\rangle = a|+\rangle \tag{3.2}$$

and a spin state is thus a two-dimensional object (a spinor) in such a basis. An alternative, overcomplete, basis for spin states is the so-called "coherent state basis" that, for the previous simple system, is simply given by the following bra's and ket's

$$
\begin{aligned}
|\xi\rangle &= e^{a^\dagger \xi}|0\rangle = (1 + a^\dagger \xi)|0\rangle & \rightarrow & \quad a|\xi\rangle = \xi|\xi\rangle \\
\langle \bar\eta| &= \langle 0|e^{\bar\eta a} = \langle 0|(1 + \bar\eta a) & \rightarrow & \quad \langle \bar\eta|a^\dagger = \langle \bar\eta|\bar\eta
\end{aligned}
\tag{3.3}
$$

and can be generalized to an arbitrary set of pairs of fermionic generators; see Appendix B for details. Coherent states (3.3) satisfy the following properties

$$\langle \bar\eta|\xi\rangle = e^{\bar\eta \xi} \tag{3.4}$$

$$\int d\bar\eta d\xi \; e^{-\bar\eta \xi}|\xi\rangle\langle \bar\eta| = \mathbb{1} \tag{3.5}$$

$$\int d\xi \; e^{(\bar\lambda - \bar\eta)\xi} = \delta(\bar\lambda - \bar\eta) \tag{3.6}$$

$$\int d\bar\eta \; e^{\bar\eta(\rho\xi)} = \delta(\rho - \xi) \tag{3.7}$$

$$\mathrm{tr}A = \int d\bar\eta d\xi \; e^{-\bar\eta \xi}\langle -\bar\eta|A|\xi\rangle = \int d\xi d\bar\eta \; e^{\bar\eta \xi}\langle \bar\eta|A|\xi\rangle \,. \tag{3.8}$$

Let us take $|\phi\rangle$ as initial state, then the evolved state will be

$$|\phi(t)\rangle = e^{-it\mathbb{H}}|\phi\rangle \tag{3.9}$$

that in the coherent state representation becomes

$$\phi(\bar\lambda; t) \equiv \langle \bar\lambda|\phi(t)\rangle = \langle \bar\lambda|e^{-it\mathbb{H}}|\phi\rangle = \int d\bar\eta d\eta e^{-\bar\eta \eta}\langle \bar\lambda|e^{-it\mathbb{H}}|\eta\rangle\phi(\bar\eta; 0) \tag{3.10}$$

where in the last equality we have used property (3.5). The integrand $\langle \bar\lambda|e^{-it\mathbb{H}}|\eta\rangle$ in (3.10) assumes the form of a transition amplitude as in the bosonic case. It is thus possible to

represent it with a fermionic path integral. In order to do that let us first take the trivial case $\mathbb{H} = 0$ and insert N decompositions of identity $\int d\bar{\xi}_i d\xi_i \; e^{-\bar{\xi}_i \xi_i} |\xi_i\rangle\langle\bar{\xi}_i| = \mathbb{1}$. We thus get

$$\langle \bar{\lambda}|\eta\rangle = \int \prod_{i=1}^{N} d\bar{\xi}_i d\xi_i \; \exp\left[\bar{\lambda}\xi_N - \sum_{i=1}^{N} \bar{\xi}_i (\xi_i - \xi_{i-1})\right], \quad \bar{\xi}_N \equiv \bar{\lambda}, \; \xi_0 \equiv \eta \tag{3.11}$$

that in the large N limit can be written as

$$\langle \bar{\lambda}|\eta\rangle = \int_{\xi(0)=\eta}^{\bar{\xi}(1)=\bar{\lambda}} D\bar{\xi}D\xi \; e^{iS[\xi,\bar{\xi}]} \tag{3.12}$$

with

$$S[\xi,\bar{\xi}] = i\left(\int_0^1 d\tau \, \bar{\xi}\dot{\xi}(\tau) - \bar{\xi}\xi(1)\right) . \tag{3.13}$$

In the presence of a nontrivial hamiltonian $\mathbb{H}$ the latter becomes

$$\langle \bar{\lambda}|e^{-it\mathbb{H}}|\eta\rangle = \int_{\xi(0)=\eta}^{\bar{\xi}(1)=\bar{\lambda}} D\bar{\xi}D\xi \; e^{iS[\xi,\bar{\xi}]} \tag{3.14}$$

$$S[\xi,\bar{\xi}] = \int_0^1 d\tau \left(i\bar{\xi}\dot{\xi}(\tau) - H(\xi,\bar{\xi})\right) - i\bar{\xi}\xi(1) \tag{3.15}$$

that is the path integral representation of the fermionic transition amplitude. Here a few comments are in order: (a) the fermionic path integral resembles more a bosonic phase space path integral than a configuration space one. (b) The boundary term $\bar{\xi}\xi(1)$, that naturally comes out from the previous construction, plays a role when extremizing the action to get the equations of motion; namely, it cancels another boundary term that comes out from partial integration. It also plays a role when computing the trace of an operator: see below. (c) The generalization from the above naive case to (3.14) is not a priori trivial, because of ordering problems. In fact $\mathbb{H}$ may involve mixing terms between a and $a^\dagger$. However result (3.14) is guaranteed in that form (i.e. the quantum $\mathbb{H}(a, a^\dagger)$ is replaced by $H(\xi,\bar{\xi})$ without the addition of counterterms) if the hamiltonian operator $\mathbb{H}(a, a^\dagger)$ is written in (anti-)symmetric form: for the present simple model it simply means $\mathbb{H}_S(a, a^\dagger) = c_0 + c_1 a + c_2 a^\dagger + c_3(aa^\dagger - a^\dagger a)$. In general the hamiltonian will not have such form and it is necessary to order it $\mathbb{H} = \mathbb{H}_S + $ "counterterms", where "counterterms" come from anticommunting a and $a^\dagger$ in order to put $\mathbb{H}$ in symmetrized form. The present ordering is called Weyl-ordering. For details about Weyl ordering in bosonic and fermionic path integrals see [3].

