CHIPP PhD Winter School, Jan. 2013, Grindelwald, Switzerland

Neutrino Physics

Eligio Lisi
INFN, Bari, Italy

Lecture II (2h)

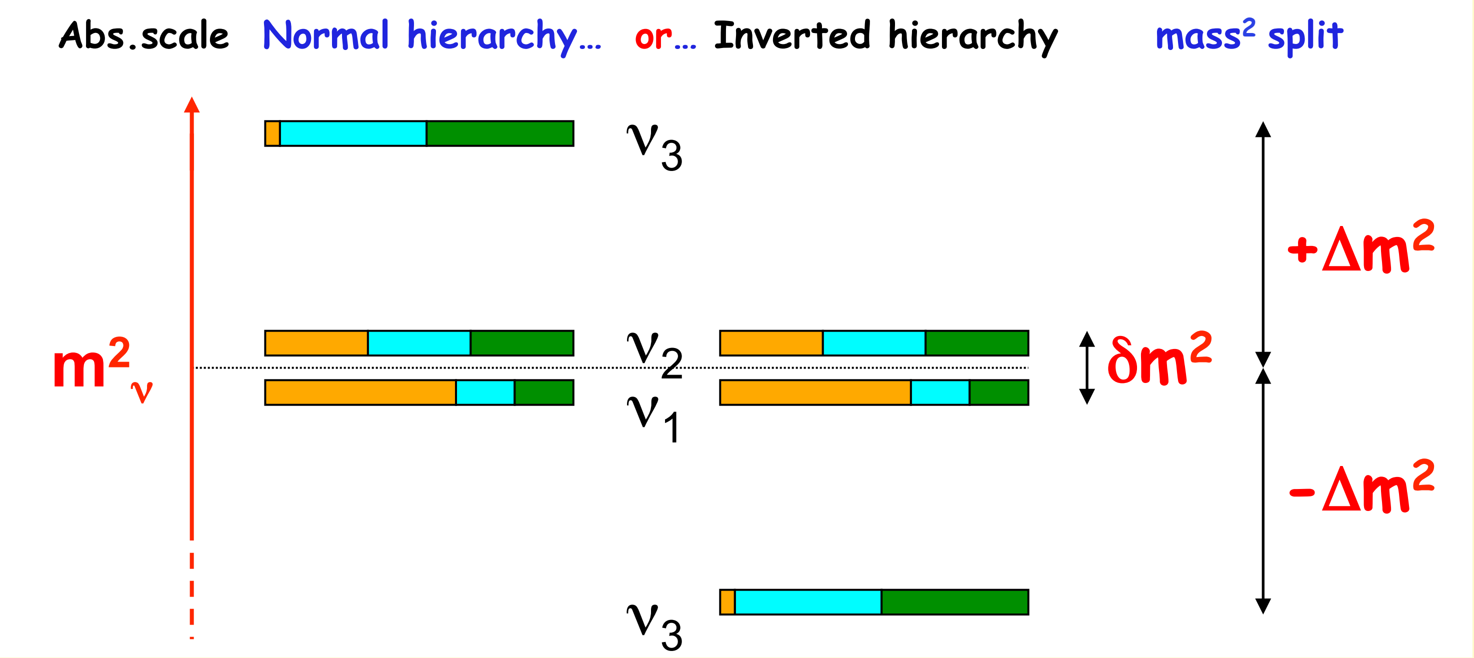
Lesson II - 1st part (~ 1h):

* (3ν) absolute mass observables:

- beta decay
- $0\nu\beta\beta$ decay (+ notes on ν spinors)
- precision cosmology
- current phenomenology

Recap: 3ν framework in just one slide (1 digit accuracy)

Flavors = e μ τ



Knowns:

$\delta m^2 \sim 8 \times 10^{-5}$ eV^2

$\Delta m^2 \sim 2 \times 10^{-3}$ eV^2

$\sin^2\theta_{12} \sim 0.3$

$\sin^2\theta_{23} \sim 0.5$

$\sin^2\theta_{13} \sim 0.02$

Unkowns:

δ (CP)

sign(Δm^2)

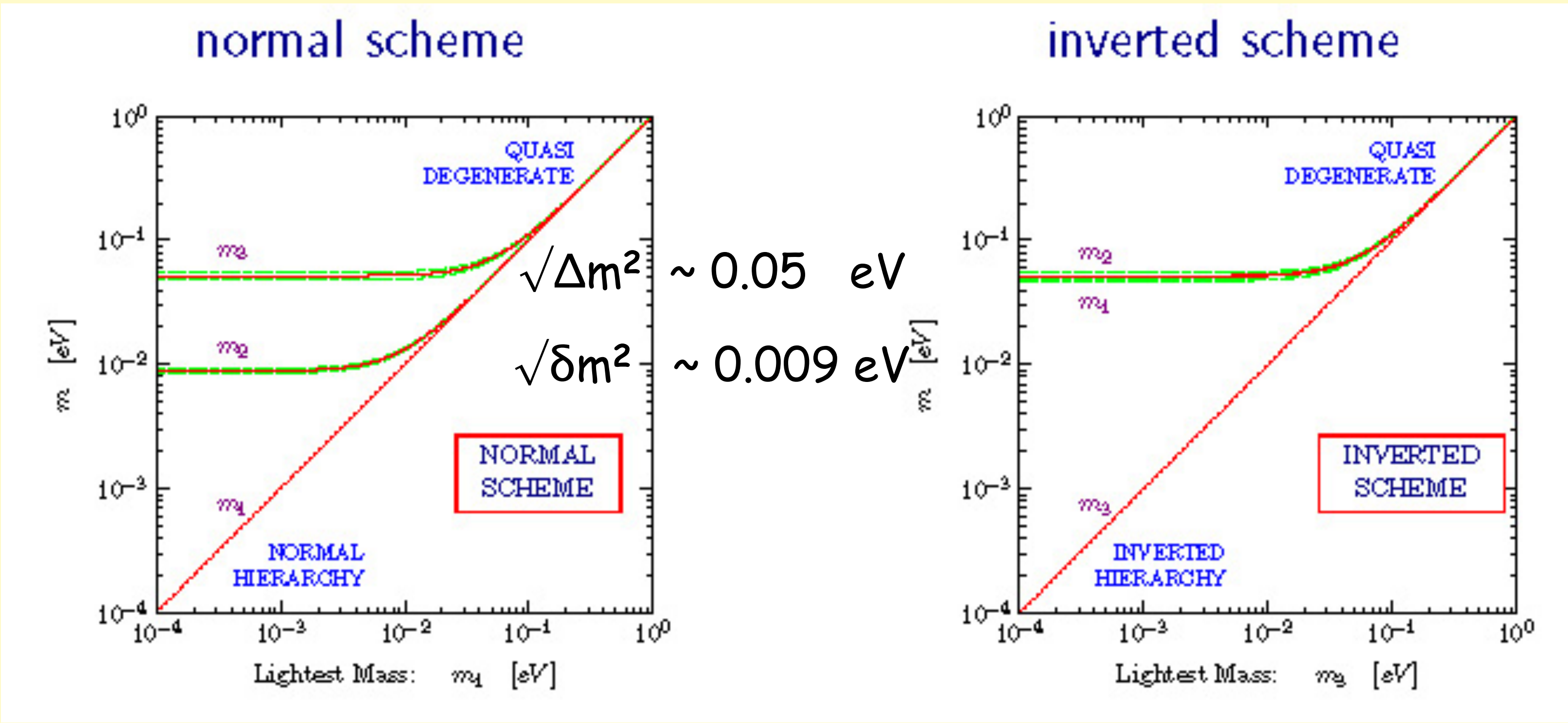
octant($\sin^2\theta_{23}$)

absolute mass scale

Dirac/Majorana nature

Oscillations constrain neutrino mixings and mass splittings but not the absolute mass scale.

E.g., can take the lightest neutrino mass as free parameter:



However, the lightest neutrino mass is not really an "observable"

We know three realistic observables to attack ν masses →

The "weapon":

One spear:

Three prongs:

ν oscillations

β decay

$0\nu2\beta$ decay

cosmology

The three prongs of the "trident": (m_β, $m_{\beta\beta}$, Σ)

1) **β decay**: $m^2_i \neq 0$ can affect spectrum endpoint. Sensitive to the "effective electron neutrino mass":

$$m_\beta = \left[c_{13}^2 c_{12}^2 m_1^2 + c_{13}^2 s_{12}^2 m_2^2 + s_{13}^2 m_3^2\right]^{\frac{1}{2}}$$

2) **$0\nu\beta\beta$ decay**: Can occur if $m^2_i \neq 0$ and $\nu=\bar{\nu}$ (Majorana, not Dirac) Sensitive to the "effective Majorana mass" (and phases):

$$m_{\beta\beta} = \left|c_{13}^2 c_{12}^2 m_1 + c_{13}^2 s_{12}^2 m_2 e^{i\phi_2} + s_{13}^2 m_3 e^{i\phi_3}\right|$$

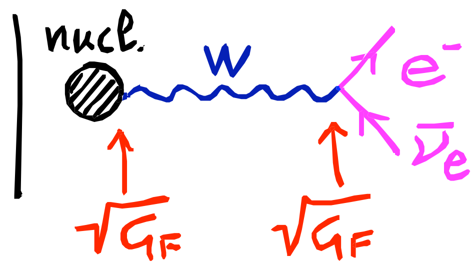
3) **Cosmology**: $m^2_i \neq 0$ can affect large scale structures in (standard) cosmology constrained by CMB + other data. Sensitive to:

$$\Sigma = m_1 + m_2 + m_3$$

Beta decay: Classic kinematic search for neutrino mass.
Look at high-energy endpoint Q of spectrum.

β-decay

rate: $d\Gamma \propto G_F^2 \times (\text{phase sp.})$

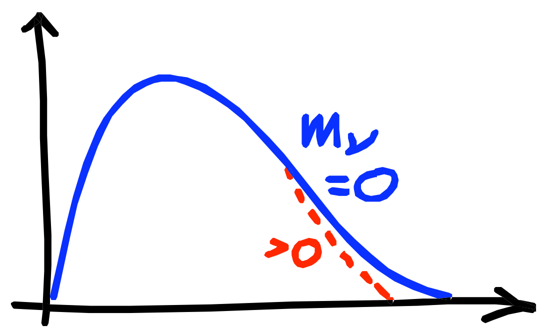


$|\ldots|^2 \propto G_F^2$

energy spectrum:

$$\frac{d\Gamma}{dE_e} \propto G_F^2\, p_e E_e (Q-E_e)^2 \qquad (m_\nu = 0)$$

$$G_F^2\, p_e E_e (Q-E_e)\sqrt{(Q-E_e)^2 + m_\nu^2} \qquad (>0)$$

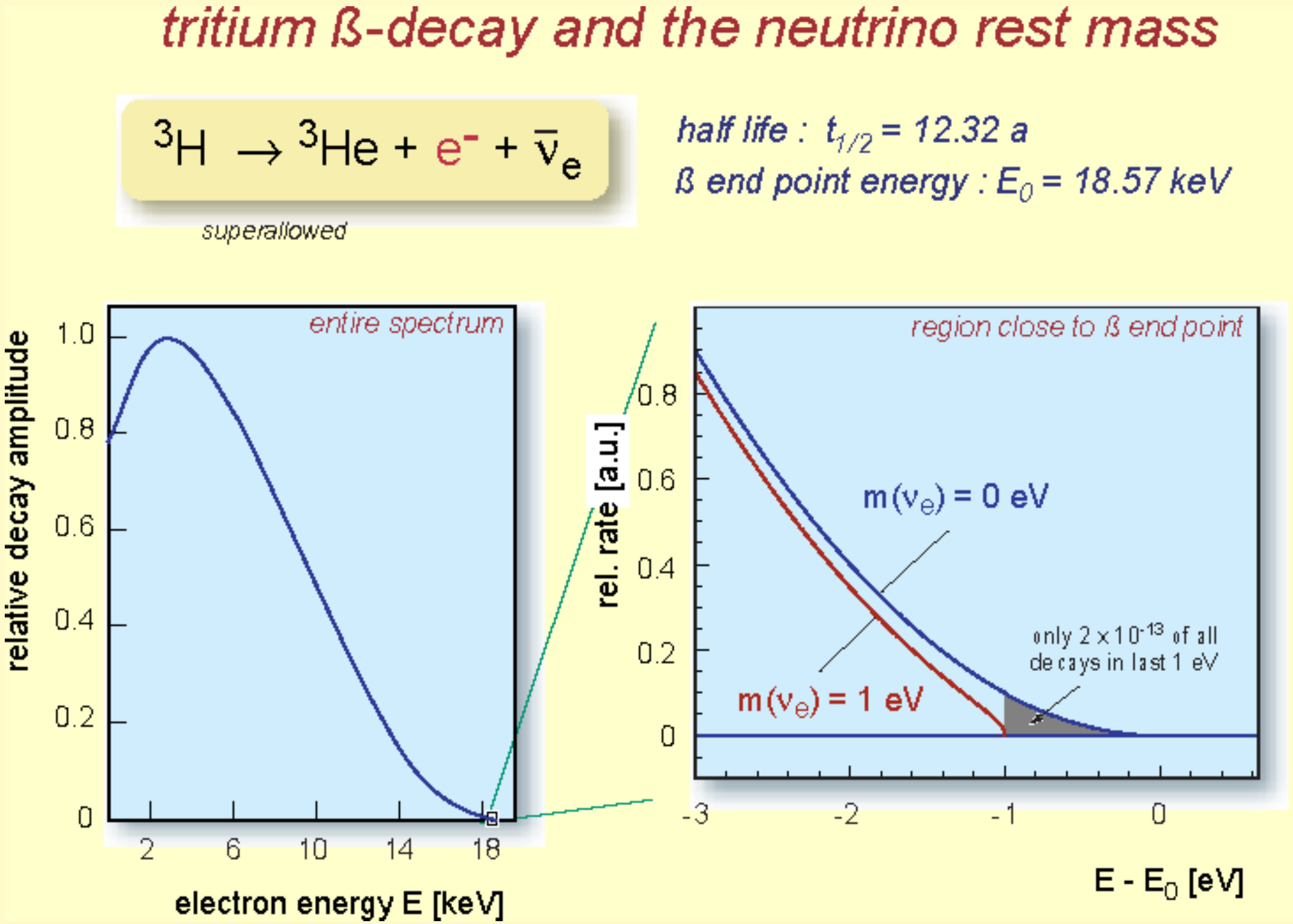


μ-decay

$$\Gamma_\mu = \frac{1}{\tau_\mu} \propto G_F^2 m_\mu^5$$

"defines" G_F

Tritium: low-Q, fast decays



Need good energy resolution

For just one (electron) neutrino family: sensitivity to $m^2(\nu_e)$ (obsolete)

For three neutrino families ν_i, and individual masses experimentally <u>unresolved</u> in beta decay: sensitivity to the sum of $m^2(\nu_i)$, weighted by squared mixings $|U_{ei}|^2$ with the electron neutrino. Observable:

$$m_\beta = \left[c_{13}^2 c_{12}^2 m_1^2 + c_{13}^2 s_{12}^2 m_2^2 + s_{13}^2 m_3^2\right]^{\frac{1}{2}}$$

(so-called "effective electron neutrino mass")

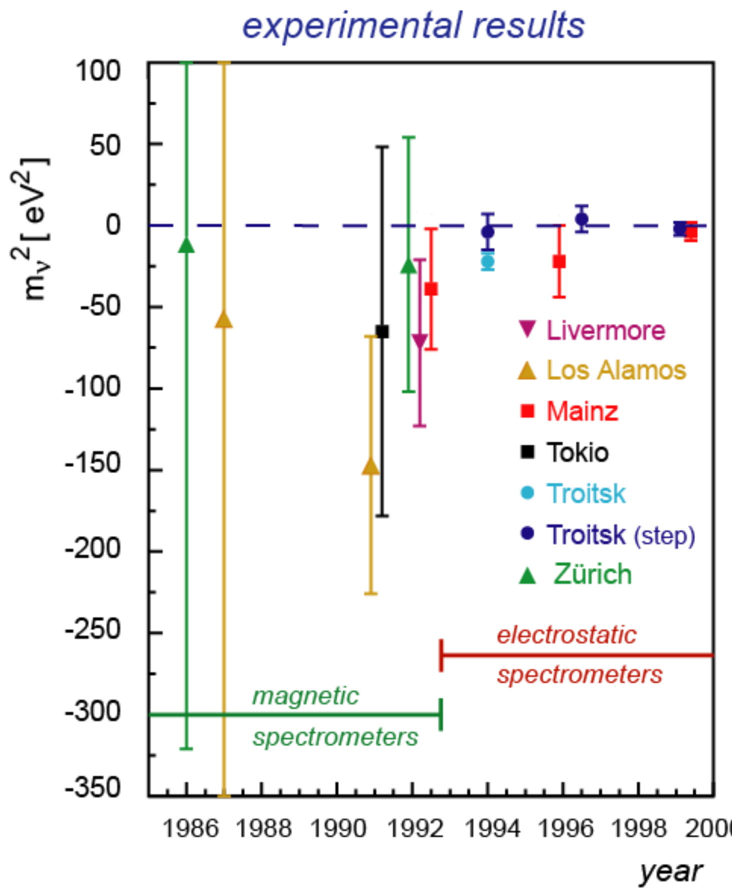
Note: mass state with largest electron flavor component is ν_1:

$$|U_{e1}|^2 \approx \cos^2\theta_{12} \approx 0.7$$

… and we can't exclude that ν_1 is ~massless in normal hierarchy.

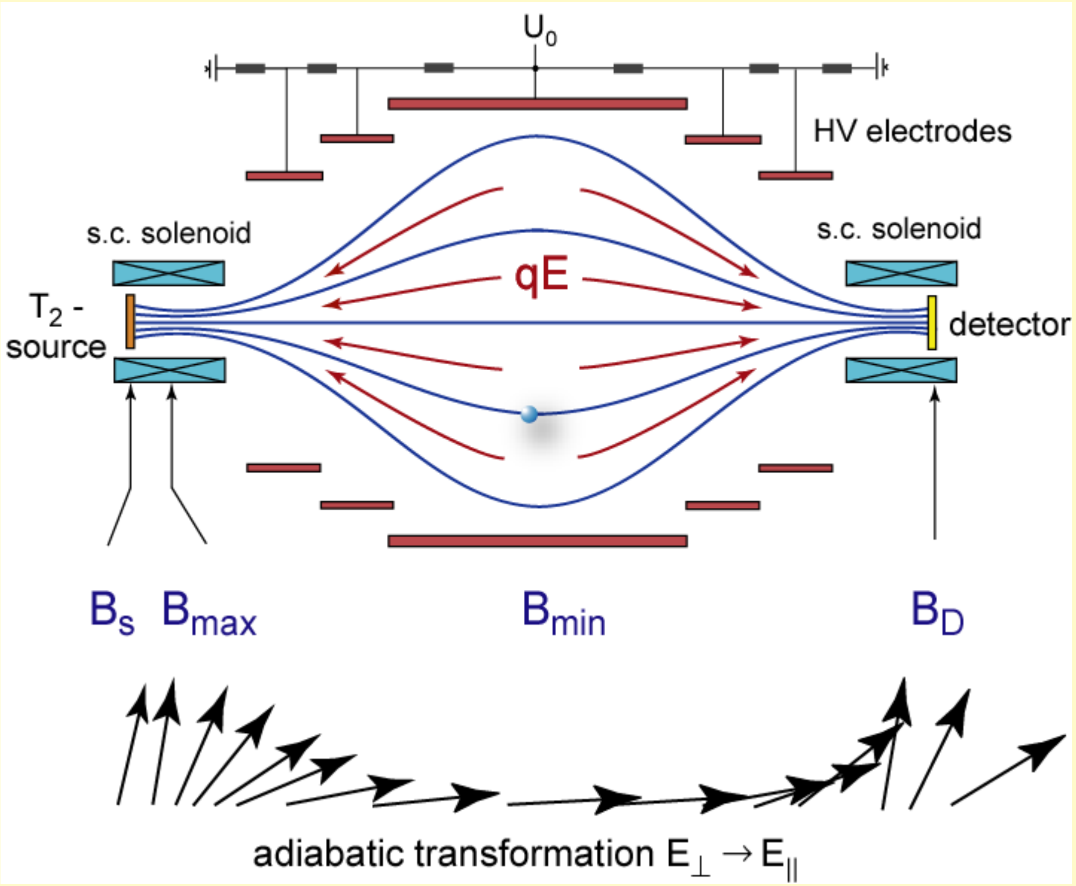
History plot for tritium

	m_ν
ITEP *T_2 in complex molecule* *magn. spectrometer (Tret'yakov)*	17-40 eV
Los Alamos *gaseous T_2 - source* *magn. spectrometer (Tret'yakov)*	< 9.3 eV
Tokio *T - source* *magn. spectrometer (Tret'yakov)*	< 13.1 eV
Livermore *gaseous T_2 - source* *magn. spectrometer (Tret'yakov)*	< 7.0 eV
Zürich *T_2 - source impl. on carrier* *magn. spectrometer (Tret'yakov)*	< 11.7 eV
Troitsk (1994-today) *gaseous T_2 - source* *electrostat. spectrometer*	**< 2.2 eV**
Mainz (1994-today) *frozen T_2 - source* *electrostat. spectrometer*	< 2.3 eV



Latest bounds at the level of ~2 eV

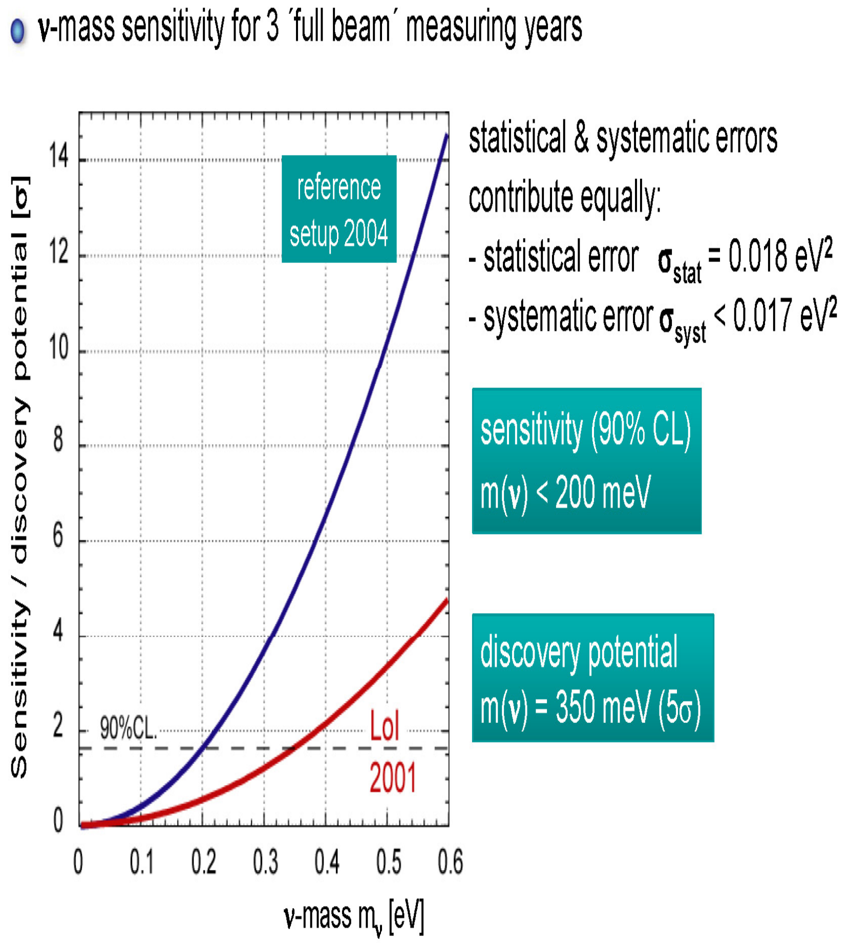
In construction: KATRIN experiment



Magnetic Adiabatic Collimation with an Electrostatic Filter

Probably the “ultimate” spectrometer of this kind....