Let us now compute the trace of the evolution operator. It yields

$$\text{tr } e^{-it\mathbb{H}} = \int d\eta d\bar{\lambda} \; e^{\bar{\lambda}\eta}\langle \bar{\lambda}|e^{-it\mathbb{H}}|\eta\rangle = \int d\eta \int_{\xi(0)=\eta}^{\bar{\xi}(1)=\bar{\lambda}} D\bar{\xi}D\xi d\bar{\lambda} \; e^{\bar{\lambda}(\eta+\xi(1))}e^{i\int_0^1 (i\bar{\xi}\dot{\xi}-H)} \tag{3.16}$$

then the integral over $\bar{\lambda}$ give a Dirac delta that can be integrated with respect to η. Hence,

$$\text{tr } e^{-it\mathbb{H}} = \int_{\xi(0)=-\xi(1)} D\bar{\xi}D\xi \; e^{i\int_0^1 (i\bar{\xi}\dot{\xi}-H(\xi,\bar{\xi}))} \tag{3.17}$$

where we notice that the trace in the fermionic variables corresponds to a path integral with anti-periodic boundary conditions (ABC), as opposed to the periodic boundary conditions of the bosonic case. Finally, we can rewrite the latter by using real (Majorana) fermions defined as

$$\xi = \frac{1}{\sqrt{2}}\left(\psi^1 + i\psi^2\right), \quad \bar{\xi} = \frac{1}{\sqrt{2}}\left(\psi^1 - i\psi^2\right) \tag{3.18}$$

and

$$\text{tr } e^{-it\mathbb{H}} = \int_{ABC} D\psi \; e^{i\int_0^1 (\frac{i}{2}\psi_a\dot{\psi}^a - H(\psi))} \; . \tag{3.19}$$

In particular

$$2 = \text{tr}\mathbb{1} = \int_{ABC} D\psi \; e^{i\int_0^1 \frac{i}{2}\psi_a\dot{\psi}^a}, \quad a = 1, 2 \; . \tag{3.20}$$

For an arbitrary number of fermionic operator pairs a_i, $a_i^\dagger$, $i = 1, ..., l$, the latter of course generalizes to

$$2^{D/2} = \text{tr}\mathbb{1} = \int_{ABC} D\psi \; e^{i\int_0^1 \frac{i}{2}\psi_a\dot{\psi}^a}, \quad a = 1, ..., D = 2l \tag{3.21}$$

that sets the normalization of the fermionic path integral with anti-periodic boundary conditions. The latter fermionic action plays a fundamental role in the description of relativistic spinning particles, that is the subject of the next section.

3.1 Problems for Section 3

(1) Show that the ket and bra defined in (3.3) are eigenstates of a and $a^\dagger$ respectively.

(2) Demonstrate properties (3.4)-(3.7).

(3) Test property (3.8) using $A = \mathbb{1}$.

(4) Obtain the equations of motion from action (3.15) and check that the boundary terms cancel.

4 Relativistic particles: bosonic particles and $O(N)$ spinning particles

We consider a generalization of the previous results to relativistic particles in flat space. In order to do that we start analyzing particle models at the classical level, then consider their quantization, in terms of canonical quantization and path integrals.

4.1 Bosonic particles: symmetries and quantization. The worldline formalism

For a nonrelativistic free particle in d-dimensional space, at the classical level we have

$$S[\mathbf{x}] = \frac{m}{2}\int_0^t d\tau \; \dot{\mathbf{x}}^2, \quad \mathbf{x} = (x^i), \; i = 1, ..., d \tag{4.1}$$

that is invariant under a set of continuous global symmetries that correspond to an equal set of conserved charges.

- time translation $\delta x^i = \xi \dot{x}^i \;\longrightarrow\; E = \frac{m}{2}\dot{\mathbf{x}}^2$, the energy

- space translations $\delta x^i = a^i \;\longrightarrow\; P^i = m\dot{x}^i$, linear momentum

- spatial rotations $\delta x^i = \theta^{ij} x^j \;\longrightarrow\; L^{ij} = m(x^i\dot{x}^j - x^j\dot{x}^i)$, angular momentum

- Galilean boosts $\delta x^i = v^i t \;\longrightarrow\; x_0^i : \; x^i = x_0^i + P^i t$, center of mass motion.

These symmetries are isometries of a one-dimensional euclidean space (the time) and a three-dimensional euclidean space (the space). However the latter action is not, of course, Lorentz invariant.

A Lorentz-invariant generalization of the free-particle action can be simply obtained by starting from the Minkowski line element $ds^2 = -dt^2 + d\mathbf{x}^2$. For a particle described by $\mathbf{x}(t)$ we have $ds^2 = -(1 - \dot{\mathbf{x}}^2)dt^2$ that is Lorentz-invariant and measures the (squared) proper time of the particle along its path. Hence the action (referred to as geometric action) for the massive free particle reads

$$S[\mathbf{x}] = -m \int_0^t dt\sqrt{1 - \dot{\mathbf{x}}^2} \tag{4.2}$$

that is, by construction, invariant under the Poincarè group of transformations

- $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu , \quad x^\mu = (t, x^i) , \; \Lambda \in SO(1,3) , \; a^\mu \in \mathbb{R}^{1,3}$

the isometry group of Minkowski space. Conserved charges are four-momentum, angular momentum and center of mass motion (from Lorentz boosts). The above action can be reformulated by making x^0 a dynamical field as well in order to render the action explicitly Lorentz-invariant. It can be achieved by introducing a gauge symmetry. Hence

$$S[x] = -m \int_0^1 d\tau \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \tag{4.3}$$

where now $\dot{} = \frac{d}{d\tau}$, and $\eta_{\mu\nu}$ is the Minkowski metric. The latter is indeed explicitly Lorentz-invariant as it is written in four-tensor notation and it also gauge invariant upon the reparametrization $\tau \to \tau'(\tau)$. Action (4.2) can be recovered upon gauge choice $x^0 = t\tau$. Yet another action for the relativistic particle can be obtained by introducing a gauge fields, the einbein e, that renders explicit the above gauge invariance.

$$S[x, e] = \int_0^1 d\tau \left(\frac{1}{2e}\dot{x}^2 - \frac{m^2 e}{2} \right) . \tag{4.4}$$

For an infinitesimal time reparametrization

$$\delta\tau = -\xi(\tau) , \quad \delta x^\mu = \xi \dot{x}^\mu , \quad \delta e = \left(e\xi \right)^\bullet \tag{4.5}$$

we have $\delta S[x, e] = \int d\tau \left(\xi L \right)^\bullet = 0$. Now a few comments are in order: (a) action (4.3) can be recovered by replacing e with its on-shell expression; namely,

$$0 = m^2 + \frac{1}{e^2}\dot{x}^2 \quad \Rightarrow \quad e = \frac{\sqrt{-\dot{x}^2}}{m} \; ; \tag{4.6}$$

(b) unlike the above geometric actions, expression (4.4), that is known as Brink-di Vecchia-Howe action, is also suitable for massless particles; (c) equation(4.4) is quadratic in x and therefore is more easily quantizable. In fact we can switch to phase-space action by taking $p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \dot{x}_\mu / e$ (e has vanishing conjugate momentum, it yields a constraint)

$$S[x, p, e] = \int_0^1 d\tau \Big[p_\mu \dot{x}^\mu - e\frac{1}{2}\big(p^2 + m^2\big) \Big] \tag{4.7}$$

which is like a standard (nonrelativistic) hamiltonian action, with hamiltonian $\mathcal{H} = e\frac{1}{2}\big(p^2 + m^2\big) \equiv eH_0$ and phase space constraint $H_0 = 0$. The constraint H_0 works also as gauge symmetry generator $\delta x^\mu = \{x^\mu, \xi H_0\} = \xi p^\mu$ and, by requiring that $\delta S = 0$, one gets $\delta e = \dot{\xi}$. Here $\{\ ,\ \}$ are Poisson brackets.

Upon canonical quantization the dynamics is governed by a Schrödinger equation with hamiltonian operator $\mathcal{H}$ and the constraint is an operatorial constraint that must be imposed on physical states

$$i\partial_\tau |\phi(\tau)\rangle = \mathcal{H}|\phi(\tau)\rangle = eH_0|\phi(\tau)\rangle = 0 \tag{4.8}$$
$$\Rightarrow \quad (p^2 + m^2)|\phi\rangle \tag{4.9}$$

with $|\phi\rangle$ being τ independent. In the coordinate representation (4.9) is nothing but Klein-Gordon equation. In conclusion the canonical quantization of the relativistic, 1d-reparametrization invariant particle action (4.4) yields Klein-Gordon equation for the wave function. This is the essence of the "worldline formalism" that uses (the quantization) of particle models to obtain results in quantum field theory (see [5] for a review of the method). Another important comment here is that the local symmetry (1d reparametrization) ensures the propagation of physical degrees of freedom; i.e. it guarantees unitarity. Before switching to path integrals let us consider the coupling to external fields: in order to achieve that, one needs to covariantize the particle momentum in H_0. For a coupling to a vector field

$$p_\mu \quad \rightarrow \quad \pi_\mu = p_\mu - qA_\mu \quad \Rightarrow \quad \{\pi_\mu, \pi_\nu\} = qF_{\mu\nu} \tag{4.10}$$
$$H_0 = \frac{1}{2}\Big(\eta^{\mu\nu}\pi_\mu\pi_\nu + m^2\Big) \tag{4.11}$$

and

$$S[x, p, e; A_\mu] = \int_0^1 d\tau \Big[p_\mu \dot{x}^\mu - e\frac{1}{2}\big(\pi^2 + m^2\big) \Big] \tag{4.12}$$

with q being the charge of the particle and $F_{\mu\nu}$ the vector field strength. Above the vector field is in general nonabelian $A_\mu = A_\mu^a T_a$ and $T_a \in$ Lie algebra of a gauge group G. For the bosonic particle, (minimal) coupling to gravity is immediate to achieve, and amounts to the replacement $\eta_{\mu\nu} \rightarrow g_{\mu\nu}(x)$; for a spinning particle it would be rather more involved (see e.g. [6]), but will not be treated here. In order to switch to configuration space we just solve for π_μ in (4.12), $\pi_\mu = \eta_{\mu\nu}\dot{x}^\nu / e$ and get

$$S[x, e; A_\mu] = \int_0^1 d\tau \Big[\frac{1}{2e}\dot{x}^2 - \frac{e}{2}m^2 + q\dot{x}^\mu A_\mu \Big] \tag{4.13}$$

so, although the hamiltonian involves a term quadratic in A_μ, the coupling between particle and external vector field in configuration space is linear. Moreover, for an abelian vector field, action (4.13) is gauge invariant upon $A_\mu \to A_\mu + \partial_\mu \alpha$. For a nonabelian vector field, whose gauge transformation is $A_\mu \to U^{-1}(A_\mu - i\partial_\mu)U$, with $U = e^{i\alpha}$ action (4.13) is not gauge invariant; however in the path integral the action enters in the exponent so it is possible to give the following gauge-invariant prescription

$$e^{iS[x,e;A_\mu]} \quad \longrightarrow \quad \mathrm{tr}\left(\mathcal{P}e^{iS[x,e;A_\mu]}\right) \tag{4.14}$$

i.e. we replace the simple exponential with a Wilson line. Here $\mathcal{P}$ defines the "path ordering" that, for the worldline integral $e^{iq\int_0^1 \dot{x}^\mu A_\mu}$, is nothing but the "time-ordering" mentioned in footnote 1; namely

$$\mathcal{P}e^{iq\int_0^1 \dot{x}^\mu A_\mu} = 1 + iq\int_0^1 d\tau \; \dot{x}^\mu A_\mu + (iq)^2 \int_0^1 d\tau_1 \; \dot{x}^{\mu_1} A_{\mu_1} \int_0^{\tau_1} d\tau_2 \; \dot{x}^{\mu_2} A_{\mu_2} + \cdots \tag{4.15}$$

that transforms covariantly $\mathcal{P}e^{iq\int_0^1 \dot{x}^\mu A_\mu} \to U^{-1}\mathcal{P}e^{iq\int_0^1 \dot{x}^\mu A_\mu}U$, so that the trace is gauge-invariant, and that, for abelian fields, reduces to the conventional expansion for the exponential.