KATRIN sensitivity



Mainz + Troitsk: m_β < 2 eV

KATRIN: O(10) improvement

Examples of prospective results at KATRIN (±1σ, [eV]):

m_β = 0.35±0.07 (5σ discovery)

m_β = 0.30±0.10 (3σ evidence)

m_β = 0 ±0.12 (<0.2 at 90% CL)

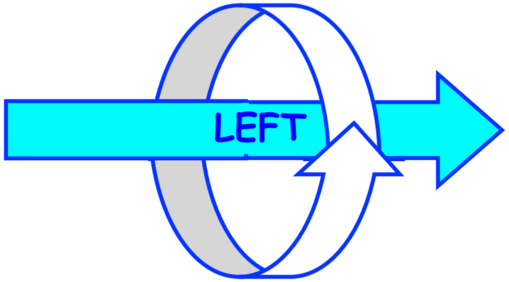
[Need new ideas to go below ~0.2 eV]

Neutrinoless double beta decay - Basics

Weak interactions are chiral (= not mirror-symmetric):

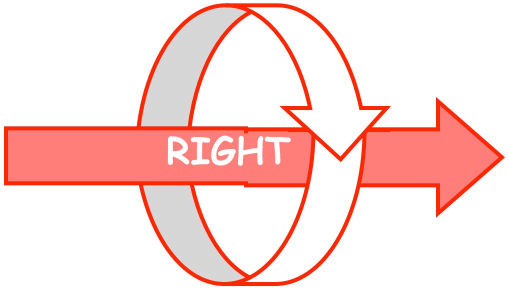
Neutrinos are created in
a left-handed (LH) state

ν



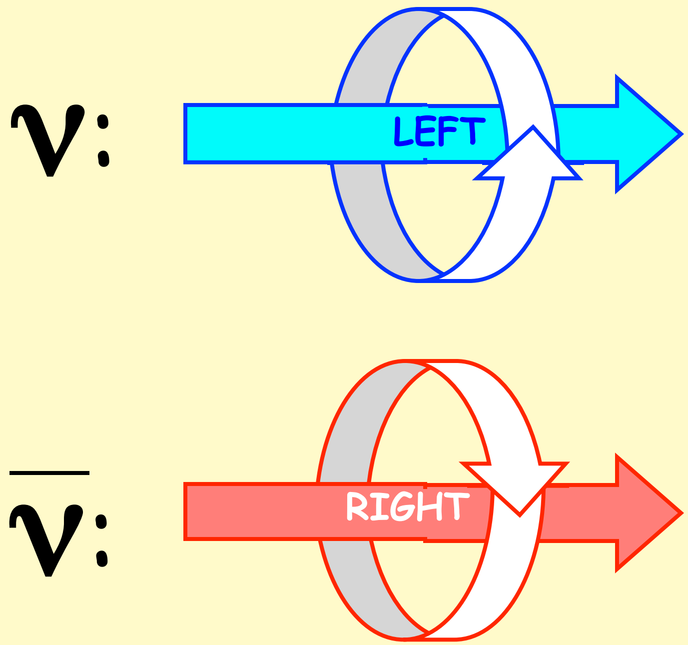
Anti-nus are created in
a right-handed (RH) state

$\bar{\nu}$



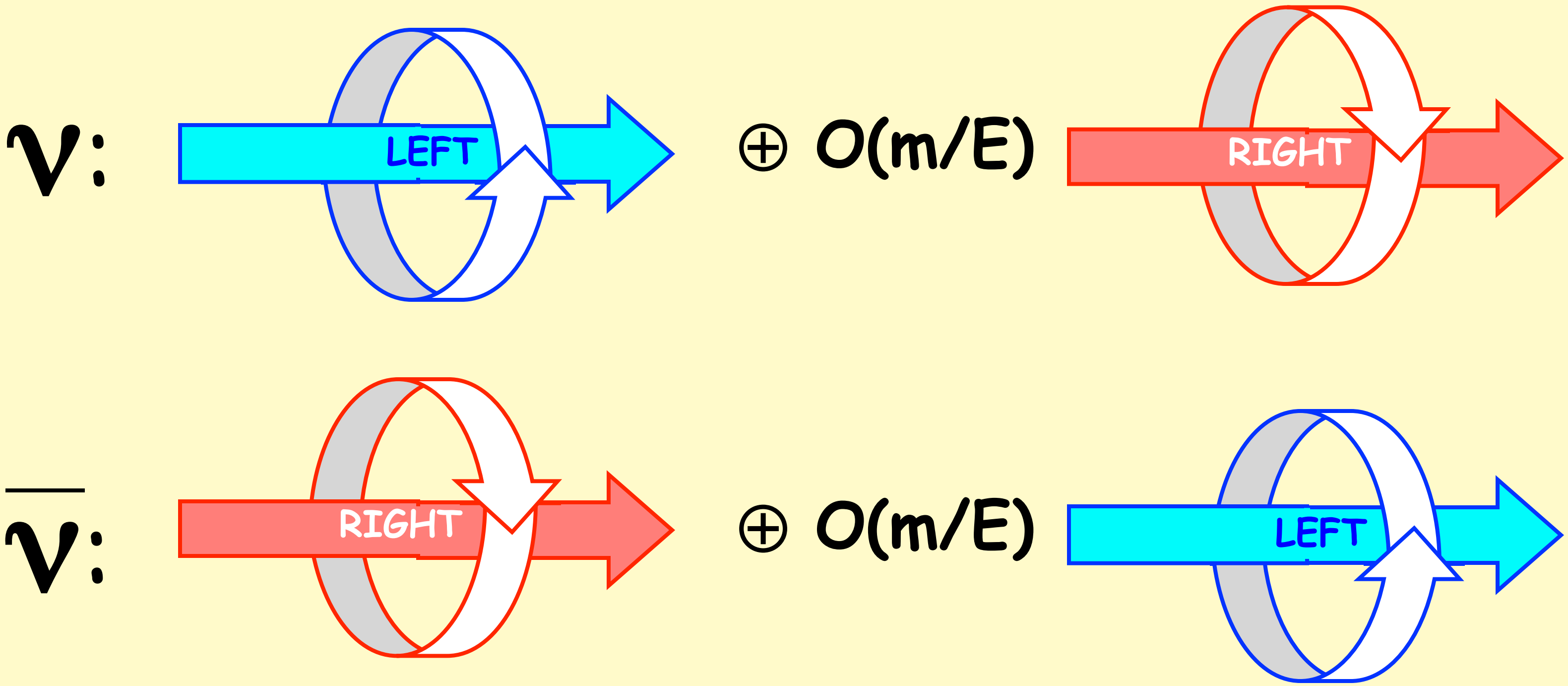
Neutrinos couldn't see themselves in a mirror… like vampires!

For massless neutrinos: handedness is a constant of motion



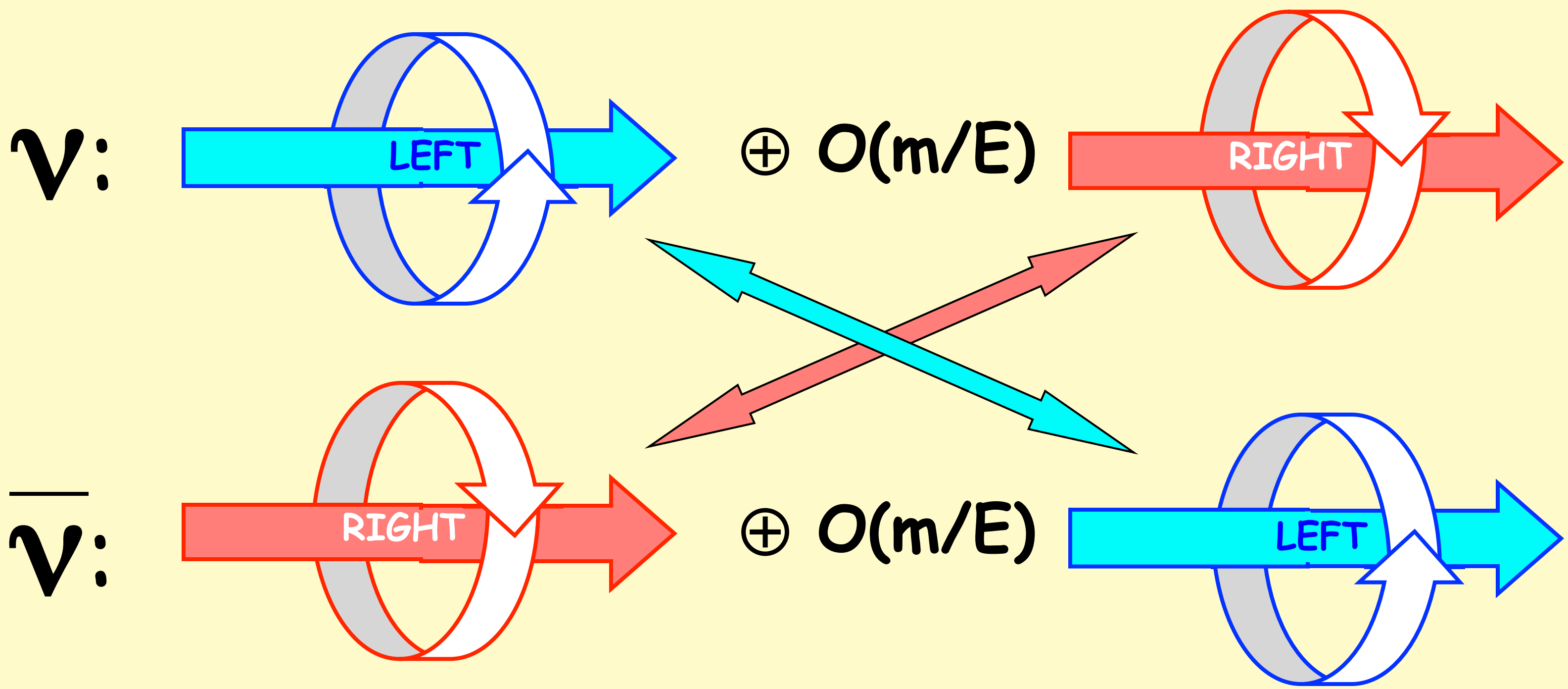
2 independent d.o.f.: massless (“Weyl”) 2-spinor

But: massive ν can develop the "wrong" handedness at O(m/E)
(the Dirac equation mixes RH and LH states for $m_\nu \neq 0$):



If these 4 d.o.f. are independent: massive ("Dirac") 4-spinor
[→ Distinction between neutrinos and antineutrinos, as for electrically charged fermions. Can define a "lepton number"]

But, for neutral fermions, 2 components might be identical !



Massive ("Majorana") 4-spinor with 2 independent d.o.f.

[No distinction between neutrinos and antineutrinos, up to a phase:
A *very* neutral particle: no electric charge, no leptonic number…]

Exercise : The $\nu = \bar{\nu}$ paradox for Majorana neutrinos.

- We can define ν_e as the particle emitted in β^+ decay : $(A,Z) \to (A,Z-1)\, e^+ \nu_e$
- We can define $\bar{\nu}_e$ as the particle emitted in β^- decay : $(A,Z) \to (A,Z+1)\, e^- \bar{\nu}_e$

- The following reactions have been experimentally observed :

$$\boxed{\nu_e + n \to p + e^-} \qquad \boxed{\bar{\nu}_e + p \to n + e^+}$$

- The following reactions have not been experimentally observed :

$$\boxed{\bar{\nu}_e + n \to p + e^-} \qquad \boxed{\nu_e + p \to n + e^+}$$

- This makes sense if ν's are Dirac, since $\nu_e \neq \bar{\nu}_e$. One can attach a "leptonic number" to the doublets (ν_e, e^-) and $(\bar{\nu}_e, e^+)$, which is conserved in the observed reactions $(\Delta L = 0)$ and would be violated in the other two $(\Delta L = 2)$.

- Try to make sense of this fact for Majorana ν's $(\nu = \bar{\nu})$

Solution – For Majorana neutrinos ($\nu = \bar{\nu}$) the unobserved reactions can actually take place! But they are suppressed by many orders of magnitude, $\mathcal{O}(m/E)$.

Indeed, for Majorana ν, we are just naming:

"ν_e" = LH component of ν state ,

"$\bar{\nu}_e$" = RH component of ν state .

The initial ν state produced in β^+ decay is LH (weak inter.) and is thus "ν_e". While propagating, it remains dominantly LH, but can develop a small RH component ("$\bar{\nu}_e$") at $\mathcal{O}(m/E)$. Then, also the reaction $\bar{\nu}_e + n \to p + e^-$ can occur in principle, but it is so suppressed to be practically unobservable. In other words, lepton number violation ($\Delta L = 2$) is allowed in principle, but suppressed at $\mathcal{O}(m/E)$ in practice.

Summary of options for neutrino spinor field:

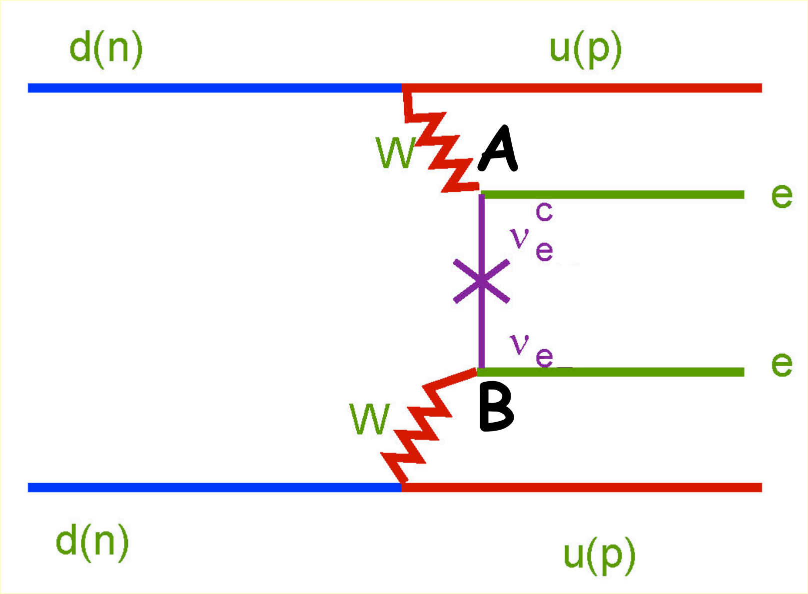
m=0, Weyl:	$\psi = \psi_R$ or $\psi = \psi_L$	massless field with 2 d.o.f.
m≠0, Majorana:	$\psi = \psi_R + \psi_R^c = \psi^c$ or $\psi = \psi_L + \psi_L^c = \psi^c$	massive field with 2 d.o.f.
m≠0, Dirac:	$\psi = \psi_R + \psi_L \neq \psi_c$	massive field with 4 d.o.f.

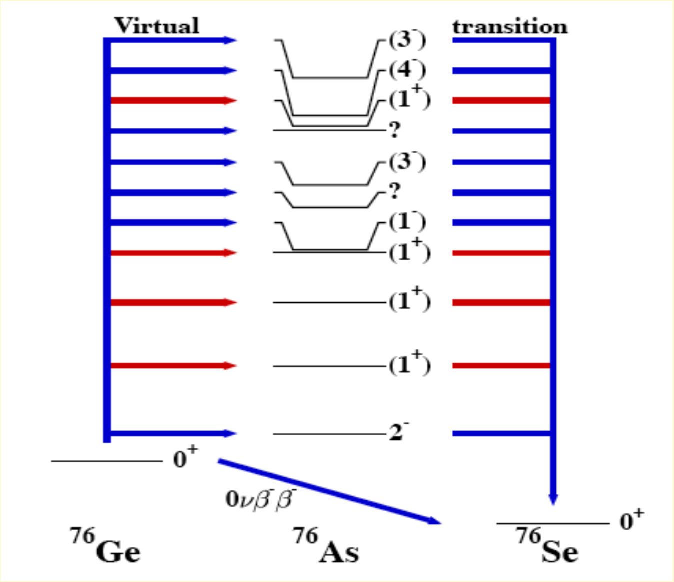
Conjugation operator: $\psi^c = \mathcal{C}(\psi) = i\gamma^2\psi^*$, $\psi_{\text{antiparticle}} = \mathcal{C}(\psi_{\text{particle}})$

[Later: Majorana masses and "see-saw" mechanism to explain their smallness]

Experiments: "unique" experimental handle to Majorana neutrinos →

Neutrinoless double beta decay: (A,Z) → (A,Z+2)+2e





Can occur only for Majorana neutrinos. Intuitive picture:

1) A RH antineutrino is emitted at point "**A**" together with an electron
2) If it is massive, at O(m/E) it develops a LH component (not possible if Weyl)
3) If neutrino=antineutrino, this component is a LH neutrino (not possible if Dirac)
4) The LH (Majorana) neutrino is absorbed at "**B**" where a 2nd electron is emitted

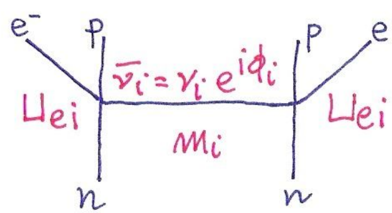
[EW part is "simple". Nuclear physics part is rather complicated and uncertain.]

Exercise : Probability of $0\nu\beta\beta$ decay and $m_{\beta\beta}$ (effective Majorana mass)

Amplitude of $0\nu\beta\beta$ decay :

- Depends on ν_e mixings U_{ei} with ν_i
- Is proportional to ν_i masses m_i (being a m/E effect)
- Depends on generalized Majorana conditions (and phases): $\bar{\nu}_i = \nu_i e^{i\phi_i}$

Probability of decay :

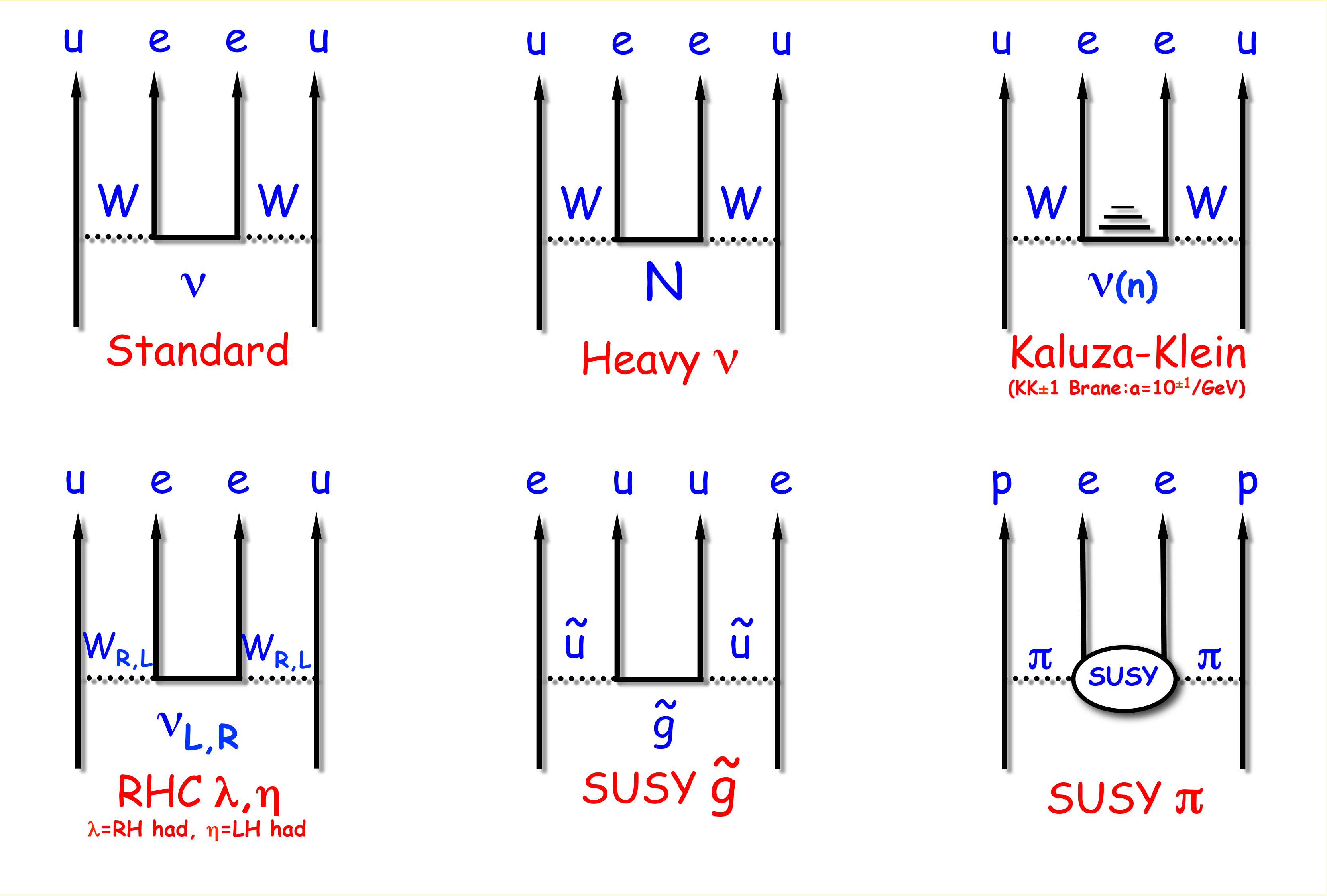
$\left| \sum_{i=1}^{3}$  $\right|^2 \propto \left| \sum_{i=1}^{3} U_{ei}^2 m_i e^{i\phi_i} \right|^2 \equiv m_{\beta\beta}^2$ ← effective Majorana mass in $0\nu\beta\beta$ decay

Since there is an absolute value, only 2 out of 3 Majorana phases are physical ("relative phases") ; typical notation:

$$m_{\beta\beta} = \left| c_{13}^2 c_{12}^2 m_1 + c_{13}^2 s_{12}^2 m_2 e^{i\phi_2} + s_{13}^2 m_3 e^{i\phi_3} \right|$$

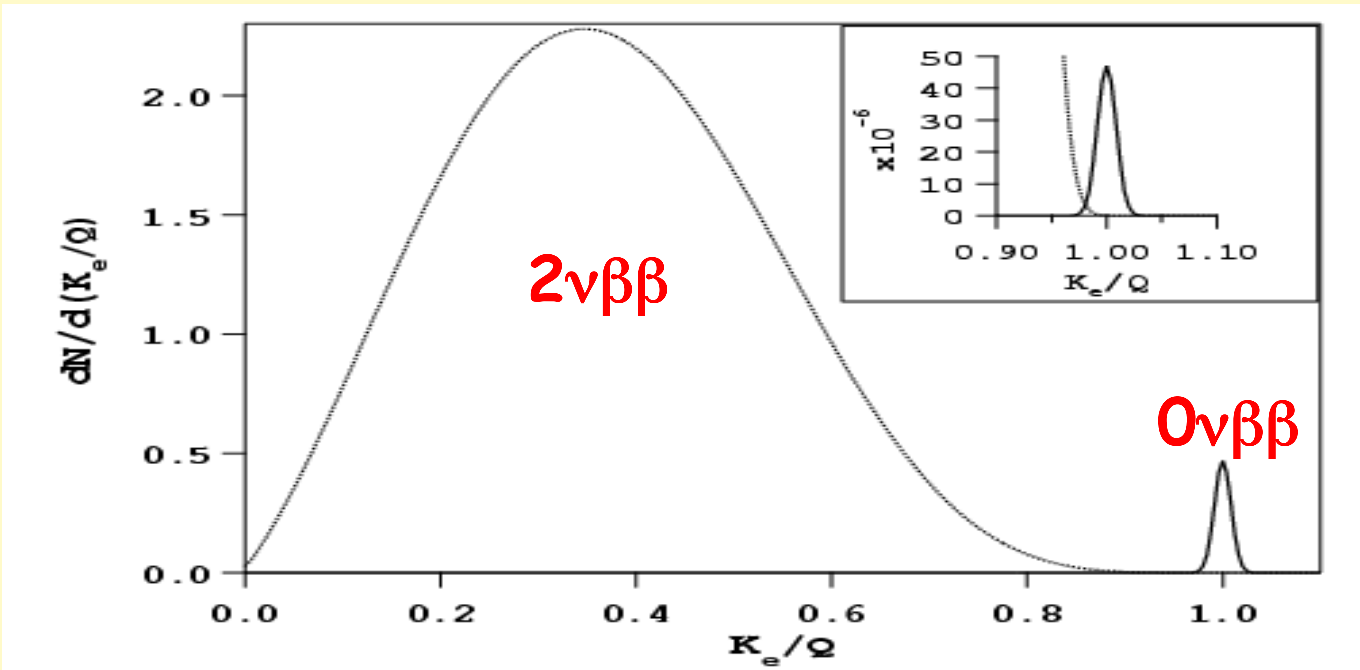
Note that $e^{i\phi_{2,3}}$ may be also equal to -1 and, in general, may induce cancellations in $m_{\beta\beta}$.