Let us now consider a path integral for the action (4.13). For convenience we consider its Wick rotated ($i\tau \to \tau$) version (we also change $q \to -q$)

$$S[x,e;A_\mu] = \int_0^1 d\tau\left[\frac{1}{2e}\dot{x}^2 + \frac{e}{2}m^2 + iq\dot{x}^\mu A_\mu\right] \tag{4.16}$$

for which the path integral formally reads

$$\int \frac{DxDe}{\mathrm{Vol\,(Gauge)}} \; e^{-S[x,e;A_\mu]} \tag{4.17}$$

where "Vol (Gauge)" refers to the fact that we have to divide out all configurations that are equivalent upon gauge symmetry, that in this case reduces to 1d reparametrization. The previous path integral can be taken over two possible topologies: on the *line* where $x(0) = x'$ and $x(1) = x$, and on the *circle* for which bosonic fields have periodic boundary conditions, in particular $x(0) = x(1)$. However, such path integrals can be used to compute more generic tree-level and (multi-)loop graphs [7, 8].

4.1.1 QM Path integral on the line: QFT propagator

Worldline path integrals on the line are linked to quantum field theory propagators. In particular, for the above 1d-reparametrization invariant bosonic model, coupled to external abelian vector field, one obtains the full propagator of scalar quantum electrodynamics (QED), i.e. a scalar propagator with insertion of an arbitrary number of couplings to A_μ.

On the line we keep fixed the extrema of $x(\tau)$ and the gauge parameter is thus constrained as $\xi(0) = \xi(1) = 0$, and the einbein can be gauge-away to an arbitrary positive constant $\hat{e} \equiv 2T$ where

$$2T \equiv \int_0^1 d\tau \; e, \quad \delta(2T) = \int_0^1 d\tau \; (e\xi)^\bullet = 0 \tag{4.18}$$

and therefore

$$De = dT D\xi \tag{4.19}$$

where $D\xi$ is the measure of the gauge group. Moreover there are no Killing vector as $(\hat{e}\xi)^\bullet = 0$ on the line has only a trivial solution $\xi = 0$. Hence the gauge-fixed path integral reads

$$\left\langle \phi(x)\phi(x') \right\rangle_A = \int_0^\infty dT \int_{x(0)=x'}^{x(1)=x} Dx \ e^{-S[x,2T;A_\mu]} \tag{4.20}$$

and

$$S[x, 2T; A_\mu] = \int_0^1 d\tau \left(\frac{1}{4T} \dot{x}^2 + Tm^2 + iq\dot{x}^\mu A_\mu \right) \tag{4.21}$$

is the gauge-fixed action. For $A_\mu = 0$ it is easy to convince oneself that (4.20) reproduces the free bosonic propagator; in fact

$$\int_0^\infty dT \int_{x(0)=x'}^{x(1)=x} Dx \ e^{-\int_0^1 (\frac{\dot{x}^2}{4T}+Tm^2)} = \int_0^\infty dT \ \langle x| e^{-T(\mathbb{p}^2+m^2)} |x'\rangle = \langle x| \frac{1}{\mathbb{p}^2 + m^2} |x'\rangle \ . \tag{4.22}$$

In perturbation theory, about trivial vector field background, with perturbation by $A_\mu(x) = \sum_i \epsilon_\mu^i e^{ip_i \cdot x}$, i.e. sum of external photons, expression (4.20) is nothing but the sum of the following Feynman diagrams

$$\int_0^\infty dT \int_{x(0)=x'}^{x(1)=x} Dx \ e^{-S[x,2T;A_\mu]} = \text{———} + \text{———}$$

$$+ \ \text{———} + \text{———} + \cdots \tag{4.23}$$

as two types of vertices appear in scalar QED

$$iqA_\mu(\bar{\phi}\partial_\mu\phi - \phi\partial_\mu\bar{\phi}) \quad \longrightarrow \quad \text{⤙} \quad , \quad q^2 A_\mu A^\mu \bar{\phi}\phi \quad \longrightarrow \quad \text{✕} \tag{4.24}$$

i.e. the linear vertex and the so-called "seagull" vertex. It is interesting to note that the previous expression for the propagator of scalar QED was already proposed by Feynman in his famous "Mathematical formulation of the quantum theory of electromagnetic interaction," [9] where he also included the interaction with an arbitrary number of virtual photons emitted and re-absorbed along the trajectory of the scalar particle.

4.1.2 QM Path integral on the circle: one loop QFT effective action

Worldline path integrals on the circle are linked to quantum field theory one loop effective actions. With the particle model of (4.16) it yields the one loop effective action of scalar QED. The gauge fixing goes similarly to previous case, except that on the circle we have periodic conditions $\xi(0) = \xi(1)$. This leaves a non-trivial solution, $\xi = $ constant, for the Killing equation that corresponds to the arbitrariness on the choice of origin of the circle. One takes care of this further symmetry, dividing by the length of the circle itself. Therefore

$$\Gamma[A_\mu] = \int_0^\infty \frac{dT}{T} \int_{PBC} Dx \; e^{-S[x,2T;A_\mu]} \tag{4.25}$$

yields the worldline representation for the one loop effective action for scalar QED. Perturbatively the latter corresponds to the following sum of one particle irreducible Feynman diagrams

$$\Gamma[A_\mu] = \sum \quad \tag{4.26}$$

i.e. it corresponds to the sum of one-loop photon amplitudes. The figure above is meant to schematically convey the information that scalar QED effective action involves both types of vertices. Further details about the many applications of the previous effective action representation will be given by Christian Schubert in his lectures set [10].