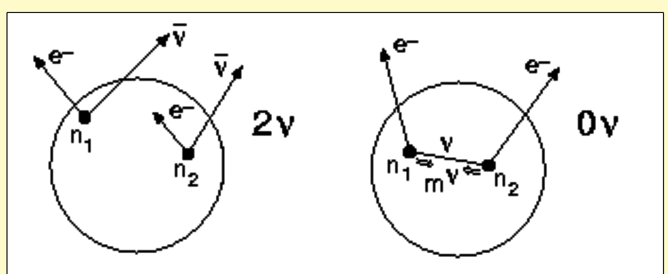
Warning: previous expression invalid for nonstandard 0νββ decays



Experimentally: Look at sum energy of both electrons

Need to see the 0νββ line emerge above bkgd, at endpoint of spectrum from "conventional" 2νββ decay.



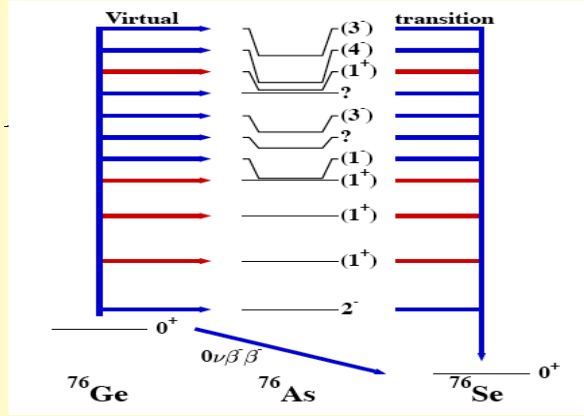


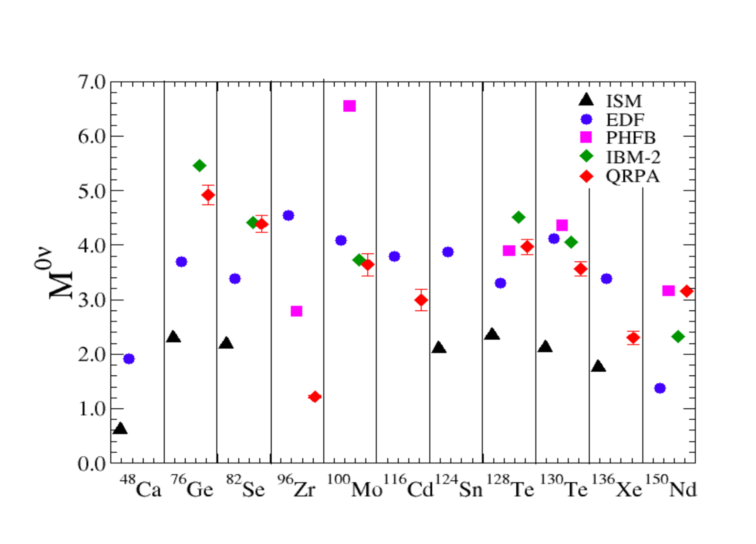
What sets the uncertainty of $m_{\beta\beta}$?

In case of positive signal, a major concern is the accuracy of the **nuclear matrix element |M|**, rather than the expt. uncertainty on the decay half life:

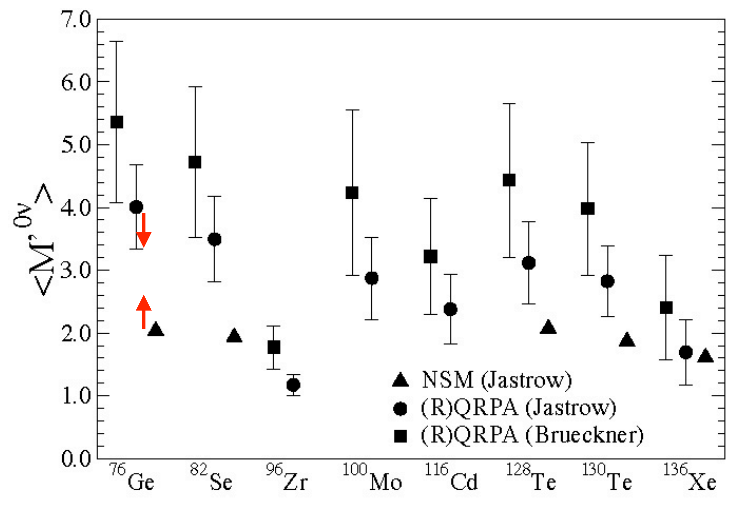
$$T_i^{-1} = G_i \, |M_i'|^2 \, m_{\beta\beta}^2$$

Half-life Phase space Matrix element





Luckily, independent nuclear physics models converge better than it could be hoped only a few years ago …

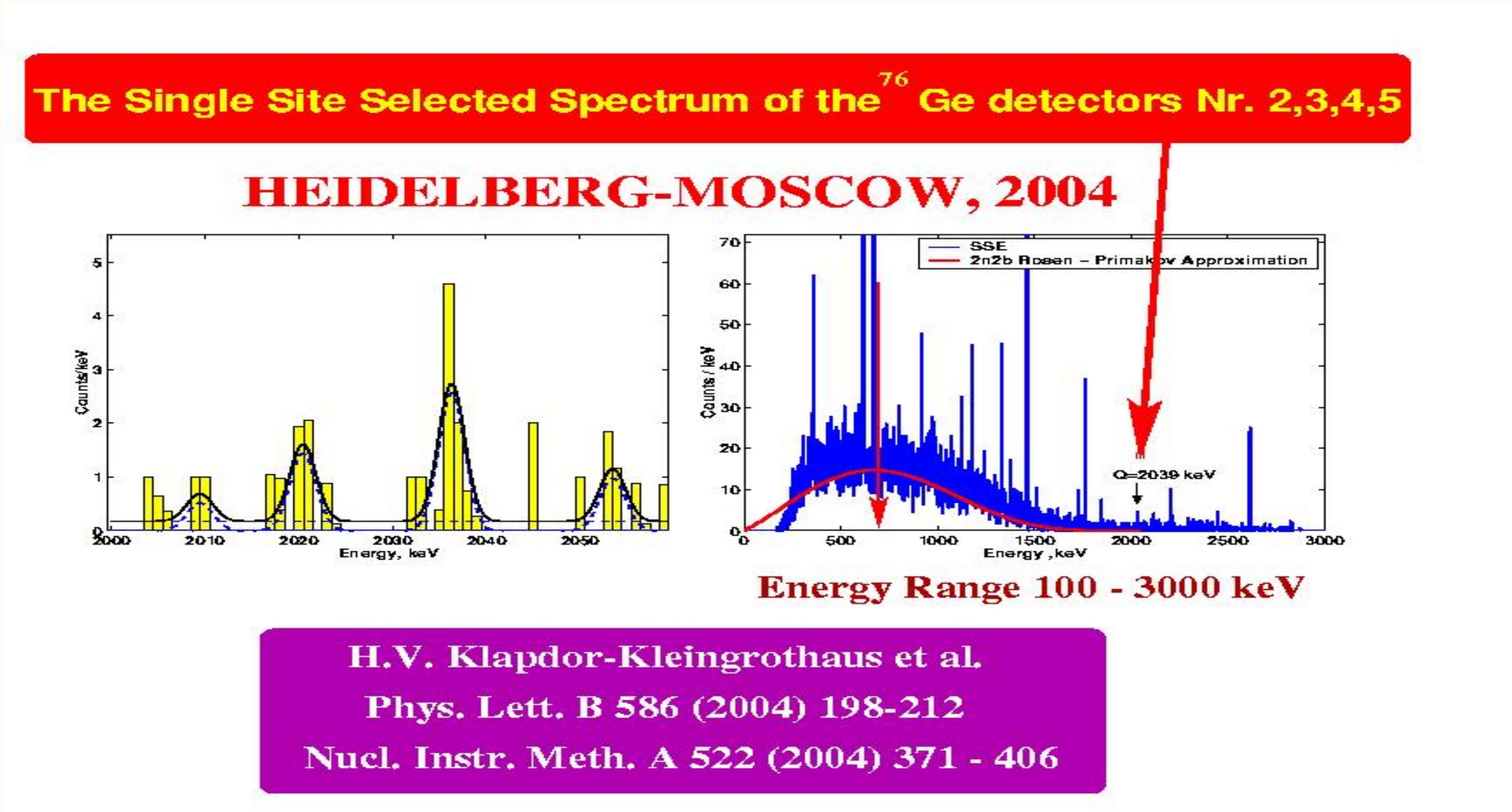


from: Simkovic

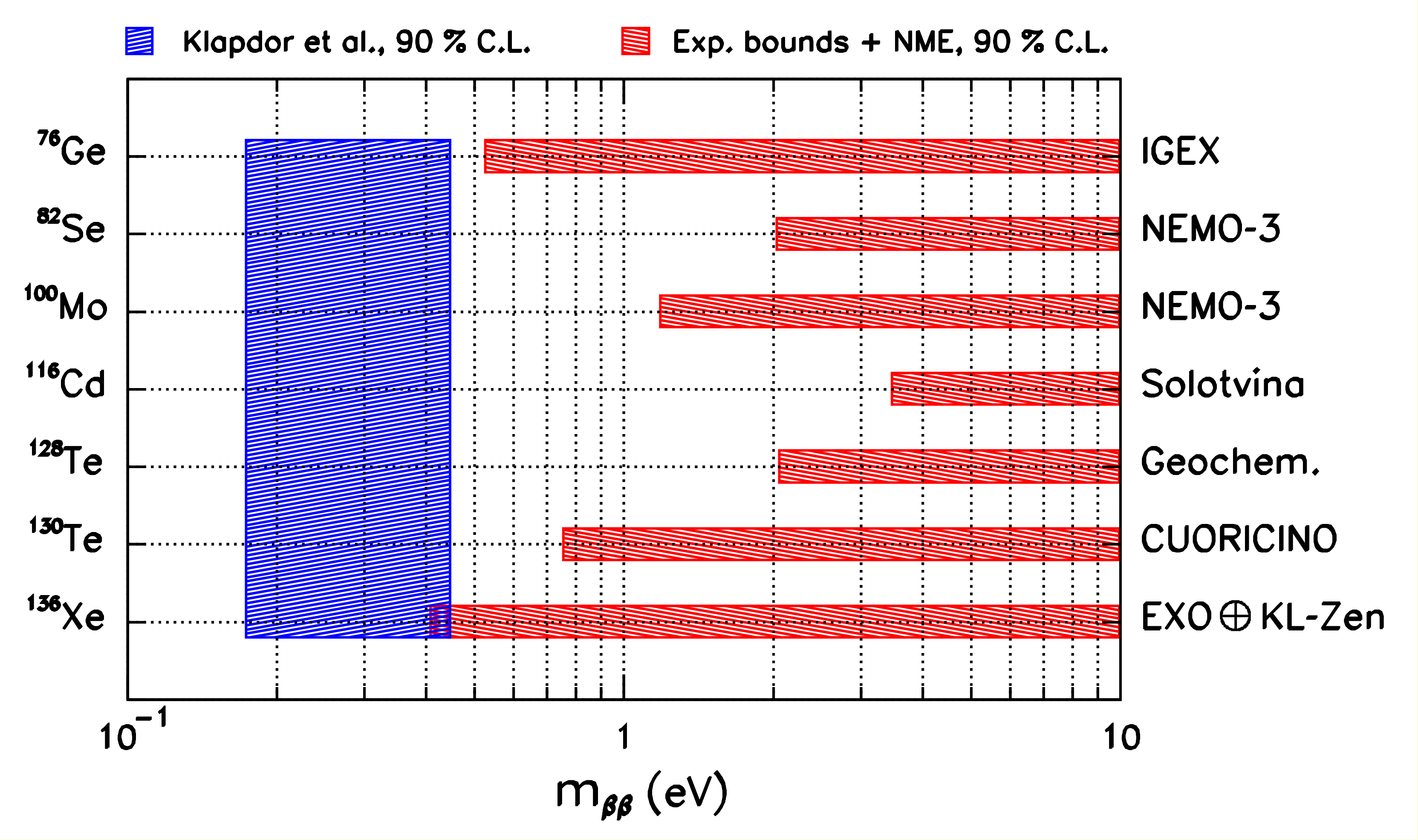
… especially when using the same theo. inputs for comparison (e.g, same description of short range nucleon repulsion) and exploiting additional data

BUT: errors remain large for each candidate nucleus.

$0\nu\beta\beta$ search: No signal observed so far, except in the most sensitive experiment to date (Heidelberg-Moscow): 6σ signal claimed by (part of) the experimental collaboration. Still hotly debated.

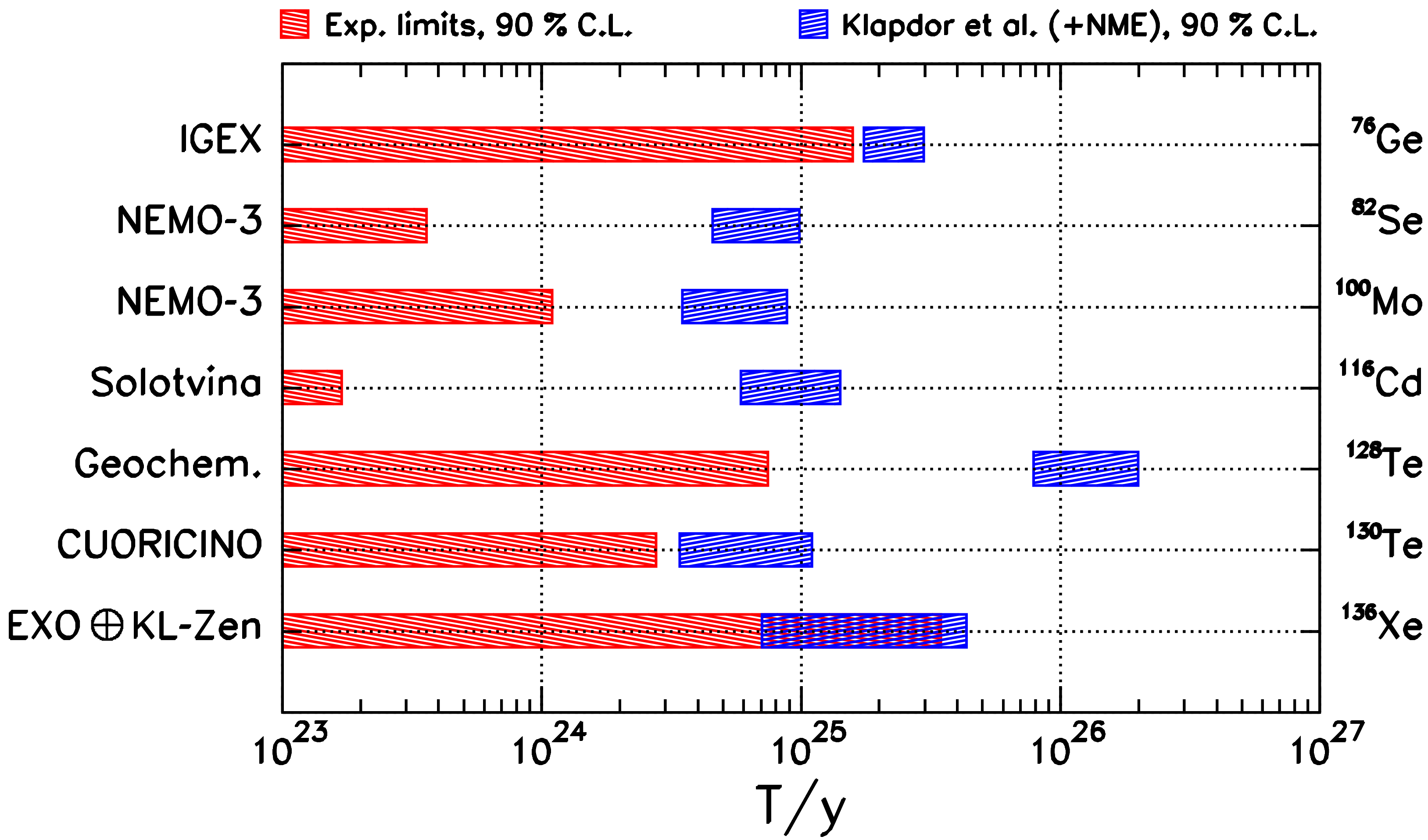


Claim versus current limits (in terms of Majorana mass)



arXiv:1301.1587

Claim versus current limits (in terms of half-life)



[Claim partly disfavored by EXO + KamLAND-Zen data]

Cosmology: a "modern" probe

Standard big bang cosmology predicts a relic neutrino background with total number density 336/cm^3 and temper. $T_\nu \sim 2$ K $\sim 1.7 \times 10^{-4}$ eV $\ll \sqrt{\delta m^2}, \sqrt{\Delta m^2}$.

→At least two relic neutrino species are nonrelativistic today (we can't exclude the lightest to be ~massless)

→Their total mass contributes to the normalized energy density as $\Omega_\nu \approx \Sigma / 50$ eV, where

$$\Sigma = m_1 + m_2 + m_3$$

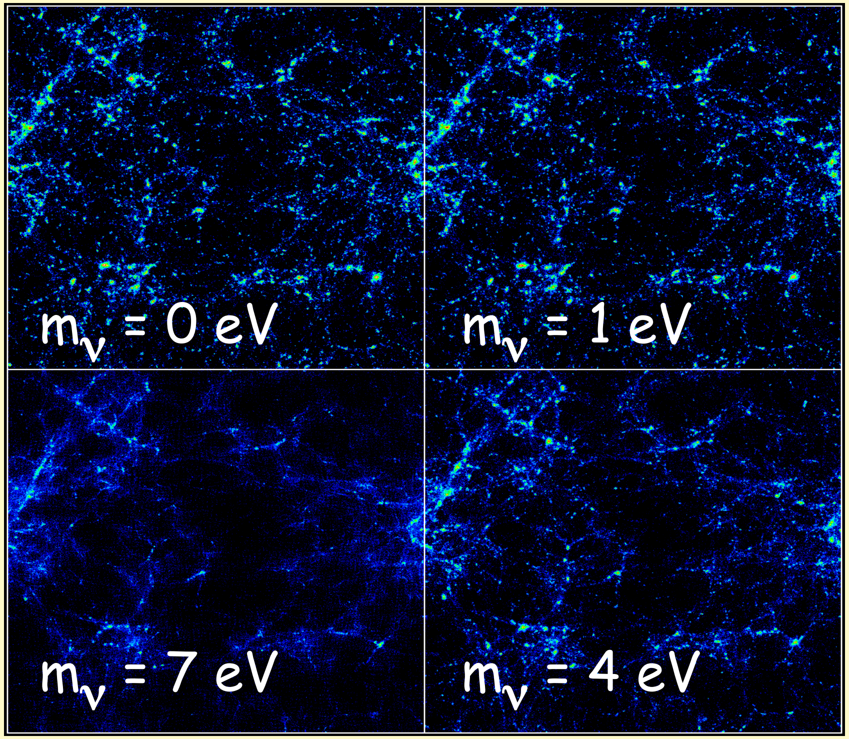
→So, if we just impose that neutrinos do not saturate the total matter density, $\Omega_\nu < \Omega_m \approx 0.25$, we get

m_i < 4 eV – not bad!

Much better bounds can be derived from neutrino effects on structure formation.

Massive neutrinos are difficult to cluster because of their relatively high velocities: they suppress matter fluctuations on scales smaller than their mass-dependent free-streaming scale.

→ Get mass-dependent suppression of small-scale structures



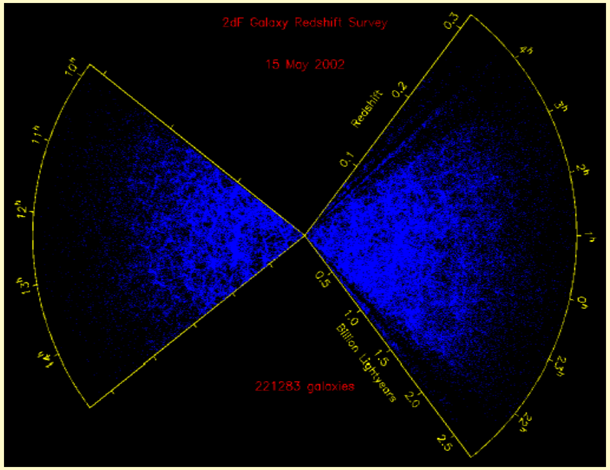
(E..g., Ma 1996)

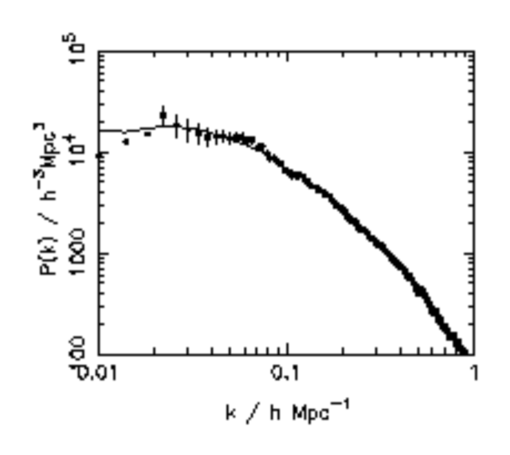
Constraints from CMB also help removing degeneracies.

Observations:

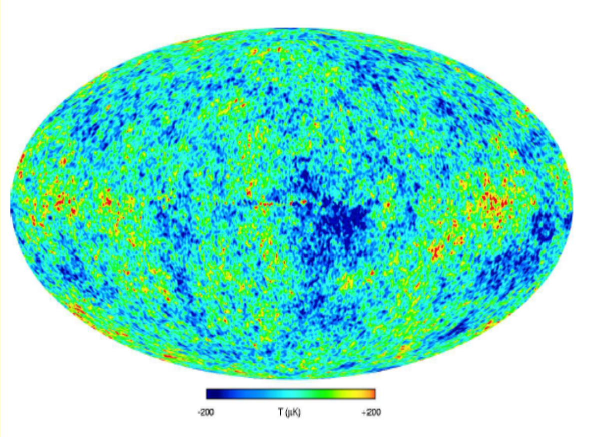
Spectra:

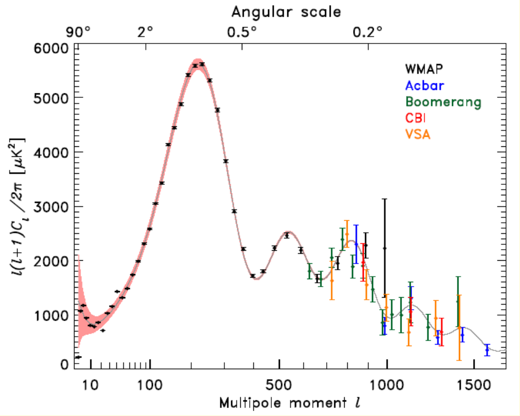
LSS



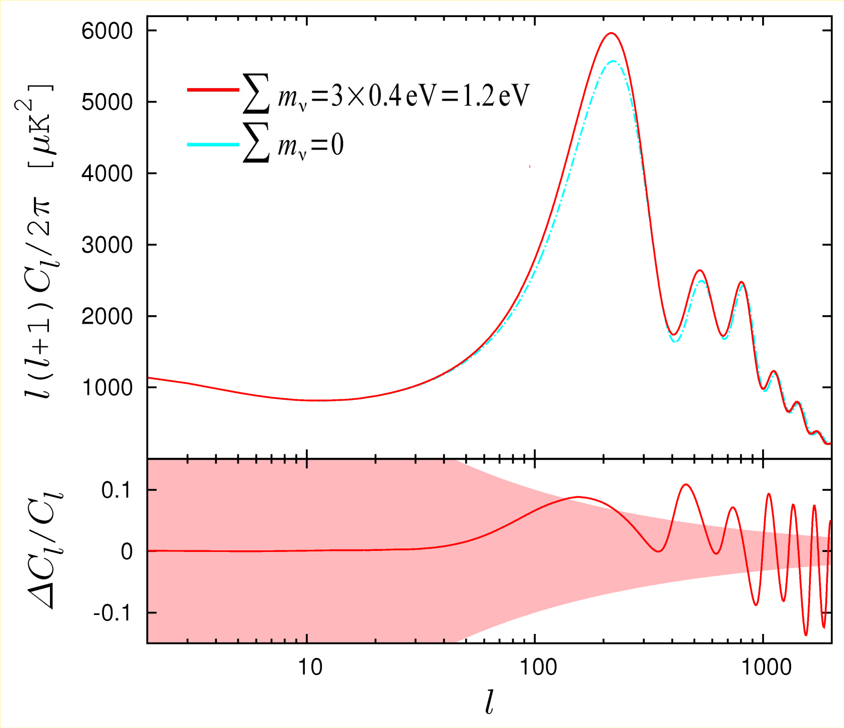


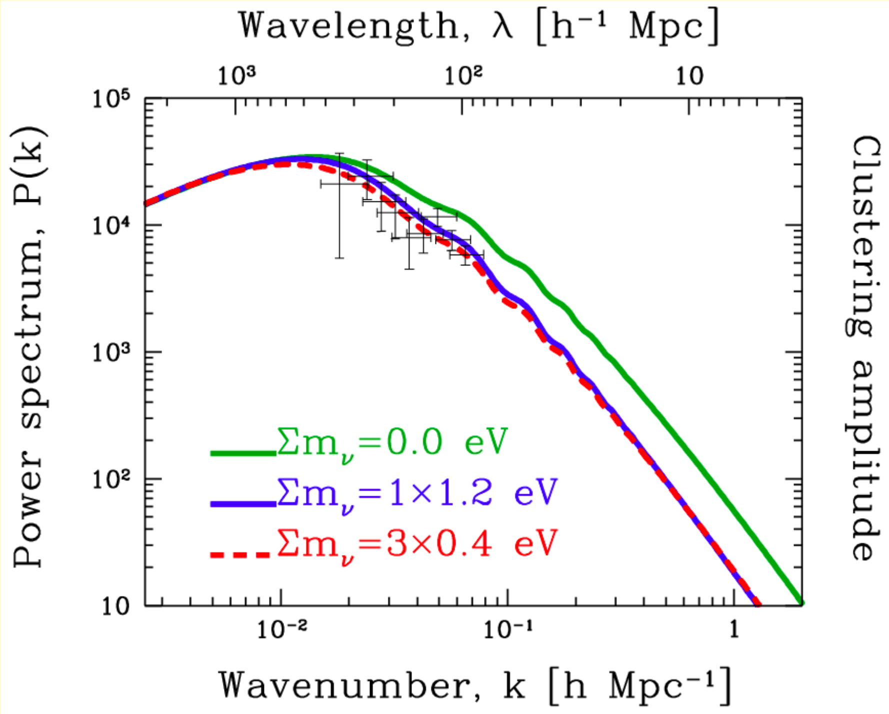
CMB





Spectral effect of massive neutrinos (e.g., from Y.Y.Y. Wong)





Significant progress after WMAP and recent galaxy surveys

Just an example of recent limits on the sum of ν masses from various data sets (assuming the "flat ΛCDM model"):
[from latest WMAP-9y data release, dec. 2012]

TABLE 8
NEUTRINO MASS[a]

Parameter	*WMAP*	+eCMB	+eCMB+BAO	+eCMB+BAO+H_0
New parameter				
$\sum m_\nu$ (eV)	< 1.3 (95% CL)	< 1.5 (95% CL)	< 0.56 (95% CL)	< 0.44 (95% CL)
Related parameters				
σ_8	$0.706^{+0.077}_{-0.076}$	$0.660^{+0.066}_{-0.061}$	$0.750^{+0.044}_{-0.042}$	0.770 ± 0.038
$\Omega_c h^2$	$0.1157^{+0.0048}_{-0.0047}$	0.1183 ± 0.0044	0.1133 ± 0.0026	0.1132 ± 0.0025
Ω_Λ	$0.641^{+0.065}_{-0.068}$	$0.586^{+0.080}_{-0.076}$	0.695 ± 0.013	0.707 ± 0.011
$10^9 \Delta^2_{\mathcal{R}}$	2.48 ± 0.12	2.59 ± 0.12	$2.452^{+0.075}_{-0.074}$	2.438 ± 0.074
n_s	0.962 ± 0.016	0.947 ± 0.014	0.9628 ± 0.0086	$0.9649^{+0.0085}_{-0.0083}$

[a] A complete list of parameter values for this model may be found at http://lambda.gsfc.nasa.gov/.

In general, upper limits range
from: **"conservative"** (only CMB data, dominated by WMAP 9y), **<1.2 eV**
to: **"aggressive"** (all relevant cosmological data), **<0.2 eV**

Intermediate upper limits around Σ < 0.6 eV have gained large consensus.
More stringent limits require more "faith" in current control of syst.'s.

The trident… in action

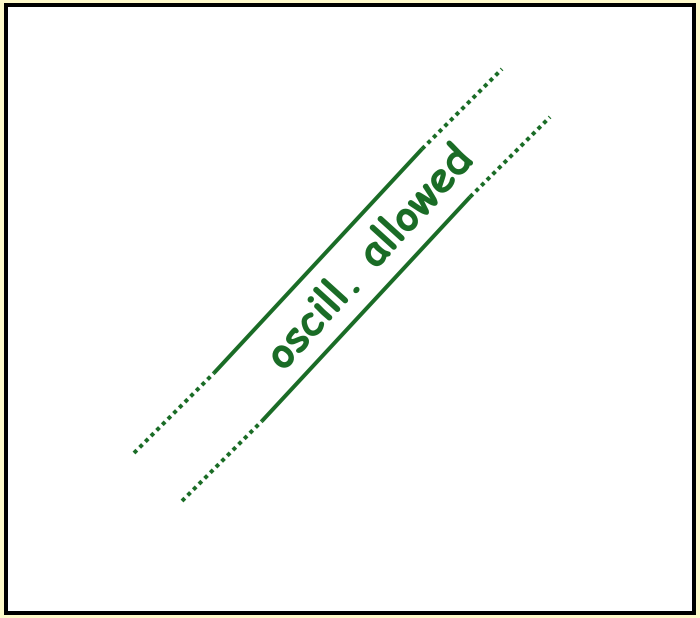
$$m_\beta = \left[c_{13}^2 c_{12}^2 m_1^2 + c_{13}^2 s_{12}^2 m_2^2 + s_{13}^2 m_3^2\right]^{\frac{1}{2}}$$

$$m_{\beta\beta} = \left|c_{13}^2 c_{12}^2 m_1 + c_{13}^2 s_{12}^2 m_2 e^{i\phi_2} + s_{13}^2 m_3 e^{i\phi_3}\right|$$

$$\Sigma = m_1 + m_2 + m_3$$

Interplay: **Oscillations** fix the **mass^2 splittings**, and thus induce **positive correlations** between any pair of the three observables **(m_β, $m_{\beta\beta}$, Σ)**, e.g.:

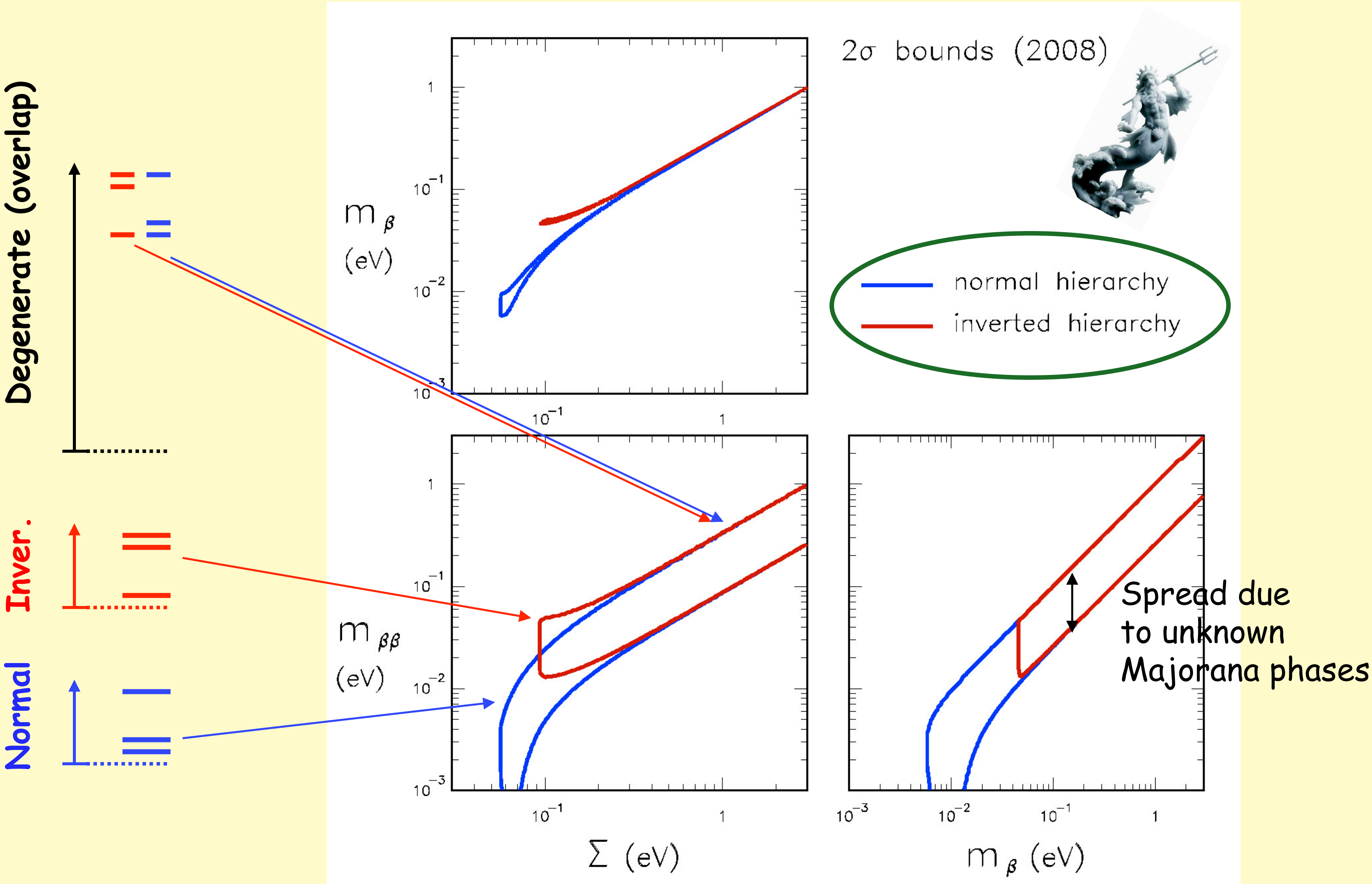
$m_{\beta\beta}$



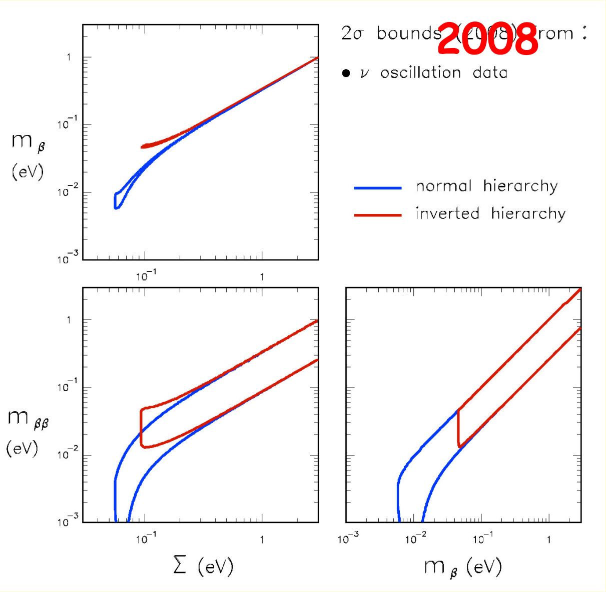
Σ

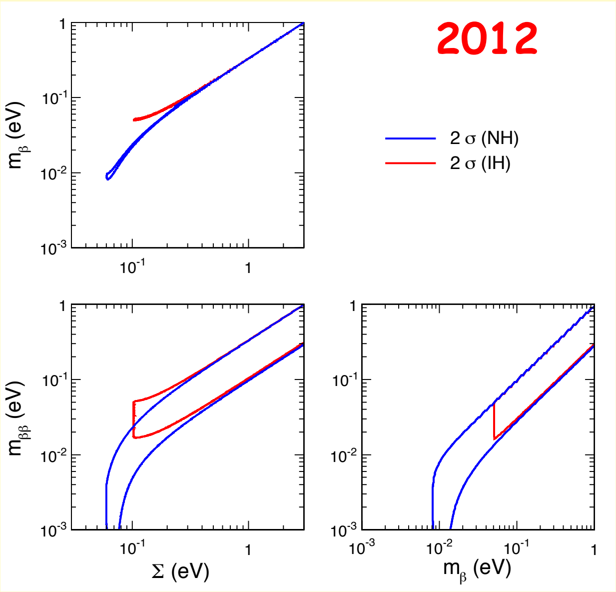
i.e., if one observable increases, the other one (typically) must increase to match mass splitting

The "spear" (oscill. data) sets the "hunting direction" in the **(m_β, $m_{\beta\beta}$, Σ)** parameter space:

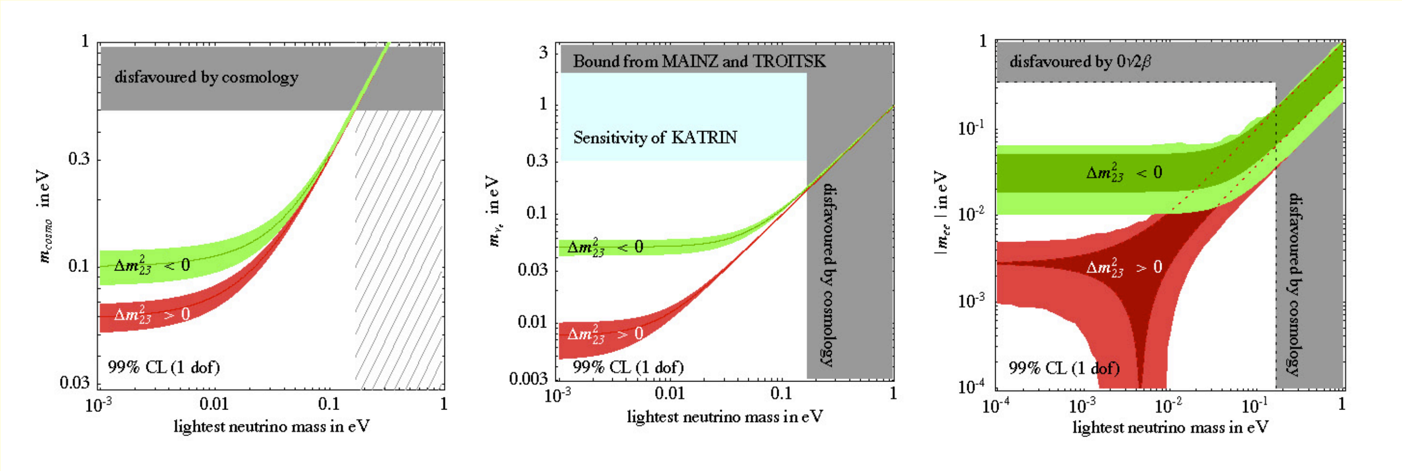


Footnote 1 - Slightly thinner bands in recent years (progress in oscillation parameters). Majorana phase uncertainty remains dominant in sub-plots with $m_{\beta\beta}$.



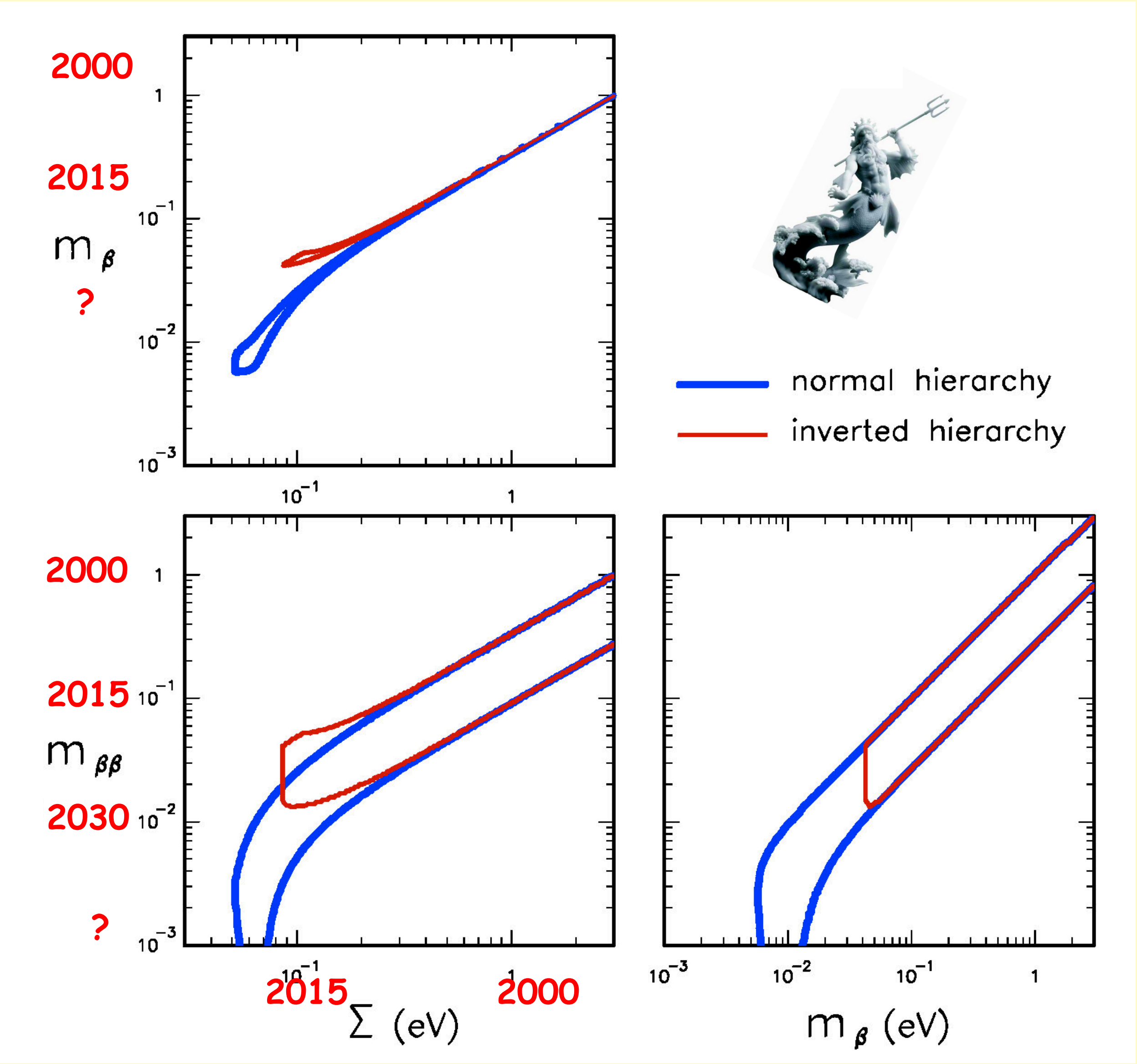


Footnote 2 - Previous plots project away the "unobservable" lightest neutrino mass from graphs like:



Taken from Strumia and Vissani, 2006

"Moore's law" in this field: factor of ~10 improvement every ~15 years



Such "logarithmic progress" seems to be:

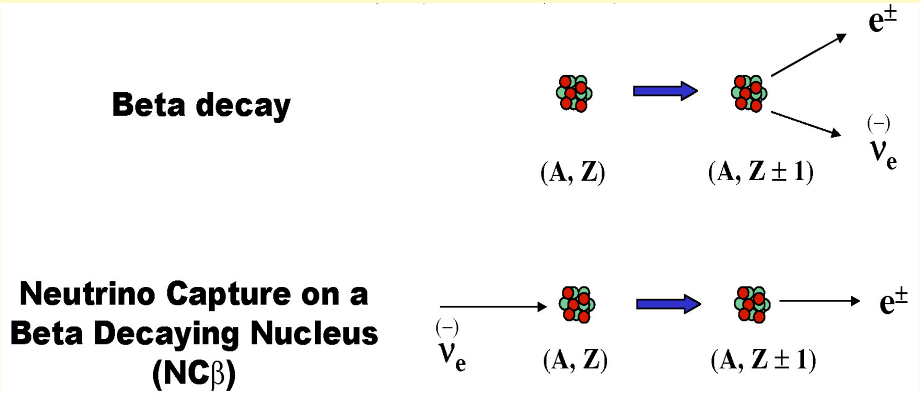
- maybe slowing for β decay (after KATRIN)

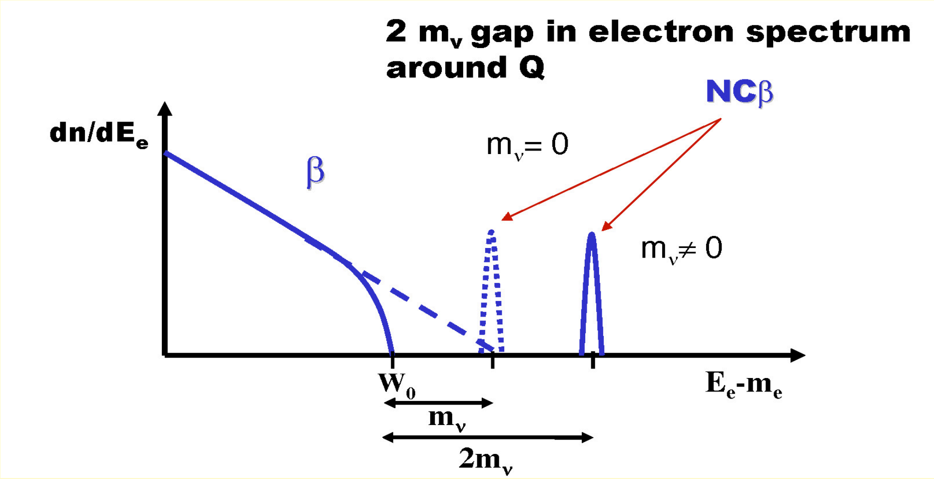
- continuing for $0\nu2\beta$ decay

- "accelerating" for cosmology: the only probe where the ultimate goal ($\Sigma_{min} = \sqrt{\Delta m^2} \approx 0.05$ eV) is claimed to be reachable

You have good chances to see first successful results within your career!