4.2 Spinning particles: symmetries and quantization. The worldline formalism

We can extend the phase space bosonic form by adding fermionic degrees of freedom. For example we can add Majorana worldline fermions that carry a space-time index μ and an internal index i and get

$$I_{sf}[x,p,\psi] = \int_0^1 d\tau \left(p_\mu \dot{x}^\mu + \frac{i}{2} \psi_{\mu i} \dot{\psi}_i^\mu \right) , \quad i = 1, \ldots, N . \tag{4.27}$$

The latter expression is invariant under the following set of continuous global symmetries, with their associated conserved Noether charges

- time translation: $\delta x^\mu = \xi p^\mu$, $\delta p_\mu = \delta \psi_i^\mu = 0 \longrightarrow H_0 = \frac{1}{2} p_\mu p^\mu$

- supersymmetries: $\delta x^\mu = i\epsilon_i \psi_i^\mu$, $\delta p^\mu = 0$, $\delta \psi_i^\mu = -\epsilon_i p^\mu \longrightarrow Q_i = p_\mu \psi_i^\mu$

- $O(N)$ rotations: $\delta x^\mu = \delta p_\mu = 0$, $\delta \psi_i^\mu = \alpha_{ij} \delta \psi_j^\mu \longrightarrow J_{ij} = i\psi_{\mu i} \psi_j^\mu$

with arbitrary constant parameters ξ, ϵ_i, α_{ij}, and $\alpha_{ij} = -\alpha_{ij}$. Conserved charged also work as symmetry generators $\delta z = \{z, g\}$ with $z = (x, p, \psi)$ and $G = \Xi^A G_A \equiv \xi H_0 + i\epsilon_i Q_i + \frac{1}{2}\alpha_{ij} J_{ij}$, and $\{ , \}$ being graded Poisson brackets; in flat space the generators G_A satisfies

a first-class algebra $\{G_A, G_B\} = C_{AB}^C G_C$, see [6] for details. Taking the parameters to be time-dependent we have that the previous symplectic form transforms as

$$\delta I_{sf}[x, p, \psi] = \int_0^1 d\tau \Big(\dot{\xi} H_0 + i \dot{\epsilon}_i Q_i + \frac{1}{2} \dot{\alpha}_{ij} J_{ij} \Big) \tag{4.28}$$

so that we can add gauge fields $E = (e, \chi_i, a_{ij})$ and get the following locally-symmetric particle action

$$S[x, p, \psi, E] = \int_0^1 d\tau \Big(p_\mu \dot{x}^\mu + \frac{i}{2} \psi_{\mu i} \dot{\psi}_i^\mu - e H_0 - i \chi_i Q_i - \frac{1}{2} a_{ij} J_{ij} \Big) \ . \tag{4.29}$$

This is a spinning particle model with gauged $O(N)$-extended supersymmetry. The fact that the symmetry algebra is first class ensures that (4.29) is invariant under the local symmetry generated by $G = \Xi^A(\tau) G_A$, provided the fields E transform as

$$\begin{aligned} \delta e &= \dot{\xi} + 2i \chi_i Q_i \\ \delta \chi_i &= \dot{\epsilon}_i - a_{ij} Q_j + \alpha_{ij} \chi_j \\ \delta a_{ij} &= \dot{\alpha}_{ij} + \alpha_{im} a_{mj} - a_{im} \alpha_{mj} \end{aligned} \tag{4.30}$$

from which it is clear that they are gauge fields.

Upon canonical quantization Poisson brackets turn into (anti-)commutators $[p_\mu, x^\nu] = -i \delta_\mu^\nu$, $\{\psi_i^\mu, \psi_j^\nu\} = \delta_{ij} \eta^{\mu\nu}$, where $\{\ ,\ \}$ now represent anti-commutators. One possible representation of the previous fermionic algebra, that is nothing but a multi-Clifford algebra, is the spin-basis, where ψ_i^μ are represented as Gamma-matrices. So, in the spin-basis and in bosonic coordinate representation the wave function is a multispinor $\phi_{\alpha_1 \cdots \alpha_N}(x)$ where ψ_i^μ acts as Gamma-matrix on the i−th α−index. First class constraints again act *à la Dirac-Gupta-Bleuler* on the wave function. In particular the susy constraints

$$Q_i |\phi\rangle = 0 \quad \longrightarrow \quad \big(\gamma^\mu \big)_{\alpha_i \tilde{\alpha}_i} \partial_\mu \phi_{\alpha_1 \cdots \tilde{\alpha}_i \cdots \alpha_N}(x) = 0 \tag{4.31}$$

amount to N massless Dirac equations, whereas the $O(N)$ constraints

$$J_{ij} |\phi\rangle = 0 \quad \longrightarrow \quad \big(\gamma^\mu \big)_{\alpha_i \tilde{\alpha}_i} \big(\gamma_\mu \big)_{\alpha_j \tilde{\alpha}_j} \phi_{\alpha_1 \cdots \tilde{\alpha}_i \cdots \tilde{\alpha}_j \cdots \alpha_N}(x) = 0 \tag{4.32}$$

are "irreducibility" constraints, i.e. they impose the propagation of a field that is described by a single Young tableau of $SO(1, D-1)$, with $N/2$ columns and $D/2$ rows. The previous set of constraints yields Bargmann-Wigner equations for spin-$N/2$ fields in flat space. For generic N only particle models in even dimensions are non-empty, whereas for $N \leq 2$ the $O(N)$ constraints are either trivial or abelian and the corresponding spinning particle models can be extended to odd-dimensional spaces.

Coupling to external fields is now much less trivial. Coupling to gravity can be achieved by covariantizing momenta, and thus susy generators; however, for $N > 2$, in a generically curved background the constraints algebra ceases to be first class. For conformally flat spaces the algebra turns into a first-class non-linear algebra that thus describes the propagation of spin-$N/2$ fields in such spaces [6].