β decay: need new ideas to go beyond KATRIN (calorimetry?). Very far future … a possible observation of the relic neutrino bkgd ?





(Cocco, Mangano & Messina)

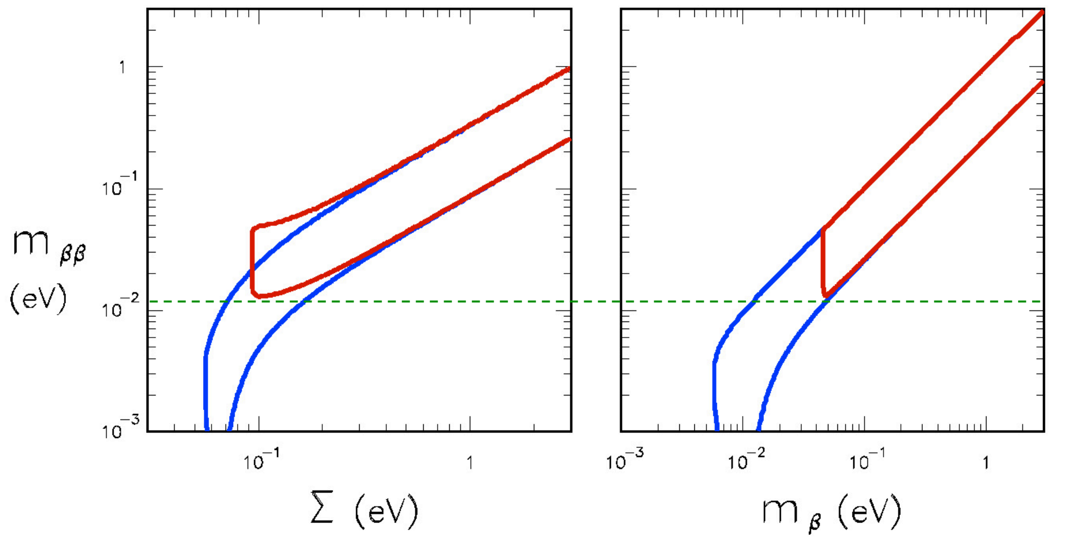
Double-beta decay:
Progress expected from many experiments in the next decade:

(from Rodejohann 2012)

Table 4. Details of the most advanced experiments. Given are life-time sensitivity and the expected limit on $\langle m_{ee} \rangle$, using the NME compilation from figure 5. Note that the range of nuclear matrix elements leads to a range for the expected sensitivity on $\langle m_{ee} \rangle$.

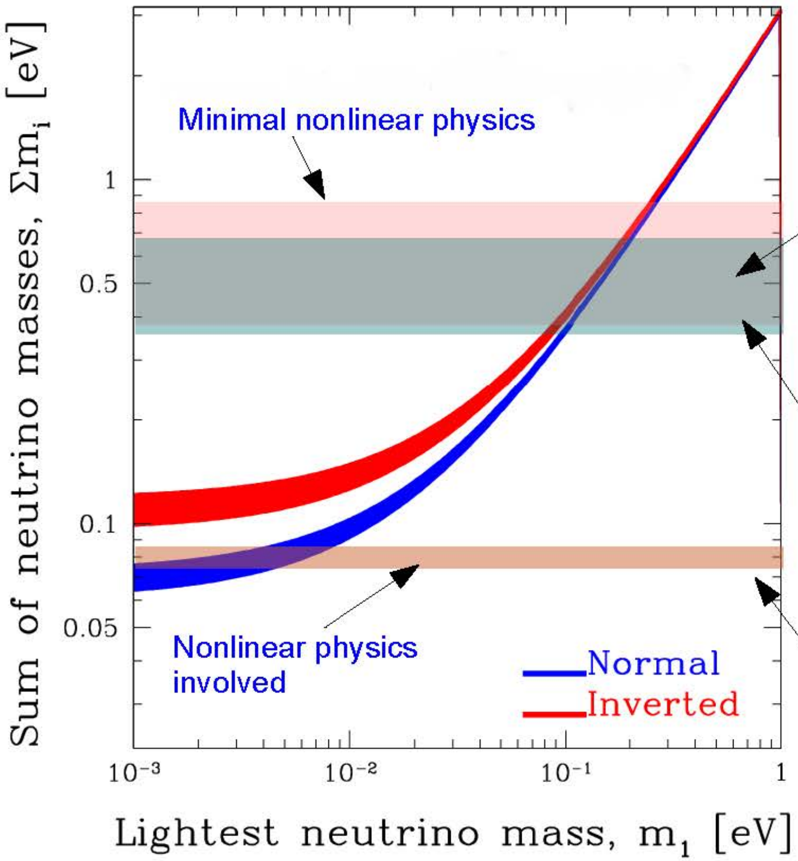
Experiment	Isotope	Mass [kg]	Sensitivity $T_{1/2}^{0\nu}$ [yrs]	Status	Start of data-taking	Sensitivity $\langle m_\nu \rangle$ [eV]
GERDA	^{76}Ge	18	3×10^{25}	running	$\sim$ 2011	0.17-0.42
		40	2×10^{26}	construction	$\sim$ 2012	0.06-0.16
		1000	6×10^{27}	R&D	$\sim$ 2015	0.012-0.030
CUORE	^{130}Te	200	$6.5 \times 10^{26*}$	construction	$\sim$ 2013	0.018-0.037
			$2.1 \times 10^{26**}$			0.03-0.066
MAJORANA	^{76}Ge	30-60	$(1-2) \times 10^{26}$	construction	$\sim$ 2013	0.06-0.16
		1000	6×10^{27}	R&D	$\sim$ 2015	0.012-0.030
EXO	^{136}Xe	200	6.4×10^{25}	running	$\sim$ 2011	0.073-0.18
		1000	8×10^{26}	R&D	$\sim$ 2015	0.02-0.05
SuperNEMO	^{82}Se	100-200	$(1-2) \times 10^{26}$	R&D	$\sim$ 2013-15	0.04-0.096
KamLAND-Zen	^{136}Xe	400	4×10^{26}	running	$\sim$ 2011	0.03-0.07
		1000	10^{27}	R&D	$\sim$ 2013-15	0.02-0.046
SNO+	^{150}Nd	56	4.5×10^{24}	construction	$\sim$ 2012	0.15-0.32
		500	3×10^{25}	R&D	$\sim$ 2015	0.06-0.12

... might cover the whole range for inverted hierarchy:



Progress expected in **cosmology**: (from Y.Y.Y. Wong)

Present constraints and future sensitivities...



CMB (WMAP7+ACBAR+BICEP+QuaD)
+ LSS (SDSS-HPS)
+ H_0+SNIa

$$\sum m_\nu < 0.36 \rightarrow 0.76\,\mathrm{eV} \quad (95\%\,\mathrm{CI})$$

depending on the model complexity

Hannestad, Mirizzi, Raffelt & Y³W 2010
Gonzalez-Garcia et al. 2010
de Putter et al. 2011, etc.

Planck alone (1 year) 2013

$$\sum m_\nu < 0.38 \rightarrow 0.84\,\mathrm{eV} \quad (95\%\,\mathrm{CI})$$

Perotto et al. 2006

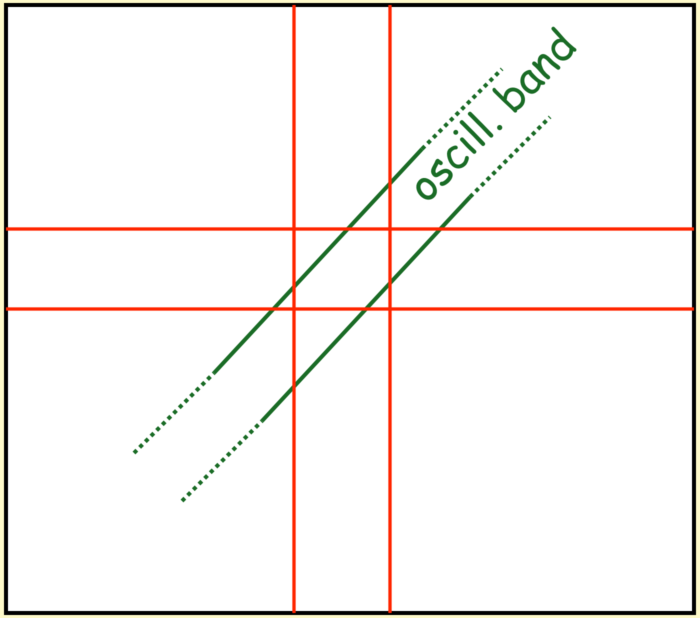
Planck+Weak lensing (**Euclid**) 2020+

$$\sum m_\nu < 0.074 \rightarrow 0.086\,\mathrm{eV} \quad (95\%\,\mathrm{CI})$$

Hannestad, Tu & Y³W 2006

Generic expectations: **In the absence of new physics** (beyond 3ν masses and mixing), **any two data** among **(m_β, $m_{\beta\beta}$, Σ)** are expected to **cross the oscillation band**

$m_{\beta\beta}$



Σ

This requirement provides either an important consistency check or, if not realized, an indication for new physics (barring expt mistakes)

⇒ Data accuracy/reliability/redundance are crucial

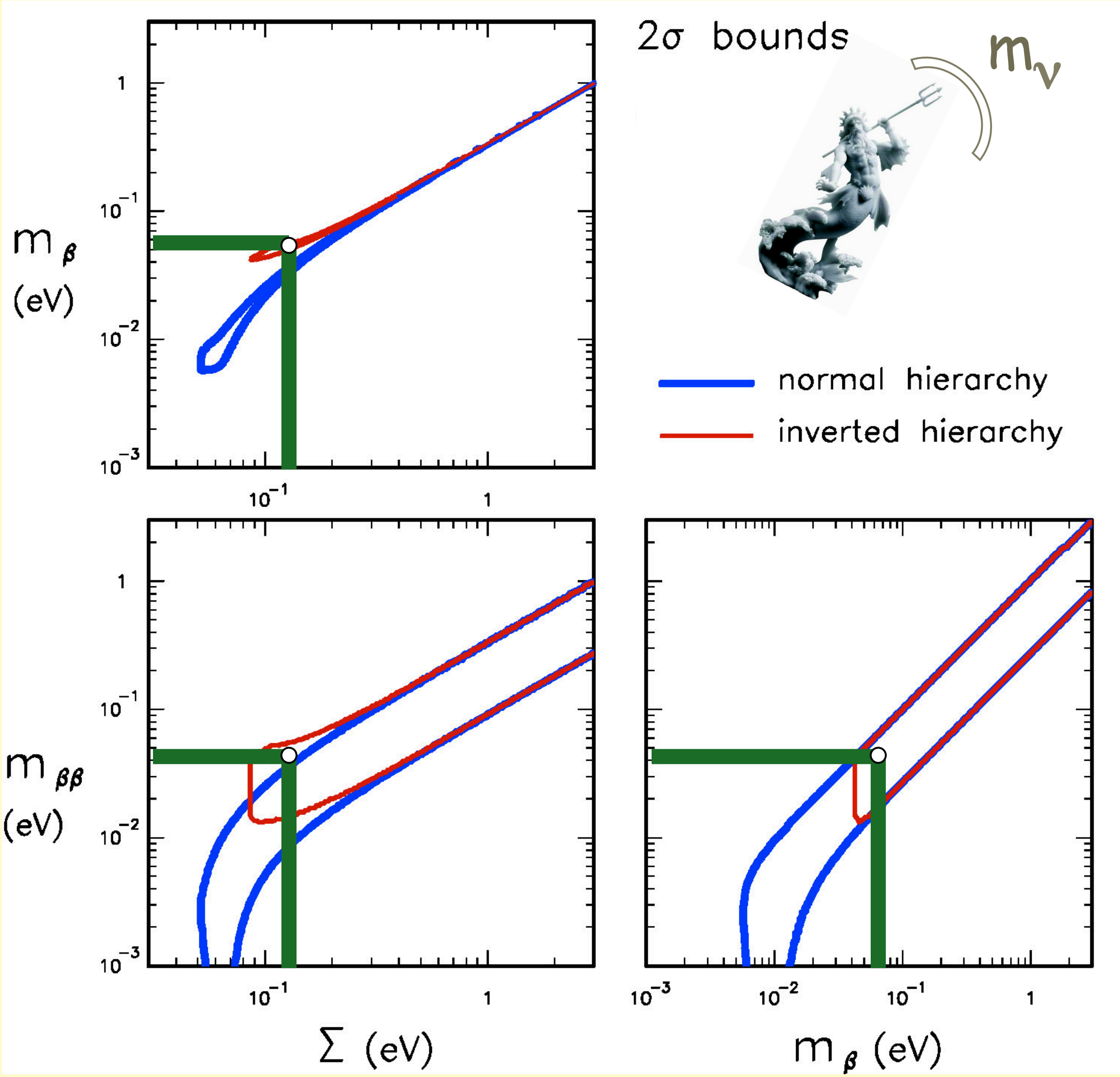
With "dreamlike" data one could, e.g.

Determine the mass scale…

Check 3ν consistency …

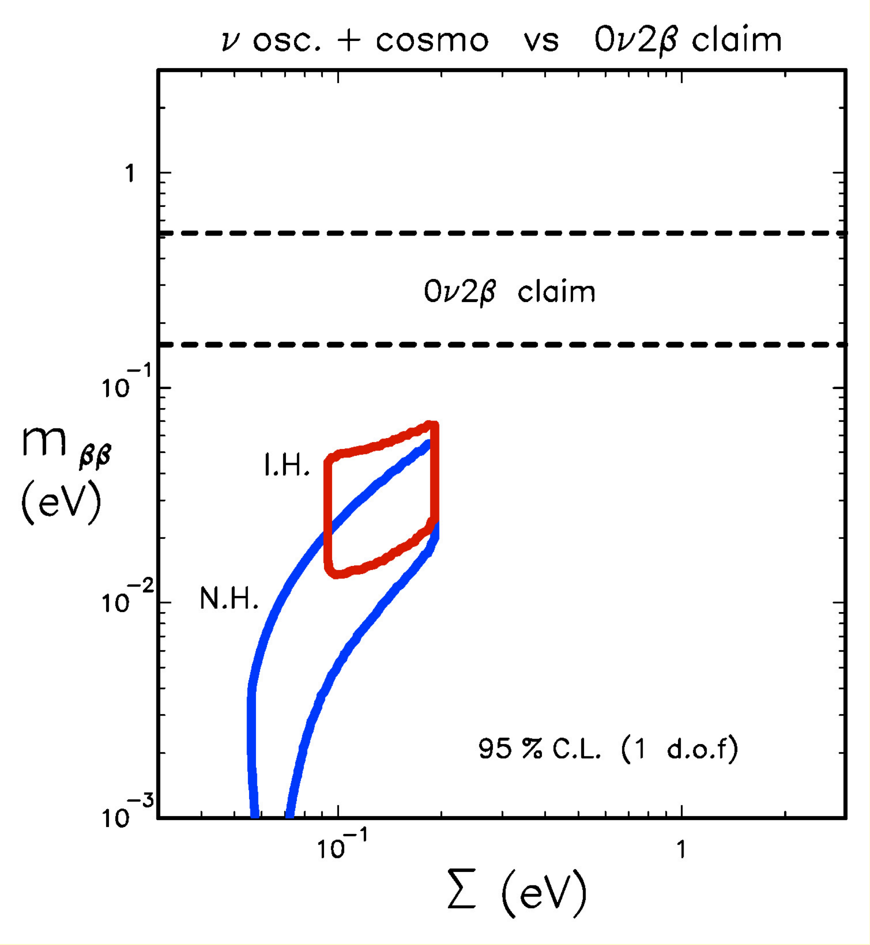
Identify the hierarchy …

Probe the Majorana phase(s) …



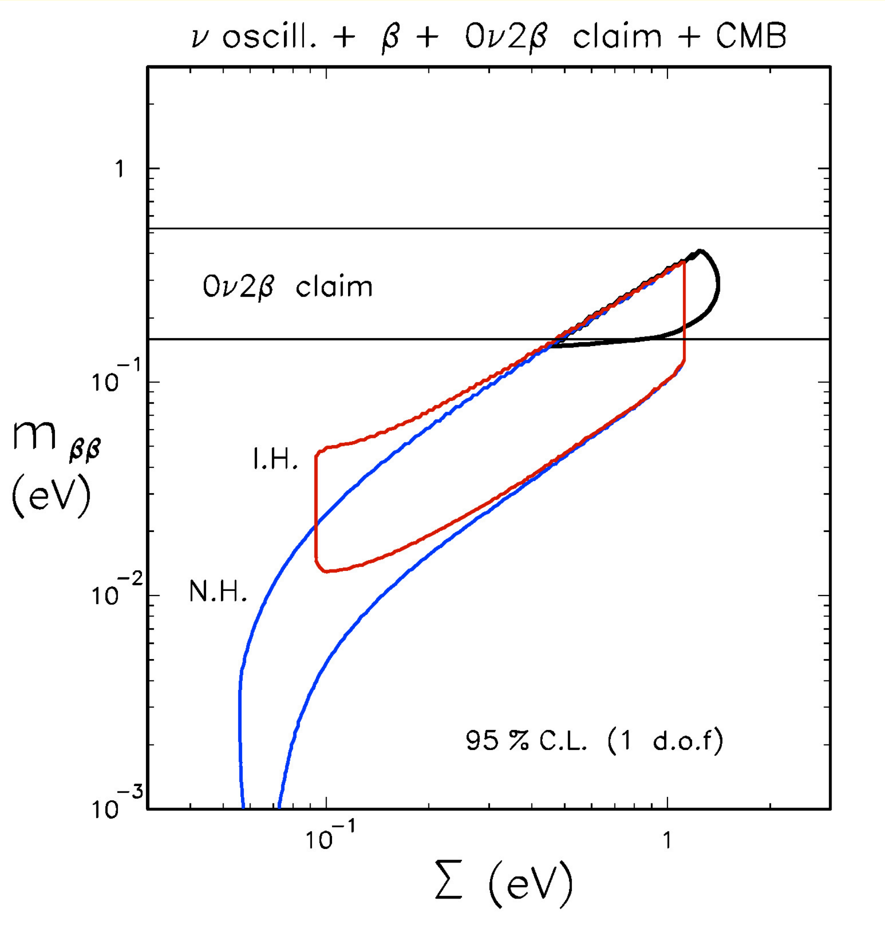
But the available data do not yet lead to definite conclusions.
Beta decay: no yet very constraining. Double beta vs cosmology: different possibilities. E.g.,

Cosmo-“aggressive”



The tighest cosmo bounds are not compatible with Klapdor’s claim. Then, either one of the two is wrong, or there is new physics beyond the standard model (of particle physics and/or of cosmology)

Cosmo-“conservative”



The safest cosmo bounds can be made compatible with Klapdor’s claim, with no new physics required. Then, the combination of data (black wedge) would prefer degenerate neutrino masses, ~few x 10^{-1} eV

Let's entertain the possibility that the "true" answer is just at or around the corner… For instance, that neutrinos are Majorana, with nearly degenerate and relatively large masses:

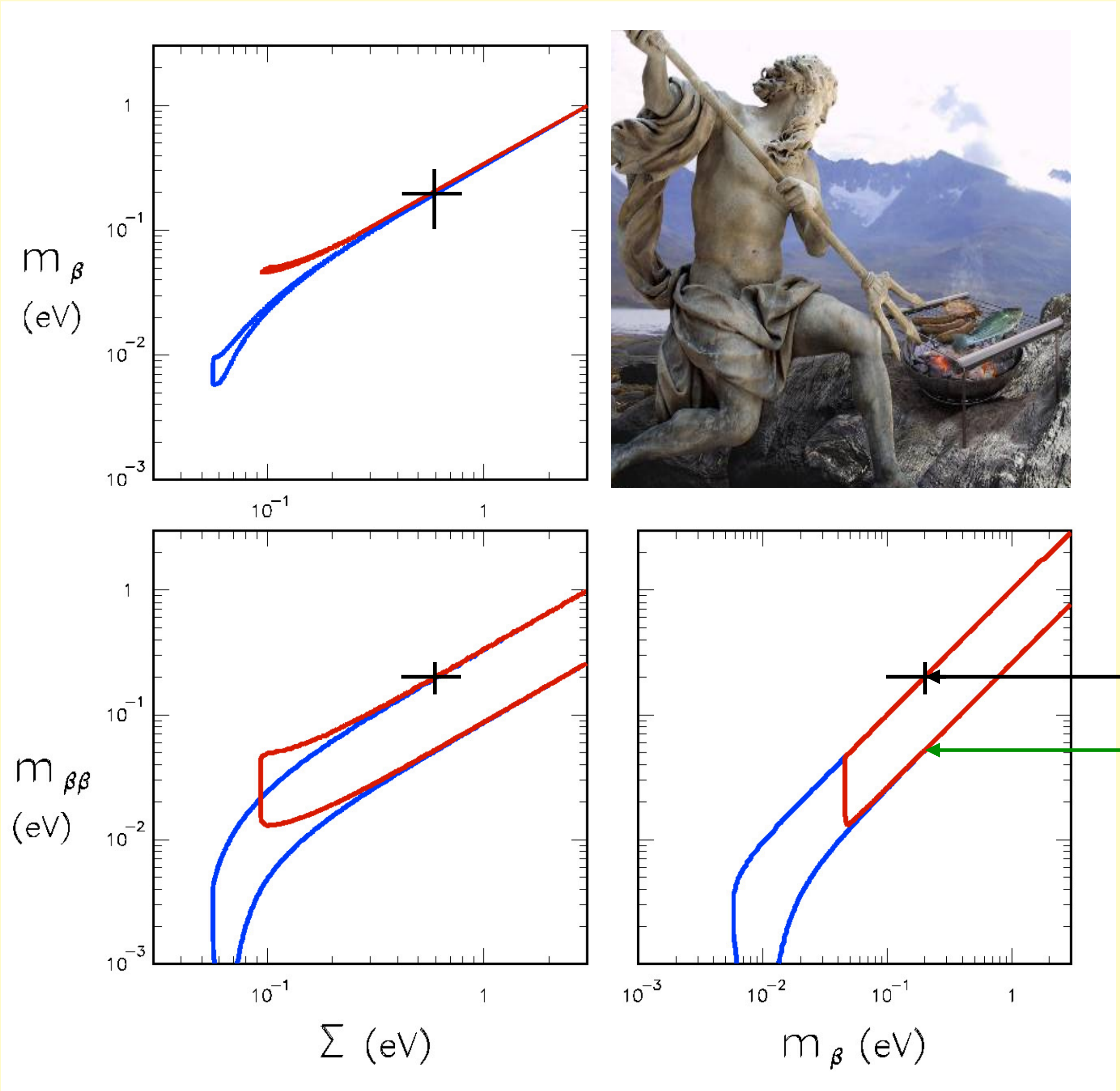
$m_1 \sim m_2 \sim m_3 \sim 0.2$ eV .

Then we might reasonably hope to observe soon all three nonoscillation signals in current/next generation experiments, e.g.,

$$
\begin{aligned}
m_{\beta\beta} &\simeq 0.2(1 \pm 0.3)\ \mathrm{eV} \\
\Sigma &\simeq 0.6(1 \pm 0.3)\ \mathrm{eV} \\
m_{\beta} &\simeq 0.2(1 \pm 0.5)\ \mathrm{eV}
\end{aligned}
$$

in which case…

…The absolute neutrino mass would be established within ~25% uncertainty, and one Majorana phase (ϕ_2) would be constrained…



$\exp(i\phi_2) = +1$

$\exp(i\phi_2) = -1$
(disfavored)

Neutrino masses are also a window to new physics...

Dirac and Majorana mass terms + See saw (1 family)

- Dirac mass terms are of the form $m\bar{\psi}\psi$ (4 dof ψ)
- Majorana " " " " " " $\frac{1}{2}m\bar{\psi}\psi$ (2 dof ψ)

Three possibilities:

Dirac : $\psi = \psi_L + \psi_R \rightarrow \bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L$

Majorana (L) : $\psi = \psi_L + \psi_L^c \rightarrow \bar{\psi}\psi = \bar{\psi}_L\psi_L^c + \bar{\psi}_L^c\psi_L$

Majorana (R) : $\psi = \psi_R + \psi_R^c \rightarrow \bar{\psi}\psi = \bar{\psi}_R\psi_R^c + \bar{\psi}_R^c\psi_R$

} absent for charged fermions!

Most general mass term for one neutrino family:

$$\boxed{\begin{aligned} & m_D(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L) + \tfrac{1}{2}m_L(\bar{\psi}_L\psi_L^c + \bar{\psi}_L^c\psi_L) + \tfrac{1}{2}m_R(\bar{\psi}_R\psi_R^c + \bar{\psi}_R^c\psi_R) \\ &= \frac{1}{2}\left[\bar{\psi}_L + \bar{\psi}_L^c,\ \bar{\psi}_R + \bar{\psi}_R^c\right]\begin{bmatrix} m_L & m_D \\ m_D & m_R \end{bmatrix}\begin{bmatrix} \psi_L + \psi_L^c \\ \psi_R + \psi_R^c \end{bmatrix} \quad \text{(matrix form)} \end{aligned}}$$

In the above eq. in matrix form, the basis fields $\psi_L + \psi_L^c$ and $\psi_R + \psi_R^c$ are Majorana. Therefore, in general, diagonalization will produce mass eigenvectors which are also Majorana, despite the presence of a Dirac mass term (unless special cancellations occur).

Explicit diagonalization of $M = \begin{bmatrix} m_L & m_D \\ m_D & m_R \end{bmatrix}$

- Trace $\quad T = \mathrm{Tr}\, M = m_L + m_R$
- Determ. $\quad D = \det M = m_L m_R - m_D^2$
- Eigenvalues : $m_\pm = \frac{1}{2}\left(T \pm \sqrt{T^2 - 4D}\right)$
- Diagonalization angle (not a mixing angle !) : $\sin 2\theta = \dfrac{m_D}{\sqrt{T^2 - 4D}} \qquad \cos 2\theta = \dfrac{m_L - m_R}{\sqrt{T^2 - 4D}}$
- Diagonalizing rotation : $\begin{bmatrix} m_+ & 0 \\ 0 & m_- \end{bmatrix} = \begin{bmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{bmatrix} \begin{bmatrix} m_L & m_D \\ m_D & m_R \end{bmatrix} \begin{bmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{bmatrix}$
- Eigenvectors : $[\nu_1, \nu_2] \begin{bmatrix} m_L & m_D \\ m_D & m_R \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} = [\nu_1', \nu_2'] \begin{bmatrix} m_+ & 0 \\ 0 & m_- \end{bmatrix} \begin{bmatrix} \nu_1' \\ \nu_2' \end{bmatrix}$, $\begin{bmatrix} \nu_1' \\ \nu_2' \end{bmatrix} = \begin{bmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}$
- If one mass eigenvalue <0 , then redefine field : $\psi \to \gamma_5 \psi$, so that $m \to -m$.
- Special case : $M = \begin{bmatrix} 0 & m \\ m & \Lambda \end{bmatrix}$ with $m \ll \Lambda$. Then:

Eigenvectors (ν fields) :	Eigenvalues (masses)
$\nu_{heavy} \simeq (\nu_R + \nu_R^c) + \frac{m}{\Lambda}(\nu_L + \nu_L^c)$	Λ
$\nu_{light} \simeq -(\nu_L + \nu_L^c) + \frac{m}{\Lambda}(\nu_R + \nu_R^c)$	$(-)\,\frac{m^2}{\Lambda} \ll m$

$\leftarrow$ "**see-saw** mechanism"

The existence of a singlet neutrino ν_R is predicted in many extensions of the EW Standard Model. E.g., in the $\underline{16}$ representation of $SO(10)$:

$$\begin{pmatrix} u_L & u_L & u_L & \nu_L \\ d_L & d_L & d_L & e_L \\ u_R & u_R & u_R & \boldsymbol{\nu_R} \\ d_R & d_R & d_R & e_R \end{pmatrix}$$

→ Can get a Majorana mass term $\sim \Lambda(\bar{\nu}_R^c \nu_R + \bar{\nu}_R \nu_R^c)$, in addition to the "standard" Dirac mass term $\sim m(\bar{\nu}_L \nu_R + \bar{\nu}_R \nu_L)$, where m is naturally associated to the EW scale, while Λ is naturally associated to the scale of new physics (Λ_{GUT} ?) which characterizes the SM extension.

→ Diagonalizing $\begin{bmatrix} 0 & m \\ m & \Lambda \end{bmatrix}$ as in the previous page, one gets a light Majorana neutrino with mass $\sim \frac{m^2}{\Lambda}$, and a very heavy Majorana neutrino with mass $\sim \Lambda$. The light one (in the presence of more than one generations) enters in neutrino oscillations. Can the heavy one be "useful" as well? Yes!

→

→ There is the possibility to have **leptogenesis**.

- CP violation at high energy scales Λ might be responsible for different decay rates of heavy ν_R into charged leptons in the early universe :

$$\Gamma(\nu_R \to \ell^+ + \dots) \neq \Gamma(\nu_R \to \ell^- + \dots)$$

- This would provide an unbalance of leptons / antileptons, possibly at the origin of the matter-antimatter asymmetry of the universe.

→ Discovery of Majorana neutrinos + CP violation in the ν sector would make the see-saw + leptogenesis mechanisms more plausible (although it would be very difficult, if not impossible, to prove them).

Dirac and Majorana mass terms (more families)

- Most general case:
  - **3 LH gauge doublets** $\nu_{\alpha L}$ $(\alpha = e\mu\tau)$ "ACTIVE" in EW interactions
  - **N_s RH gauge singlets** ν_{sR} $(s = 1, \ldots, N_s)$ "STERILE" in EW interactions

  where N_s can be any number, at any mass scale (light or heavy).

- Most general mass matrix has a block form,
  $M_L = 3\times 3$, $M_D = 3\times N_s$, $M_R = N_s \times N_s$

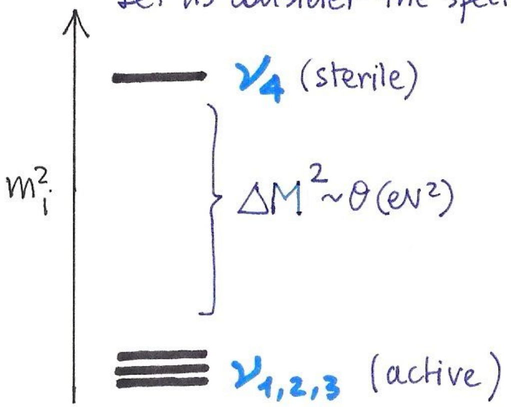
$$M = \left[\begin{array}{c:c} M_L & M_D \\ \hdashline M_D' & M_R \end{array}\right]$$

- After diagonalization, generic eigenvectors (ν fields) will be Majorana
  → expect $0\nu\beta\beta$ decay allowed
  → expect Majorana phases besides $\not{C}\not{P}$ phase

- After diagonalization, active and sterile neutrinos will be mixed
  → expect see-saw suppression of such mixing, but ... who knows?
  → U_{PMNS} not precisely unitary in general
  → active/sterile ν oscillations

Exercise: One-dominant-mass-scale oscillations for a 4th, sterile neutrino ν_s

Let us consider the spectrum:



Then, the PMNS 3x3 matrix is not unitary, being part of a larger 4x4 mixing matrix:

$$U = \begin{pmatrix} \begin{bmatrix} & U_{PMNS} & \end{bmatrix} & \begin{matrix} U_{e4} \\ U_{\mu 4} \\ U_{\tau 4} \end{matrix} \\ U_{s1} \quad U_{s2} \quad U_{s3} & U_{s4} \end{pmatrix}, \quad \text{with } |U_{s4}|^2 \simeq 1 - \text{epsilon},$$

in order not to alter too much the established 3ν phenomenology $\rightarrow |U_{\alpha 4}|^2_{\alpha = e,\mu,\tau} \ll 1$

For experiments sensitive mainly to $\Delta M^2 \sim \mathcal{O}(\mathrm{eV}^2)$, one can take the limits $\delta m^2 \to 0$ and $\Delta m^2 \to 0$, and apply the same logic as for one dominant mass scale in $3\nu \cong (2\nu) \oplus (1\nu) \Rightarrow 4\nu \simeq (3\nu) \oplus (1\nu)$. Similarly, one gets (for $\alpha, \beta = e \mu \tau$)

Disappearance $(\alpha = \beta)$: $\quad 1 - P_{\alpha\alpha} \simeq 4|U_{\alpha 4}|^2 (1 - |U_{\alpha 4}|^2) \sin^2\left(\frac{\Delta M^2 x}{4E}\right) \quad \leftarrow$ "singly" suppressed by $|U_{\alpha 4}|^2 \ll 1$

Appearance $(\alpha \neq \beta)$: $\quad P_{\alpha\beta} \simeq 4|U_{\alpha 4}|^2 |U_{\beta 4}|^2 \sin^2\left(\frac{\Delta M^2 x}{4E}\right) \quad \leftarrow$ "doubly" suppressed by both $|U_{\alpha 4}| \ll 1$ and $|U_{\beta 4}| \ll 1$

Exercise: Majorana/Dirac "confusion" in ν oscillations

For Majorana ν's, the mixing matrix U is generalized as:
$U \to U \cdot U_M$, where $U_M = \mathrm{diag}(1, e^{i\phi}, e^{i\phi'})$ contains two independent Majorana phases.

In the hamiltonian of ν oscillations (either in vacuum or in matter), the mixing matrix always appears in the form:
$U \frac{\mathcal{M}^2}{2E} U^+$, with $\mathcal{M}^2 = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$.

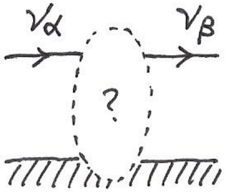
The replacement $U \to U U_M$ is then ineffective:

$$U \frac{\mathcal{M}^2}{2E} U^+ \to U U_M \frac{\mathcal{M}^2}{2E} U_M^+ U = U \frac{\mathcal{M}^2}{2E} U^+ .$$

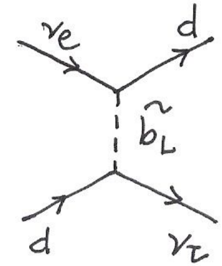
Thus, oscillations do not distinguish Dirac/Majorana neutrinos ("confusion" of the two possibilities).

New neutrino interactions

New physics beyond the SM might also be responsible for new interactions of neutrinos, e.g., FCNC:



→ an example: SUSY $\cancel{R}$:



→ May get relevant modifications of the evolution hamiltonian:

$$H = \underbrace{H_{vac} + H_{mat}}_{\text{standard}} + \underbrace{H_{new\,physics}}_{\text{nonstandard}}$$

However, new coupling are tipically expected to be $\mathcal{O}(\epsilon G_F)$ with $\epsilon \ll 1$ → effects difficult to disentangle from standard oscillations.

End of 1st part of Lecture II

Lesson II - 2nd part (~ 1h):

* Oscillations - some open problems:
  - mass hierarchy
  - CP violation and precision frontier
    (+ long exercise on $P_{e\mu}$)
  - sterile neutrinos

* Appendix for further reading:
  Notes on statistics and data analysis

Hierarchy: no current hint ! How can we get sign($\pm\Delta m^2$) ?

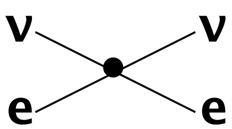
Is the state with smallest ν_e component (ν_3) lightest or heaviest?

The hierarchy, namely, sign($\pm\Delta m^2$), can be probed (in principle), via interference of Δm^2-driven oscillations with some other Q-driven oscillations, where Q is a quantity with known sign.

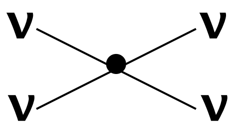
At present, the only known possibilities (barring new physics) are:

Q = δm^2 (e.g., high-precision oscillometry in vacuum)

Q = Electron density (e.g., matter effects in Earth)



Q = Neutrino density (SN $\nu-\nu$ interaction effects)

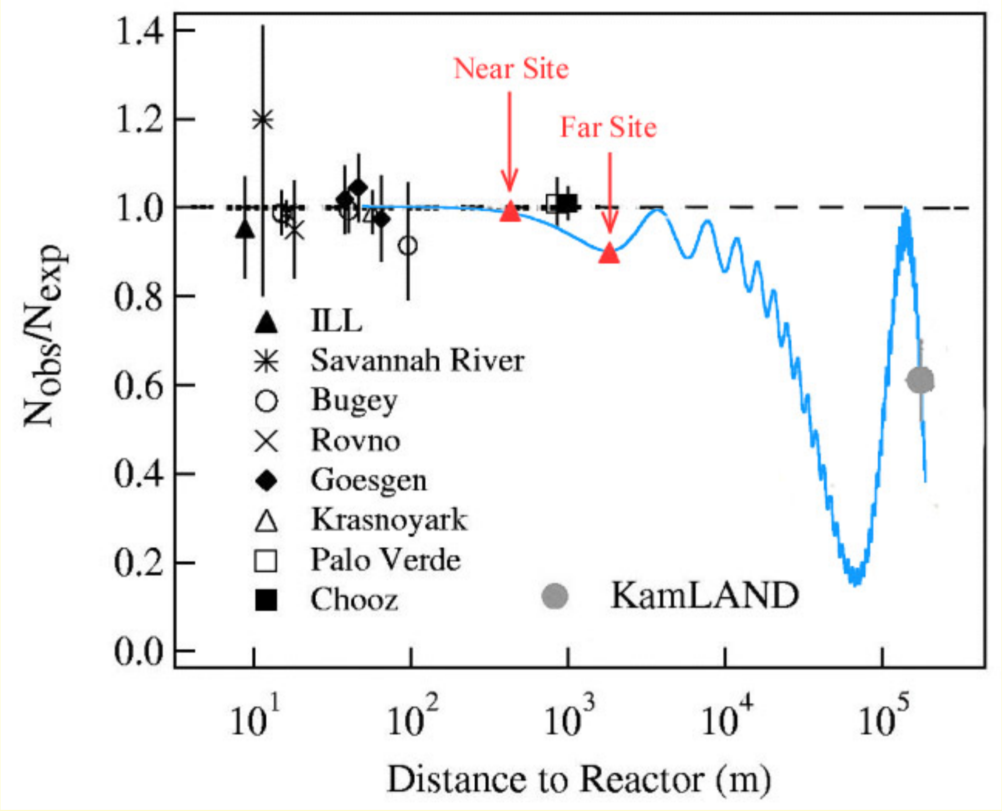


Each of them is very challenging, for rather different reasons.

All of them are worth revisiting in more detail for "large" θ_{13} !

Precision oscillometry with reactors

Short baseline O(1 km): sensitive to δm^2
Long baseline O(100 km): sensitive to Δm^2



Can "medium" baselines, ~50 km, be sensitive to both?
In principle, yes... (Petcov & Piai et al)

Full 3-neutrino oscillations at reactors :

$$
\begin{aligned}
1 - P_{ee} &= 4|U_{e1}|^2|U_{e2}|^2 \sin^2\left(\frac{\delta m^2 L}{4E}\right) \\
&+ 4|U_{e1}|^2|U_{e3}|^2 \sin^2\left(\frac{\pm\Delta m^2 + \delta m^2/2}{4E}\, L\right) \\
&+ 4|U_{e2}|^2|U_{e3}|^2 \sin^2\left(\frac{\pm\Delta m^2 - \delta m^2/2}{4E}\, L\right)
\end{aligned}
$$

"slow osc."
(long baseline)

"fast osc."
(short baseline)

Note that the "fast oscillation" terms are not invariant under swap of hierarchy, having different amplitudes.

Of course, their separation would demand very high statistics and resolution, and accurate energy scale

Exercise: Calculate the exact $P_{ee}^{3\nu}$ in vacuum (useful for reactor ν).

Let us consider normal hierarchy for definiteness. Then:

$m_2^2 - m_1^2 = \delta m^2$

$m_3^2 - m_2^2 = \Delta m^2 - \delta m^2/2$

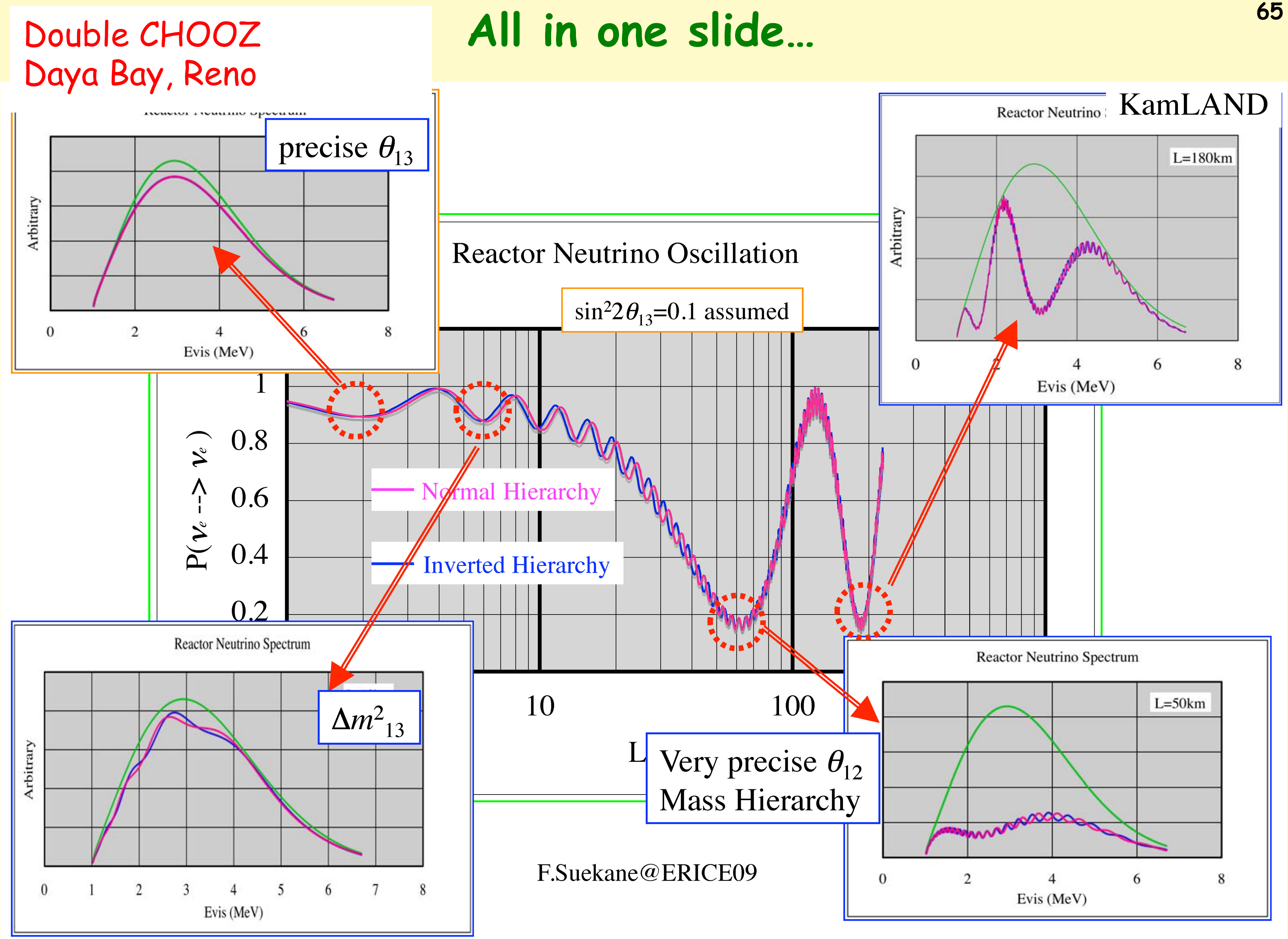
$m_3^2 - m_1^2 = \Delta m^2 + \delta m^2/2$

$\mathrm{Im}(J_{ee}^{ij}) = 0$; $\mathrm{Re}(J_{ee}^{ij}) = |U_{ei}|^2 |U_{ej}|^2 = \begin{cases} s_{12}^2 c_{12}^2 c_{13}^4 & ij = 12 \\ s_{12}^2 s_{13}^2 c_{13}^2 & ij = 23 \\ c_{12}^2 s_{13}^2 c_{13}^2 & ij = 13 \end{cases}$; Then:

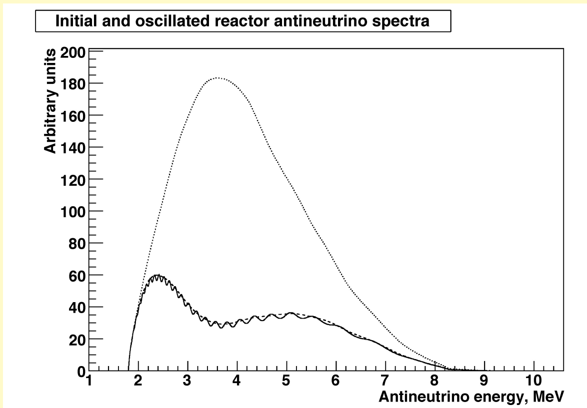
$$\boxed{\begin{aligned} P_{ee}^{3\nu} = 1 &- \cos^4\theta_{13} \sin^2 2\theta_{12} \sin^2\left(\frac{\delta m^2 x}{4E}\right) \\ &- \sin^2\theta_{12} \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2 - \frac{\delta m^2}{2}}{4E} x\right) \\ &- \cos^2\theta_{12} \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2 + \frac{\delta m^2}{2}}{4E} x\right) \end{aligned}}$$

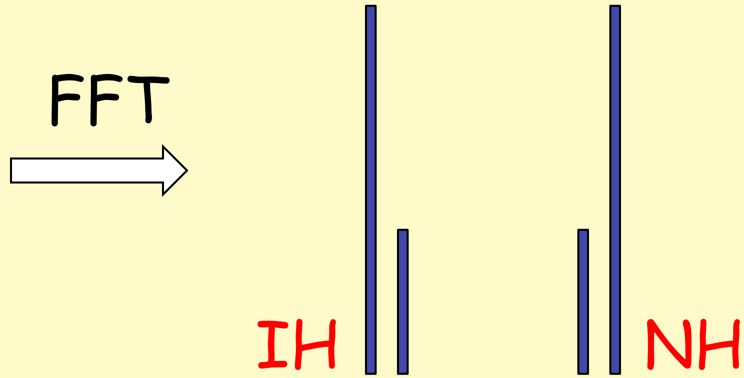
Note that $P_{ee}^{3\nu}$ is not invariant under the replacement $\Delta m^2 \to -\Delta m^2$. It would be so only for $\theta_{12} = \pi/4$ (i.e., $\sin^2\theta_{12} = \frac{1}{2} = \cos^2\theta_{12}$), which is excluded experimentally ($\sin^2\theta_{12} \simeq 0.3$).