4.2.1 $N = 1$ spinning particle: coupling to vector fields

We consider the spinning particle model with $N = 1$ that describes the first quantization of a Dirac field. For the free model, at the classical level, the constraint algebra is simply

$$\{Q, Q\} = -2iH_0, \quad \{Q, H_0\} = 0 \tag{4.33}$$

that is indeed first class. To couple the particle model to an external vector field we covariantize the momentum as in (4.10), and consequently

$$Q \equiv \pi_\mu \psi^\mu, \quad \{Q, Q\} = -2iH \tag{4.34}$$

with

$$H = \frac{1}{2}\eta^{\mu\nu}\pi_\mu\pi_\nu + \frac{i}{2}qF_{\mu\nu}\psi^\mu\psi^\nu \tag{4.35}$$

and the phase-space locally symmetric action reads

$$S[x, p, \psi, e, \chi; A_\mu] = \int_0^1 d\tau \Big[p_\mu \dot{x}^\mu + \frac{i}{2}\psi_\mu \dot{\psi}^\mu - eH - i\chi Q \Big] \tag{4.36}$$

whereas

$$S[x, \psi, e, \chi; A_\mu] = \int_0^1 d\tau \Big[\frac{1}{2e}\eta_{\mu\nu}(\dot{x}^\mu - i\chi\psi^\mu)(\dot{x}^\nu - i\chi\psi^\nu) + \frac{i}{2}\psi_\mu\dot{\psi}^\mu$$
$$+ q\dot{x}^\mu A_\mu - eq\frac{i}{2}F_{\mu\nu}\psi^\mu\psi^\nu \Big] \tag{4.37}$$

is the locally symmetric configuration space action where, along with the bosonic coupling found previously, a Pauli-type coupling between field strength and spin appears.

4.2.2 QM Path integral on the circle: one loop QFT effective action

We now consider the above spinning particle models on a path integral on the circle, i.e. we consider the one loop effective actions produced by the spin-$N/2$ fields whose first quantization is described by the spinning particle models. On the circle (fermionic) bosonic fields have (anti-)periodic boundary conditions. It is thus not difficult to convince oneself that gravitini χ_i can be gauged-away completely. For the $N = 1$ model of the previous section this yields the spinor QED effective action

$$\Gamma[A_\mu] = \int_0^\infty \frac{dT}{2T} \int_{PBC} Dx \int_{ABC} D\psi \, e^{-S[x,\psi,2T,0;A_\mu]} \tag{4.38}$$

with

$$S[x, \psi, 2T, 0; A_\mu] = \int_0^1 d\tau \Big[\frac{1}{4T}\dot{x}^2 + \frac{1}{2}\psi_\mu\dot{\psi}^\mu + iq\dot{x}^\mu A_\mu - iTqF_{\mu\nu}\psi^\mu\psi^\nu \Big] \tag{4.39}$$

being the (euclidean) gauge-fixed spinning particle action, that is globally supersymmetric. Perturbatively the previous path integral is the sum of one particle irreducible diagrams with external photons and a Dirac fermion in the loop.

For arbitrary N we will not consider the coupling to external fields as it too much of an involved topic to be covered here. The interested reader may consult the recent manuscript [11] and references therein. Let us consider the circle path integral for the free $O(N)$−extended spinning particle. The euclidean configuration space action can be obtained from (4.29) by solving for the particle momenta and Wick rotating. We thus get

$$S[x, \psi, E] = \int_0^1 d\tau \left[\frac{1}{2e} \eta_{\mu\nu} (\dot{x}^\mu - \chi_i \psi_i^\mu)(\dot{x}^\nu - \chi_i \psi_i^\nu) + \frac{1}{2} \psi_\mu \dot{\psi}^\mu - \frac{1}{2} a_{ij} \psi_{\mu i} \psi_j^\mu \right] \qquad (4.40)$$

that yields the circle path integral

$$\Gamma = \frac{1}{\text{Vol (Gauge)}} \int_{PBC} DxDeDa \int_{ABC} D\psi D\chi \, e^{-S[x,\psi,E]} \, . \qquad (4.41)$$

Using (4.30), with antiperiodic boundary conditions for fermions, gravitini can be gauged away completely, $\chi_i = 0$. On the other hand $O(N)$ gauge fields enter with periodic boundary conditions and cannot be gauged away completely. In fact, as shown in [12] they can be gauged to a skew-diagonal constant matrix parametrized by $n = [N/2]$ angular variables, θ_k. The whole effective action is thus proportional to the number of degrees of freedom of fields described by a Young tableau with n columns and $D/2$ rows. Such Young tableaux correspond to the field strengths of higher-spin fields. For $D = 4$ this involves all possible massless representations of the Poincaré group, that at the level of gauge potentials are given by totally symmetric (spinor-) tensors, whereas for $D > 4$ it corresponds to conformal multiplets only.

4.3 Problems for Section 4

(1) Use the Noether trick to obtain the conserved charges for the free particle described by the geometric action (4.2).

(2) Repeat the previous problem with action (4.3).

(3) Show that the interaction term $L_{int} = q\dot{x}^\mu A_\mu$, with $A^\mu = (\phi, \mathbf{A})$, yields the Lorentz force.

(4) Show that, with time-dependent symmetry parameters, the symplectic form transforms as (4.28).

(5) Show that action (4.39) is invariant under global susy $\delta x^\mu = \epsilon \psi^\mu$, $\delta \psi^\mu = -\frac{1}{2T} \epsilon \dot{x}^\mu$.

Acknowledgments

5 Final Comments

What included in the present manuscript is an expanded version of a set of lectures given in Morelia (Mexico) in November 2012, where we reviewed some elementary material on particle path integrals and introduced a list of (more or less) recent topics where particle path integrals can be efficiently applied. In particular we pointed out the role that some locally-symmetric relativistic particle models have in the first-quantized description of higher spin fields [13]. However the list of topics reviewed here is by no means complete. Firstly, coupling to external gravity has been overlooked (it has been partly covered in [10].) In fact, path integrals in curved spaces involve nonlinear sigma models and perturbative calculations have to be done carefully as superficial divergences appear. Although this issue is well studied and understood by now, it had been source of controversy (and errors) in the past (see [14] for a review.) Moreover, for the $O(N)$ spinning particle models discussed above, coupling to gravity is not straightforward as (for generic N) the background appears to be constrained [6] and the symmetry algebra is not linear and thus the topic is an on-going research argument (see [11] for some recent results.)