All in one slide…



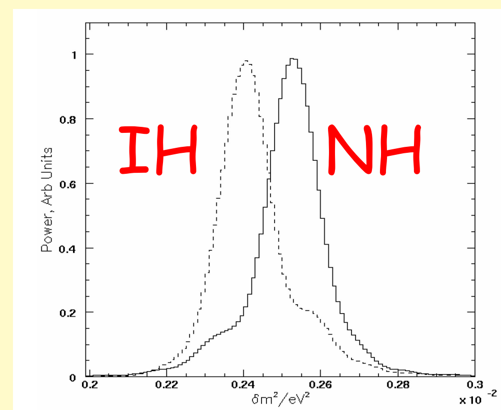
E.g.: **Hierarchy** via Fourier analysis of the fast oscillations (Learned et al.)
For perfect resolution, should find two high frequencies $\Delta m^2 \pm \delta m^2/2$ with different amplitudes:





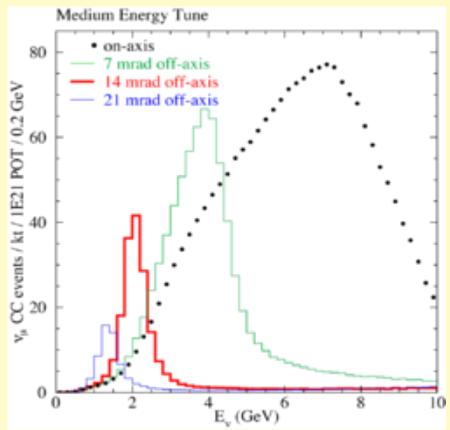
For finite resolution, the two peaks would merge, but the lowest one should still survive as a "shoulder" on the left (NH) or on the right (IH) of the dominant peak.

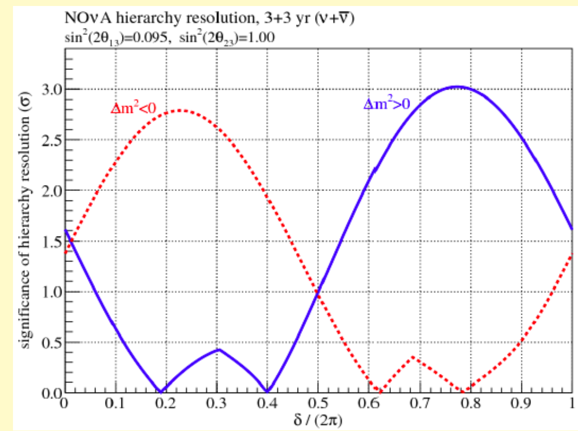
Feasibility studies focused on possible follow-up of Daya Bay and/or RENO.



A second approach to hierarchy is provided by the usual MSW effect (neutrino-matter forward scattering). Fractional variation of amplitude or phase is roughly $\pm 2\sqrt{2} G_F N_e E/(\pm\Delta m^2)$, where the first $\pm$ refers to nu/antinu and the second to NH/IH.

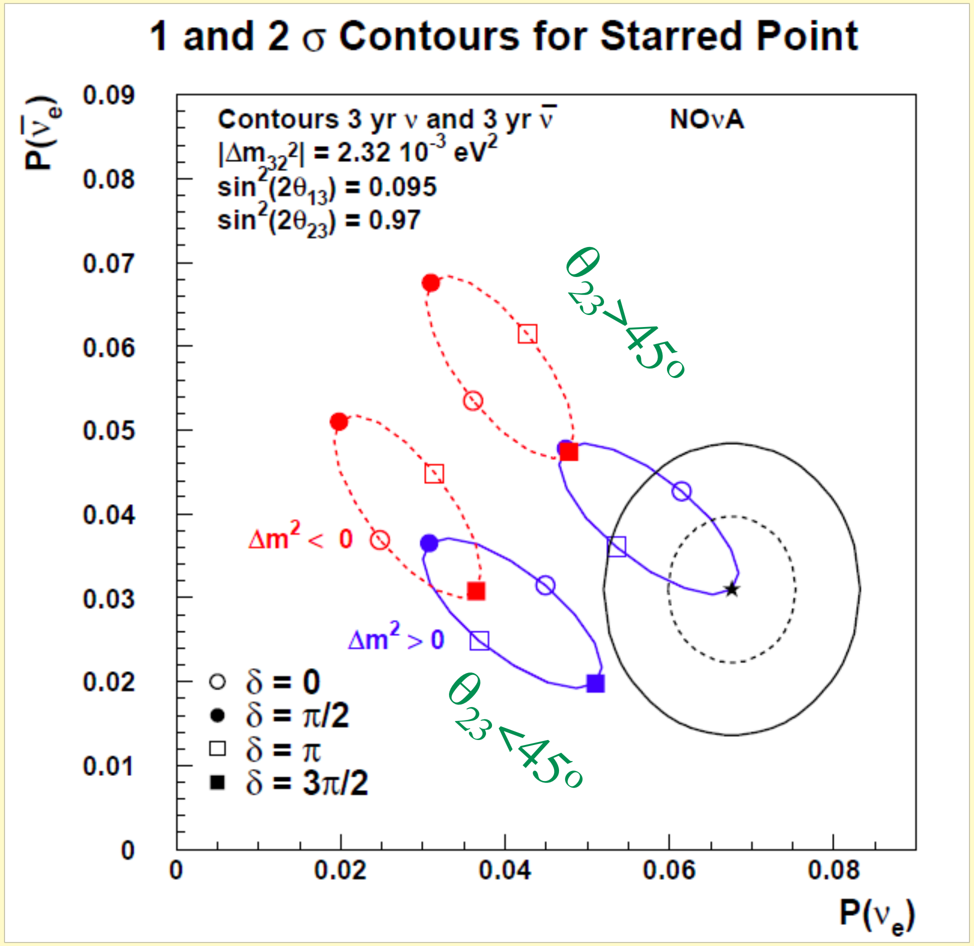
Variations can be up to ~30% in accelerator beams with relatively sharp E-spectra (off-axis) and relatively long L inside the Earth crust (optimal choice: ~oscillation maximum). E.g., NOνA:





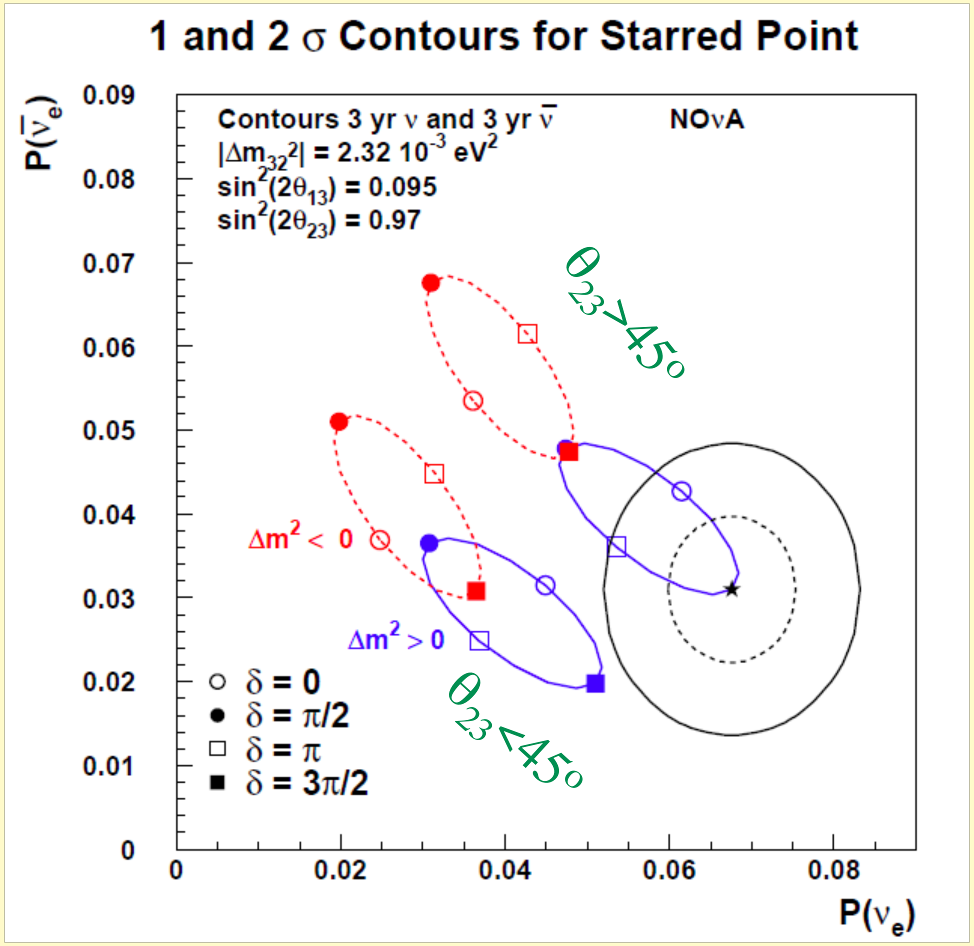
But: absolute amplitude of $\nu_\mu \to \nu_e$ (which scales as $\sin^2\theta_{13}$) has strong δ dependence. Must be lucky !

Discussion of NOvA, T2K, future LBL experiments often refers to bi-probability plots for electron flavor appearance in the neutrino and antineutrino channels:



At fixed L, E and N_e, the ellipses are parametric curves as a function of the CP phase δ (→ measurable in principle via ν-antiν comparison)

These curves are useful to address the serious problem of degeneracies: one experimental point can correspond to different hierarchies or octants or CP phases (clone solutions)

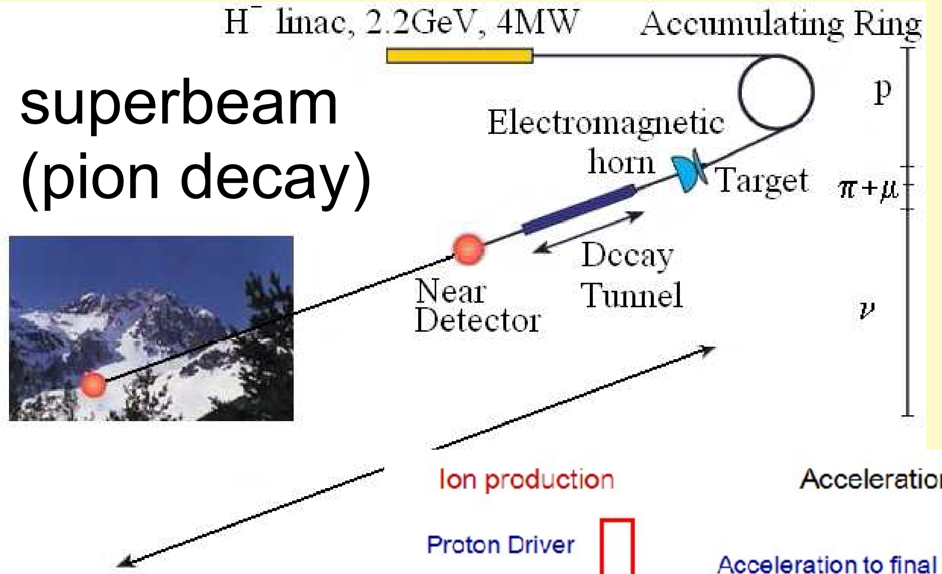


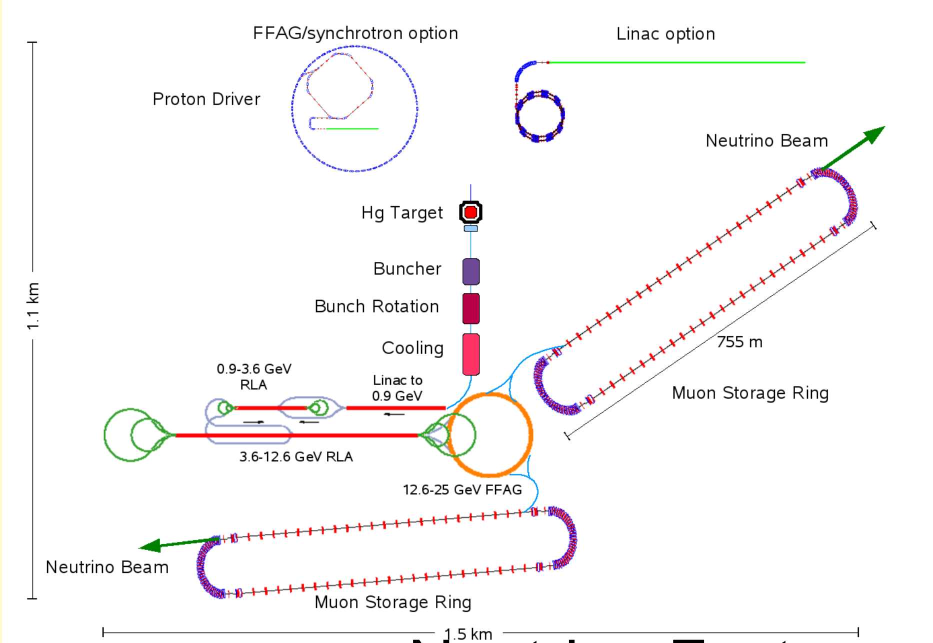
Large literature on degeneracy problem; solution requires data at different L, E and in different oscillation channels.

Intense R&D on beams...

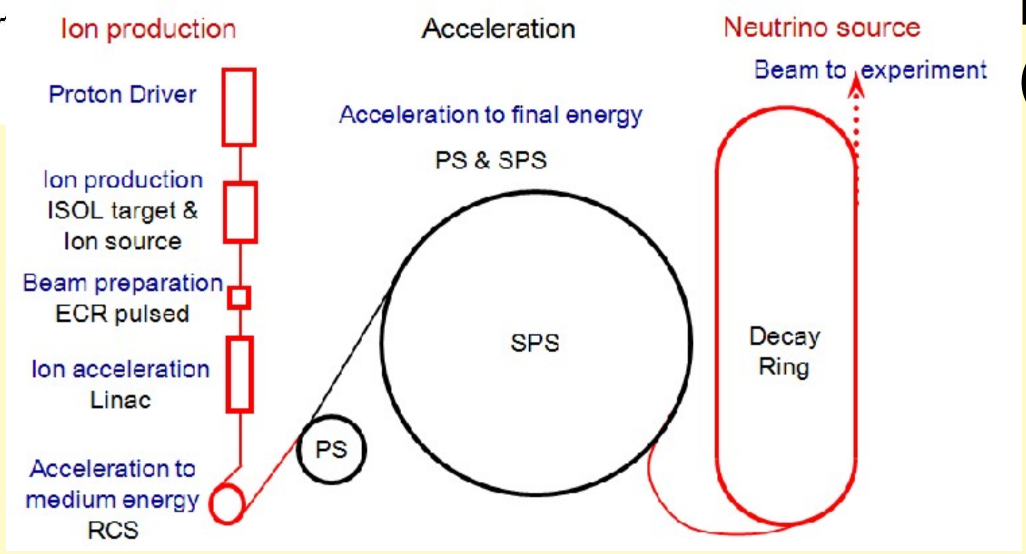
(from A. Blondel)

superbeam
(pion decay)





Neutrino Factory
(muon decay)



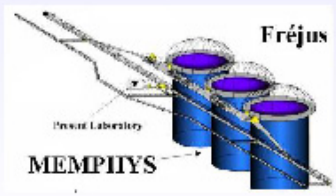
... on sites
(?) and on...

Beta-beam
ion decay 6He, 18Ne or 8Be, 8Li

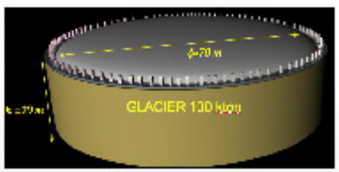
…possible big detectors, also for other purposes (p-decay, SN, solar, atmospheric, geo-nu…).

Just one example:

LAGUNA detector concepts



- MEMPHYS - MEgaton Mass PHYSics
  - 80 m heigth $\times$65 m $\varnothing$
  - $\sim$500 kt water Cherenkov detector
  - 81 000 PMTs per shaft (30% coverage)



- GLACIER - Giant Liquid Argon Charge Imaging ExpeRiment
  - 20 m heigth $\times$70 m $\varnothing$
  - $\sim$ 100 kt liquid Ar TPC
  - Light (28 000 PMTs) + charge readout

- LENA - Low Energy Neutrino Astronomy
  - 100 m long $\times$ 30 m $\varnothing$
  - $\sim$ 50 kt liquid scintillator
  - 13 500 PMTs for 30% coverage

Neutrino Champagne 2009 Teresa Marrodán Undagoitia

Discussion about beams/sites/detectors is ongoing.

Several studies on systematic comparison of different options in terms of discovery potential, **accuracy**, physics synergies, cost etc., especially towards the **goal of CP violation** searches in neutrinos; e.g.,

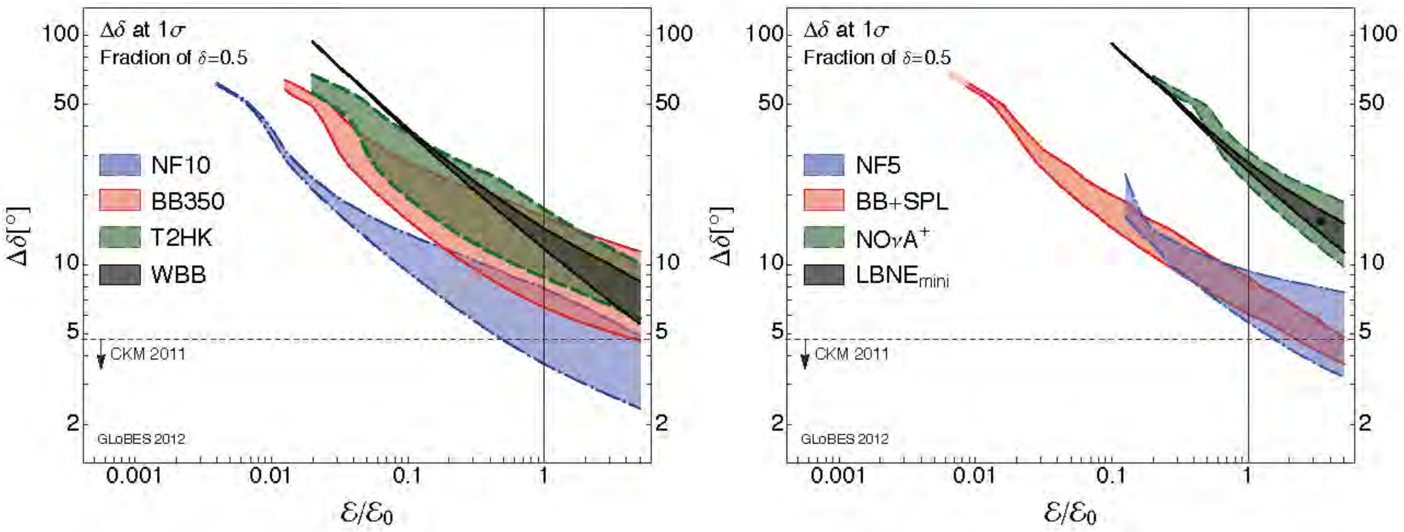


Figure 4: Error on δ (at 1σ, for $\sin^2 2\theta_{13} = 0.1$) as a function of exposure, where the bands reflect the variation in the results due to different assumptions for the systematic errors between the optimistic (lower edges) and conservative (upper edges) values in Table 2. In the left panel, the results for the benchmark setups from Table 1 are shown, while the right panel shows the results for the alternative setups. The nominal exposure $\mathcal{E}_0$, to which the exposure $\mathcal{E}$ on the horizontal axis is normalized, is the one given in Table 1. Here near detectors are included, and the results are shown for the median values of δ (fraction of δ is 50%). The current precision on the CP phase in the CKM matrix V_{CKM} is also indicated. In the right panel, the black dot indicates the luminosity for the original LBNE configuration (34 kt LAr detector [54]).

(from Coloma, Huber, Kopp, Winter)

Such studies often refer to the following appearance probability:

$$\nu_\mu \leftrightarrow \nu_e$$

$$\begin{aligned} P_{\text{app}} \simeq\ & \sin^2 2\theta_{13} \sin^2\theta_{23} \frac{\sin^2[(1-\hat{A})\Delta]}{(1-\hat{A})^2} \\ \pm\ & \alpha \sin 2\theta_{13}\, \xi \sin\delta_{\text{CP}} \sin(\Delta) \frac{\sin(\hat{A}\Delta)}{\hat{A}} \frac{\sin[(1-\hat{A})\Delta]}{(1-\hat{A})} \\ +\ & \alpha \sin 2\theta_{13}\, \xi \cos\delta_{\text{CP}} \cos(\Delta) \frac{\sin(\hat{A}\Delta)}{\hat{A}} \frac{\sin[(1-\hat{A})\Delta]}{(1-\hat{A})} \\ +\ & \alpha^2 \cos^2\theta_{23} \sin^2 2\theta_{12} \frac{\sin^2(\hat{A}\Delta)}{\hat{A}^2}, \end{aligned}$$

$$\alpha \equiv \frac{\Delta m^2_{21}}{\Delta m^2_{31}} \simeq \pm 0.03,\ \Delta \equiv \frac{\Delta m^2_{31} L}{4E},\ \xi \equiv \sin 2\theta_{12} \sin 2\theta_{23},\ \hat{A} \equiv \pm \frac{2\sqrt{2} G_F n_e E}{\Delta m^2_{31}}$$

(Cervera et al. 2000; Freund, Huber, Lindner, 2000; Freund, 2001)

- Complicated, but all interesting information there: θ_{13}, δ_{CP}, mass hierarchy (via A)

Slide from: Walter Winter

... in various different versions, e.g.,

$$
\begin{aligned}
P_{\nu_\mu \to \nu_e} = {} & 4c_{13}^2 s_{13}^2 s_{23}^2 \sin^2 \frac{\Delta m_{13}^2 L}{4E} \times \left[1 \pm \frac{2a}{\Delta m_{13}^2} (1 - 2s_{13}^2) \right] && \theta_{13}\ \text{driven} \\
& + 8c_{13}^2 s_{12} s_{13} s_{23} (c_{12} c_{23} \cos\delta - s_{12} s_{13} s_{23}) \cos \frac{\Delta m_{23}^2 L}{4E} \sin \frac{\Delta m_{13}^2 L}{4E} \sin \frac{\Delta m_{12}^2 L}{4E} && \text{CPev} \\
& \mp 8c_{13}^2 c_{12} c_{23} s_{12} s_{13} s_{23} \sin\delta \sin \frac{\Delta m_{23}^2 L}{4E} \sin \frac{\Delta m_{13}^2 L}{4E} \sin \frac{\Delta m_{12}^2 L}{4E} && \text{CPodd} \\
& + 4s_{12}^2 c_{13}^2 \{ c_{13}^2 c_{23}^2 + s_{12}^2 s_{23}^2 s_{13}^2 - 2c_{12} c_{23} s_{12} s_{23} s_{13} \cos\delta \} \sin^2 \frac{\Delta m_{12}^2 L}{4E} && \text{solar driven} \\
& \mp 8c_{12}^2 s_{13}^2 s_{23}^2 \cos \frac{\Delta m_{23}^2 L}{4E} \sin \frac{\Delta m_{13}^2 L}{4E} \frac{aL}{4E} (1 - 2s_{13}^2) && \text{matter effect (CP odd)}
\end{aligned}
$$

Note that matter effects are trivially “CP-violating” (i.e., induce a difference between neutrinos and antineutrinos) since ordinary matter contains electrons but not positrons. Future LBL experiments seeking to measure genuine CP effects due to the phase δ must “disentangle” matter effects.