Some other modern applications of particle path integrals that have not been covered here, include: the numerical worldline approach to the Casimir effect [15], AdS/CFT correspondence and string dualities [16], photon-graviton amplitudes computations [17], nonperturbative worldline methods in QFT [4, 18], QFT in curved space time [19], the worldgraph approach to QFT [8], as well as the worldline approach to QFT on manifolds with boundary [20] and to noncommutative QFT [21].

A Natural Units

In quantum field theory it is often convenient to use so called "natural units": for a generic physical quantity X, its physical units can always be express in terms of energy, angular-momentum $\times$ velocity and angular momentum as

$$[X] = E^a (Lc)^b L^c = m^\alpha l^\beta t^\gamma \tag{A.1}$$

with

$$\begin{aligned}
a &= \alpha - \beta - \gamma \\
b &= \beta - 2\alpha \\
c &= \gamma + 2\alpha \ .
\end{aligned} \tag{A.2}$$

Therefore, if velocities are measured in units of the speed of light c and angular momenta are measured in units of the Planck constant $\hbar$, and thus are numbers, we have the natural units for X given by

$$[X]\big|_{n.u.} = E^{\alpha - \beta - \gamma}\,, \quad (\hbar c)\big|_{n.u.} = 1\,, \quad \hbar\big|_{n.u.} = 1 \tag{A.3}$$

and energy is normally given in MeV. Conversion to standard units is then easily obtained by using

$$\hbar c = 1.97 \times 10^{-11} \text{ MeV} \cdot \text{cm}\,, \quad \hbar = 6.58 \times 10^{-22} \text{ MeV} \cdot \text{s} \ . \tag{A.4}$$

For example a distance expressed in natural units by $d = 1$ (MeV)$^{-1}$, corresponds to $d = 1.97 \times 10^{-11}$ cm.

B Fermionic coherent states

This is a compendium of properties of fermionic coherent states. Further details can be found for instance in [22].

The even-dimensional Clifford algebra

$$\{\psi^M, \psi^N\} = \delta^{MN}, \quad M, N = 1, \ldots, 2l \tag{B.1}$$

can be written as a set of l fermionic harmonic oscillators (the index M may collectively denote a set of indices that may involve internal indices as well as a space-time index), by simply taking complex combinations of the previous operators

$$a^m = \frac{1}{\sqrt{2}} \left(\psi^m + i\psi^{m+l} \right) \tag{B.2}$$

$$a_m^\dagger = \frac{1}{\sqrt{2}} \left(\psi^m - i\psi^{m+l} \right), \quad m = 1, \ldots, l \tag{B.3}$$

$$\{a^m, a_n^\dagger\} = \delta_n^m \tag{B.4}$$

and it can be thus represented in the vector space spanned by the 2^l orthonormal states $|\mathbf{k}\rangle = \prod_m (a_m^\dagger)^{k_m} |0\rangle$ with $a_m|0\rangle = 0$ and the vector $\mathbf{k}$ has elements taking only two possible values, $k_m = 0, 1$. This basis (often called spin-basis) yields a standard representation of the Clifford algebra, i.e. of the Dirac gamma matrices.

An alternative overcomplete basis is given by the coherent states that are eigenstates of creation or annihilation operators

$$|\xi\rangle = e^{a_m^\dagger \xi^m} |0\rangle \quad \rightarrow \quad a^m |\xi\rangle = \xi^m |\xi\rangle = |\xi\rangle \xi^m \tag{B.5}$$

$$\langle \bar{\eta}| = \langle 0| e^{\bar{\eta}_m a^m} \quad \rightarrow \quad \langle \bar{\eta}| a_m^\dagger = \langle \bar{\eta}| \bar{\eta}_m = \bar{\eta}_m \langle \bar{\eta}| . \tag{B.6}$$

Below we list some of the useful properties satisfied by these states. Using the Baker-Campbell-Hausdorff formula $e^X e^Y = e^Y e^X e^{[X,Y]}$, valid if $[X, Y] = c$-number, one finds

$$\langle \bar{\eta}|\xi\rangle = e^{\bar{\eta} \cdot \xi} \tag{B.7}$$

that in turn implies

$$\langle \bar{\eta}|a^m|\xi\rangle = \xi^m \langle \bar{\eta}|\xi\rangle = \frac{\partial}{\partial \bar{\eta}_m} \langle \bar{\eta}|\xi\rangle \tag{B.8}$$

$$\langle \bar{\eta}|a_m^\dagger|\xi\rangle = \bar{\eta}_m \langle \bar{\eta}|\xi\rangle \tag{B.9}$$

so that $\{\frac{\partial}{\partial \bar{\eta}_m}, \bar{\eta}_n\} = \delta_n^m$. Defining

$$d\bar{\eta} = d\bar{\eta}_l \cdots d\bar{\eta}_1, \quad d\xi = d\xi^1 \cdots d\xi^l \tag{B.10}$$

so that $d\bar{\eta}d\xi = d\bar{\eta}_1 d\xi^1 d\bar{\eta}_2 d\xi^2 \cdots d\bar{\eta}_l d\xi^l$, one finds the following relations

$$\int d\bar{\eta}d\xi \ e^{-\bar{\eta}\cdot\xi} = 1 \tag{B.11}$$

$$\int d\bar{\eta}d\xi \ e^{-\bar{\eta}\cdot\xi} \ |\xi\rangle\langle\bar{\eta}| = \mathbb{1} \tag{B.12}$$

where $\mathbb{1}$ is the identity in the Fock space. One can also define a fermionic delta function with respect to the measure (B.10) by

$$\delta(\bar{\eta} - \bar{\lambda}) \equiv (\bar{\eta}^1 - \bar{\lambda}^1) \cdots (\bar{\eta}^l - \bar{\lambda}^l) = \int d\xi \ e^{(\bar{\lambda}-\bar{\eta})\cdot\xi} \ . \tag{B.13}$$

Finally, the trace of an arbitrary operator can be written as

$$\mathrm{Tr}\, A = \int d\bar{\eta}d\xi \ e^{-\bar{\eta}\cdot\xi}\langle -\bar{\eta}|A|\xi\rangle = \int d\xi d\bar{\eta} \ e^{\bar{\eta}\cdot\xi}\langle\bar{\eta}|A|\xi\rangle \ . \tag{B.14}$$

As a check one may compute the trace of the identity

$$\mathrm{Tr}\, \mathbb{1} = \int d\xi d\bar{\eta} \ e^{\bar{\eta}\cdot\xi}\langle\bar{\eta}|\xi\rangle = \int d\xi d\bar{\eta} \ e^{2\bar{\eta}\cdot\xi} = 2^l \ . \tag{B.15}$$

C   Noether theorem

The Noether theorem is a power tool that relates continuous global symmetries to conserved charged. Let us consider a particle action $S[x]$: dynamics associated to this action is the same as that obtained from the action $S'[x]$ that differs from $S[x]$ by a boundary term, i.e. $S'[x] \cong S[x]$. Then, if a continuous transformation of fields $\delta x^i = \alpha\Delta x^i$, parametrized by a continuous parameter α, yields $S[x + \alpha\Delta x] = S'[x]$, $\forall x^i(\tau)$, the action is said to be symmetric upon that transformation. Moreover, taking the parameter to be time-dependent the variation of the action will be given by

$$\delta S \equiv S[x + \alpha\Delta x] - S'[x] = \int d\tau \ \dot{\alpha}Q \ . \tag{C.1}$$

The latter in general does not vanish. However it does vanish ($\forall\alpha(\tau)$) on the mass shell of the particle, i.e. imposing equations of motion. Hence

$$\frac{d}{d\tau}Q\Big|_{on-shell} = 0 \tag{C.2}$$

in other words, $Q\big|_{on-shell}$ is a conserved quantity. The formal construction that lead to (C.1) is often called the "Noether trick".

References

[1] M. Chaichian and A. Demichev, "Path integrals in physics. Vol. 1: Stochastic processes and quantum mechanics," Bristol, UK: IOP (2001) 336 p

[2] L. S. Schulman, "Techniques And Applications Of Path Integration," New York, Usa: Wiley (1981) 358p

[3] F. Bastianelli and P. van Nieuwenhuizen, "Path integrals and anomalies in curved space," Cambridge, UK: Univ. Pr. (2006) 379 p.

[4] M. Reuter, M. G. Schmidt and C. Schubert, "Constant external fields in gauge theory and the spin 0, 1/2, 1 path integrals," Annals Phys. **259** (1997) 313 [hep-th/9610191].

[5] C. Schubert, "Perturbative quantum field theory in the string inspired formalism," Phys. Rept. **355** (2001) 73 [hep-th/0101036].

[6] F. Bastianelli, O. Corradini and E. Latini, "Spinning particles and higher spin fields on (A)dS backgrounds," JHEP **0811** (2008) 054 [arXiv:0810.0188 [hep-th]].
O. Corradini, "Half-integer Higher Spin Fields in (A)dS from Spinning Particle Models," JHEP **1009** (2010) 113 [arXiv:1006.4452 [hep-th]].

[7] M. G. Schmidt and C. Schubert, "Multiloop calculations in the string inspired formalism: The Single spinor loop in QED," Phys. Rev. D **53** (1996) 2150 [hep-th/9410100].

[8] P. Dai, Y. -t. Huang and W. Siegel, "Worldgraph Approach to Yang-Mills Amplitudes from N=2 Spinning Particle," JHEP **0810** (2008) 027 [arXiv:0807.0391 [hep-th]].

[9] R. P. Feynman, "Mathematical formulation of the quantum theory of electromagnetic interaction," Phys. Rev. **80** (1950) 440.

[10] C. Schubert, lecture notes "Applications of the Quantum Mechanical Path Integral in Quantum Field Theory" given at the "School on Spinning Particles in Quantum Field Theory: Worldline Formalism, Higher Spins and Conformal Geometry," held in Morelia (Michoacán) Mexico, 19-23 November 2012.

[11] F. Bastianelli, R. Bonezzi, O. Corradini and E. Latini, "Effective action for higher spin fields on (A)dS backgrounds," arXiv:1210.4649 [hep-th], JHEP in press.

[12] F. Bastianelli, O. Corradini and E. Latini, "Higher spin fields from a worldline perspective," JHEP **0702** (2007) 072 [hep-th/0701055].

[13] D. Sorokin, "Introduction to the classical theory of higher spins," AIP Conf. Proc. **767** (2005) 172, [arXiv:hep-th/0405069].

[14] F. Bastianelli, "Path integrals in curved space and the worldline formalism," hep-th/0508205.

[15] H. Gies, K. Langfeld and L. Moyaerts, "Casimir effect on the worldline," JHEP **0306** (2003) 018 [hep-th/0303264].

[16] R. Gopakumar, "From free fields to AdS," Phys. Rev. D **70** (2004) 025009 [hep-th/0308184].

[17] F. Bastianelli and C. Schubert, "One loop photon-graviton mixing in an electromagnetic field: Part 1," JHEP **0502** (2005) 069 [gr-qc/0412095].

[18] H. Gies, J. Sanchez-Guillen and R. A. Vazquez, "Quantum effective actions from nonperturbative worldline dynamics," JHEP **0508** (2005) 067 [hep-th/0505275].

[19] T. J. Hollowood and G. M. Shore, "Causality and Micro-Causality in Curved Spacetime," Phys. Lett. B **655** (2007) 67 [arXiv:0707.2302 [hep-th]].

[20] F. Bastianelli, O. Corradini, P. A. G. Pisani and C. Schubert, "Scalar heat kernel with boundary in the worldline formalism," JHEP **0810** (2008) 095 [arXiv:0809.0652 [hep-th]].

[21] R. Bonezzi, O. Corradini, S. A. Franchino Vinas and P. A. G. Pisani, "Worldline approach to noncommutative field theory," J. Phys. A **45** (2012) 405401 [arXiv:1204.1013 [hep-th]].

[22] J. W. Negele and H. Orland, "Quantum Many Particle Systems," Redwood City, USA:

Addison-Wesley (1988) 459 p. (Frontiers in Physics, 68)