A phenomenological analysis of this equation, in the context of current and future oscillation searches seeking effects of hierarchy, octant, CP-violation, and Earth matter, would require dedicated lectures.

There are many excellent talks/lectures/reviews devoted to such studies, and to the optimization of prospective facilities in order to observe subleading effects.

However, it's very difficult to find a pedagogical derivation of the previous (approximate) formula, which is at the basis of bi-probability plots and of various optimizations.

You can find it in the next slides (longest exercise of this course).

General reduction tools for $P_{\alpha\beta}$ in matter

- 3ν Hamiltonian in flavor basis for generic $N_e(x)$ profile:

  $\tilde{H} = \frac{1}{2E} U \mathcal{M}^2 U^\dagger + V$

  $\mathcal{M}^2 = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$

  $U = O_{23}\Gamma_\delta O_{13} \Gamma_\delta^\dagger O_{12}$ with $\Gamma_\delta = \mathrm{diag}(1, 1, e^{i\delta})$ and $O_{ij}^\dagger = O_{ij}^T$

  $V(x) = \mathrm{diag}(\sqrt{2}\, G_F N_e(x), 0, 0)$

- It is easy to verify that:

  $(O_{23}\Gamma_\delta)^\dagger V (O_{23}\Gamma_\delta) = V$

  $\Gamma_\delta^\dagger O_{12} \mathcal{M}^2 O_{12}^T \Gamma_\delta = O_{12} \mathcal{M}^2 O_{12}^T$

- Let's go from the flavor basis to a new "primed flavor" basis defined by:

  $$\begin{bmatrix} (\nu_e)' \\ (\nu_\mu)' \\ (\nu_\tau)' \end{bmatrix} = (O_{23}\Gamma_\delta)^\dagger \begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} \leftarrow \text{components}, \quad \text{with } O_{23}\Gamma_\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23}e^{i\delta} \\ 0 & -s_{23} & c_{23}e^{i\delta} \end{pmatrix} \quad (\text{note } (\nu_e)' = \nu_e)$$

- Hamiltonian in the primed basis:

  $\tilde{H}' = (O_{23}\Gamma_\delta)^\dagger \tilde{H} (O_{23}\Gamma_\delta) = O_{13} O_{12} \frac{\mathcal{M}^2}{2E} (O_{13} O_{12})^T + V$

  Note that $\tilde{H}'$ does not depend on δ (and is thus real symmetric) nor on θ_{23}.

  It is thus simpler to find the evolution operator $\tilde{S}'$ in the primed basis.

- The evolution operator $\tilde{S}$ in the flavor basis is then given by:
$$\tilde{S}(x_f, x_i) = (O_{23}\Gamma_\delta)\,\tilde{S}'(x_f, x_i)\,(O_{23}\Gamma_\delta)^+$$
- In terms of matrix components:

if $\tilde{S}' = \begin{pmatrix} \tilde{S}'_{ee} & \tilde{S}'_{e\mu} & \tilde{S}'_{e\tau} \\ \tilde{S}'_{\mu e} & \tilde{S}'_{\mu\mu} & \tilde{S}'_{\mu\tau} \\ \tilde{S}'_{\tau e} & \tilde{S}'_{\tau\mu} & \tilde{S}'_{\tau\tau} \end{pmatrix}$, then $\tilde{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23}e^{i\delta} \\ 0 & -s_{23} & c_{23}e^{i\delta} \end{pmatrix} \cdot \tilde{S}' \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & -s_{23} \\ 0 & s_{23}e^{i\delta} & c_{23}e^{-i\delta} \end{pmatrix}$, namely:

$$\tilde{S}_{ee} = \tilde{S}'_{ee}$$
$$\tilde{S}_{\mu e} = \tilde{S}'_{\mu e}\,c_{23} + \tilde{S}'_{\tau e}\,s_{23}\,e^{i\delta}$$
$$\tilde{S}_{\tau e} = -\tilde{S}'_{\mu e}\,s_{23} + \tilde{S}'_{\tau e}\,c_{23}\,e^{i\delta}$$
$$\tilde{S}_{\mu\mu} = \tilde{S}'_{\mu\mu}\,c^2_{23} + \tilde{S}'_{\mu\tau}\,c_{23}\,s_{23}\,e^{-i\delta} + \tilde{S}'_{\tau\mu}\,c_{23}\,s_{23}\,e^{i\delta} + \tilde{S}'_{\tau\tau}\,s^2_{23}$$
$$\tilde{S}_{\tau\mu} = -\tilde{S}'_{\mu\mu}\,c_{23}\,s_{23} - \tilde{S}'_{\mu\tau}\,s^2_{23}\,e^{-i\delta} + \tilde{S}'_{\tau\mu}\,c^2_{23}\,e^{i\delta} + \tilde{S}'_{\tau\tau}\,c_{23}\,s_{23}$$
$$\tilde{S}_{\tau\tau} = \tilde{S}'_{\mu\mu}\,s^2_{23} - \tilde{S}'_{\mu\tau}\,c_{23}\,s_{23}\,e^{-i\delta} - \tilde{S}'_{\tau\mu}\,c_{23}\,s_{23}\,e^{i\delta} + \tilde{S}'_{\tau\tau}\,c^2_{23}$$

with $\tilde{S}_{e\mu}$, $\tilde{S}_{e\tau}$, $\tilde{S}_{\mu\tau}$ obtained from $\tilde{S}'_{\alpha\beta} \leftrightarrow \tilde{S}'_{\beta\alpha}$ <u>and</u> $+\delta \leftrightarrow -\delta$

- Note that $\tilde{S}'_{\alpha\beta} = \tilde{S}'_{\beta\alpha}$ for symmetrical matter density profiles. Proof follows.

In general, $\tilde{S}'_{\alpha\beta} \neq \tilde{S}'_{\beta\alpha}$, even if $\tilde{H}'_{\alpha\beta} = \tilde{H}'_{\beta\alpha}$ (real symmetric). Indeed, let us divide a generic profile $N_e(x)$ into N steps $\{\Delta x_i\}_{i=1\ldots N}$, where $N_e \sim$ const in each step. Then:
$$\tilde{S}' = e^{-i\tilde{H}'_N \Delta x_N}\, e^{-i\tilde{H}'_{N-1}\Delta x_{N-1}} \cdots e^{-i\tilde{H}'_2\Delta x_2}\, e^{-i\tilde{H}'_1\Delta x_1}.$$

Although $(\tilde{H}_i')^T = \tilde{H}_i'$, the transpose of $\tilde{S}'$ is not equal to $\tilde{S}'$, since the ordering of the steps is reversed from $1, \dots, N$ to $N, \dots, 1$ ("reverse" density profile):
$(\tilde{S}')^T = e^{-i\tilde{H}_1' \Delta x_1} \dots e^{-i\tilde{H}_N' \Delta x_N}$. In other words:

$\tilde{S}'_{\alpha\beta}$ [direct profile] $= \tilde{S}'_{\beta\alpha}$ [reverse profile] $\neq \tilde{S}'_{\beta\alpha}$ [direct profile]

Only if the direct and reverse profile are symmetrical (i.e., coincide upon reflection) it is $\tilde{S}'_{\alpha\beta}$ [symmetric] $= \tilde{S}'_{\beta\alpha}$ [symmetric].

This is true, in particular, for constant density: $\tilde{S}'_{\alpha\beta} = \tilde{S}'_{\beta\alpha}$.

In this case: $\tilde{S}_{\alpha\beta} = \tilde{S}_{\beta\alpha}\,(\delta \to -\delta)$. For the osc. probabilities $P_{\alpha\beta} = |\tilde{S}_{\beta\alpha}|^2 = P(\nu_\alpha \to \nu_\beta)$:

$$P_{\alpha\beta} = P_{\beta\alpha}\,(\delta \to -\delta).$$

- Further reductions come from some peculiar symmetries of $\tilde{S}_{\alpha\beta}$ under the substitutions $s_{23} \to \pm c_{23}$ and $c_{23} \to \mp s_{23}$:

$$\tilde{S}_{\tau e} = \pm \tilde{S}_{\mu e}\Big|_{\substack{s_{23} \to \pm c_{23} \\ c_{23} \to \mp s_{23}}} \implies P_{e\tau} = P_{e\mu}\Big|_{\substack{s_{23} \to \pm c_{23} \\ c_{23} \to \mp s_{23}}} \overset{\text{def}}{=} P'_{e\mu}$$

$$\tilde{S}_{\mu\tau} = \mp \tilde{S}_{\tau\mu}\Big|_{\substack{s_{23} \to \pm c_{23} \\ c_{23} \to \mp s_{23}}} \implies P_{\tau\mu} = P_{\mu\tau}\Big|_{\substack{s_{23} \to \pm c_{23} \\ c_{23} \to \mp s_{23}}} \overset{\text{def}}{=} P'_{\mu\tau}$$

$$\tilde{S}_{\mu\mu} = \pm \tilde{S}_{\tau\tau}\Big|_{\substack{s_{23} \to \pm c_{23} \\ c_{23} \to \mp s_{23}}} \implies P_{\mu\mu} = P_{\tau\tau}\Big|_{\substack{s_{23} \to \pm c_{23} \\ c_{23} \to \mp s_{23}}} \overset{\text{def}}{=} P'_{\tau\tau}$$

- The previous relations, together with the unitarity of $P_{\alpha\beta}$, allow to express all the probabilities in terms of just two, e.g., $P_{e\mu}$ and $P_{\mu\tau}$, and their transformed $P'_{e\mu} = P_{e\mu}\Big|_{\substack{s_{23}\to\pm c_{23}\\ c_{23}\to\mp s_{23}}}$ and $P'_{\mu\tau} = P_{\mu\tau}\Big|_{\substack{s_{23}\to\pm c_{23}\\ c_{23}\to\mp s_{23}}}$

  (It is equivalent to choose the upper or lower substitution).

  Explicitly:

  $P_{ee} = 1 - P_{e\mu} - P_{e\tau} = 1 - P_{e\mu} - P'_{e\mu}$

  $P_{e\tau} = P'_{e\mu}$

  $P_{\mu e} = 1 - P_{\mu\mu} - P_{\mu\tau} = 1 - P_{\mu\mu} - P_{e\mu} + P_{e\mu} - P_{\mu\tau} = P_{e\mu} + P_{\tau\mu} - P_{\mu\tau} = P_{e\mu} + P'_{\mu\tau} - P_{\mu\tau}$

  $P_{\mu\mu} = 1 - P_{e\mu} - P_{\tau\mu} = 1 - P_{e\mu} - P'_{\mu\tau}$

  $P_{\tau\mu} = P'_{\mu\tau}$

  $P_{\tau\tau} = 1 - P_{e\tau} - P_{\mu\tau} = 1 - P'_{e\mu} - P_{\mu\tau}$

  Also: $P(\bar\nu_\alpha \to \bar\nu_\beta) = P(\nu_\alpha \to \nu_\beta \,|\, V \to -V,\ \delta \to -\delta)$

  and: $P_{\alpha\beta} = P_{\beta\alpha}\,(\delta \to -\delta)$ in constant matter ($V = \text{const}$).

  Such relations reduce the calculations of $P_{\alpha\beta}(\nu, \bar\nu)$ to a few independent probabilities.

Calculation of $P(\nu_e \to \nu_\mu)$ in matter at 2nd order in δm^2, θ_{13}

- The so-called "golden channel" $\nu_e \to \nu_\mu$ is particularly important in the context of future long-baseline accelerator experiments ($E \gtrsim 1$ GeV). In this context:
- Let us show that, for constant density N_e, and at 2nd order in the small parameters δm^2 and s_{13}, $P(\nu_e \to \nu_\mu)$ takes the form:

$$\begin{cases} P(\nu_e \to \nu_\mu) \simeq X \sin^2 2\theta_{13} + Y \sin 2\theta_{13} \cos\left(\delta - \frac{\Delta m^2 x}{4E}\right) + Z\,, \quad \text{where} \\ X = \sin^2\theta_{23} \left(\frac{\Delta m^2}{A-\Delta m^2}\right)^2 \sin^2\left(\frac{A-\Delta m^2}{4E} x\right) \\ Y = \sin 2\theta_{12} \sin 2\theta_{23} \left(\frac{\delta m^2}{A}\right)\left(\frac{\Delta m^2}{A-\Delta m^2}\right) \sin\left(\frac{Ax}{4E}\right) \sin\left(\frac{A-\Delta m^2}{4E} x\right) \\ Z = \cos^2\theta_{23} \sin^2 2\theta_{12} \left(\frac{\delta m^2}{A}\right)^2 \sin^2\left(\frac{Ax}{4E}\right) \end{cases}$$

[Note that, sometimes, a further $\cos\theta_{13}$ factor is inserted in Y. This, however, is irrelevant at the stated 2nd order approximation.]

- Note that, given $P(\nu_e \to \nu_\mu)$ ("golden" channel) one can get $P(\nu_e \to \nu_\tau)$ ("silver" channel) via

$$P(\nu_e \to \nu_\tau) = P(\nu_e \to \nu_\mu)\Big|_{\substack{c_{23} \to \mp s_{23} \\ s_{23} \to \pm c_{23}}} \,.$$ Moreover:

For antineutrinos, $P(\bar\nu_e \to \bar\nu_{\mu,\tau}) = P(\nu_e \to \nu_{\mu,\tau} \mid A \to -A;\ \delta \to -\delta)$

For inverted hierarchy: just swap $\Delta m^2 \to -\Delta m^2$

For swapped flavors: $P(\nu_{\mu,\tau} \to \nu_e) = P(\nu_e \to \nu_{\mu,\tau} \mid \delta \to -\delta)$

- We start the calculation by reminding that:
  $P(\nu_e \to \nu_\mu) = P_{e\mu} = |\tilde{S}_{\mu e}|^2$ with $\tilde{S}_{\mu e} = \tilde{S}'_{\mu e} c_{23} + \tilde{S}'_{\tau e} s_{23} e^{i\delta}$, so that
  $P_{e\mu} = |\tilde{S}'_{\mu e} c_{23} + \tilde{S}'_{\tau e} s_{23} e^{i\delta}|^2 = A_{e\mu} \cos\delta + B_{e\mu} \sin\delta + C_{e\mu}$ with:
  $A_{e\mu} = 2\,\mathrm{Re}[\tilde{S}'^{*}_{\mu e} \tilde{S}'_{\tau e}]\, c_{23} s_{23}$
  $B_{e\mu} = -2\,\mathrm{Im}[\tilde{S}'^{*}_{\mu e} \tilde{S}'_{\tau e}]\, c_{23} s_{23}$
  $C_{e\mu} = |\tilde{S}'_{\mu e}|^2 c^2_{23} + |\tilde{S}'_{\tau e}|^2 s^2_{23}$
- The next "trick" is to reduce the evolution from 3ν to approximately $2\nu \oplus 1\nu$, by exploiting the expansion in two small parameters: s_{13} and δm^2.
  A term T will be called "of 1st order" if proportional to s_{13} or δm^2:
  $T \sim O_1$ if $T \propto s_{13}$ or $T \propto \delta m^2$. Analogously:
  $T \sim O_2$ if $T \propto s_{13}^2$ or $T \propto (\delta m^2)^2$ or $T \propto s_{13}\, \delta m^2$, etc.
- We shall show that $\tilde{S}'_{\mu e} \sim O_1$ and $\tilde{S}'_{\tau e} \sim O_1$. Therefore, since $P_{e\mu}$ is quadratic in $\tilde{S}'_{\mu e}$ and $\tilde{S}'_{\tau e}$, it is $P_{e\mu} \sim O_2$ as desired.
- Let's remind that, in primed basis and for normal hierarchy:
  $\tilde{H}' = O_{13} O_{12} \dfrac{\mathcal{M}^2}{2E} (O_{13} O_{12})^T + V$
  $\mathcal{M}^2 = \mathrm{diag}\left(-\frac{\delta m^2}{2}, +\frac{\delta m^2}{2}, \Delta m^2\right)$ ← up to a term $\propto \mathbb{1}$
  $V = \mathrm{diag}(\sqrt{2} G_F N_e, 0, 0)$

- In the primed basis, the evolution decouples as $3\nu = (2\nu)\oplus(1\nu)$ in two limits:

  $s_{13} \to 0 \Rightarrow O_{13} = \mathbb{1}$

  $\delta m^2 \to 0 \Rightarrow O_{12}\mathcal{M}^2 O_{12}^T = \mathcal{M}^2$

  It is then convenient to define:

  $$\tilde{H}^\ell = \lim_{s_{13}\to 0} \tilde{H}'$$

  $$\tilde{H}^h = \lim_{\delta m^2\to 0} \tilde{H}'$$

  and to study the evolution operator components $\tilde{S}'_{\mu e}$ and $\tilde{S}'_{\tau e}$ in $\tilde{H}^\ell$ and $\tilde{H}^h$.
  The task is simpler since both $\tilde{H}^\ell$ and $\tilde{H}^h$ have only one nontrivial 2×2 submatrix.

- $\tilde{H}^\ell$ in primed basis ($s_{13}\to 0$ limit):

  $$\tilde{H}^\ell = \lim_{s_{13}\to 0} \tilde{H}' = \frac{1}{2E}\left[O_{12}\begin{pmatrix} -\delta m^2/2 & & \\ & +\delta m^2/2 & \\ & & \Delta m^2 \end{pmatrix} O_{12}^T + \begin{pmatrix} A & & \\ & 0 & \\ & & 0 \end{pmatrix}\right]$$

  $$= \frac{A}{4E}\mathbb{1} + \frac{1}{4E}\begin{bmatrix} A - \cos 2\theta_{12}\,\delta m^2 & \sin 2\theta_{12}\,\delta m^2 & 0 \\ \sin 2\theta_{12}\,\delta m^2 & \cos 2\theta_{12}\,\delta m^2 - A & 0 \\ 0 & 0 & 2\Delta m^2 - A \end{bmatrix}.$$ Given this structure for $\tilde{H}^\ell$:

  In the primed basis, for $s_{13}\to 0$, the (e,μ') flavors evolve separately from the (τ') one $\to$

  $$\tilde{S}^\ell_{\tau e} = \lim_{s_{13}\to 0} \tilde{S}'_{\tau e} = 0 \quad (\text{no } \nu_e \to \nu'_\tau \text{ transitions}),\text{ thus:}$$

  $$\tilde{S}'_{\tau e} = \mathcal{O}(s_{13}) = O_1 \text{ at least.}$$

  Instead, $\tilde{S}^\ell_{\mu e}$ is nonzero. From the 2ν case in matter (already worked out) we get:

  $$\tilde{S}^\ell_{\mu e} = e^{-i\frac{A}{4E}x}\left[-i\sin 2\tilde{\theta}_{12}\sin\left(\frac{\delta \tilde{m}^2 x}{4E}\right)\right]$$ by exponentiation of $\tilde{H}^\ell$, with:

$\sin 2\tilde{\theta}_{12} = \sin 2\theta_{12} / \sqrt{(\cos 2\theta_{12} - A/\delta m^2)^2 + \sin^2 2\theta_{12}}$ and $\delta\tilde{m}^2 = \delta m^2 \sin 2\theta_{12} / \sin 2\tilde{\theta}_{12}$, implying:

$\tilde{S}^{\ell}_{\mu e} = \mathcal{O}(\delta m^2) = \mathcal{O}_1$

- $\tilde{H}^h$ in primed basis ($\delta m^2 \to 0$ limit)

$$\tilde{H}^h = \lim_{\delta m^2 \to 0} \tilde{H}' = \frac{1}{2E}\left(O_{13} \begin{bmatrix} 0 & & \\ & 0 & \\ & & \Delta m^2 \end{bmatrix} O_{13}^T + \begin{bmatrix} A & & \\ & 0 & \\ & & 0 \end{bmatrix} \right)$$

$$= \left(\frac{\Delta m^2}{4E} + \frac{A}{4E} \right) \mathbb{1} + \frac{1}{4E} \begin{bmatrix} A - \cos 2\theta_{13} \Delta m^2 & 0 & \sin 2\theta_{13} \Delta m^2 \\ 0 & -\Delta m^2 - A & 0 \\ \sin 2\theta_{13} \Delta m^2 & 0 & \cos 2\theta_{13} \Delta m^2 - A \end{bmatrix}.$$ Given this structure:

In the primed basis, for $\delta m^2 \to 0$, the (e, τ') flavors evolve separately from the μ' one $\to$

$\tilde{S}^h_{\mu e} = \lim_{\delta m^2 \to 0} \tilde{S}'_{\mu e} = 0$ (no $\nu_e \to \nu_\mu'$ transitions), thus:

$\tilde{S}'_{\mu e} = \mathcal{O}(\delta m^2) = \mathcal{O}_1$ at least.

Instead, $\tilde{S}^h_{\tau e}$ is nonzero. From the 2ν case in matter (already worked out) we get:

$\tilde{S}^h_{\tau e} = e^{-i\frac{A}{4E}x} e^{-i\frac{\Delta m^2}{4E}x} \left[-i \sin 2\tilde{\theta}_{13} \sin\left(\frac{\Delta\tilde{m}^2 x}{4E} \right) \right]$ by exponentiation of $\tilde{H}^h$, with:

$\sin 2\tilde{\theta}_{13} = \sin 2\theta_{13} \sqrt{(\cos 2\theta_{13} - A/\Delta m^2)^2 + \sin^2 2\theta_{13}}$ and $\Delta\tilde{m}^2 = \Delta m^2 \sin 2\theta_{13} / \sin 2\tilde{\theta}_{13}$, implying:

$\tilde{S}^h_{\tau e} = \mathcal{O}(s_{13}) = \mathcal{O}_1$.

- Summarizing, at $\mathcal{O}_1$ we have that:

$\tilde{S}'_{\mu e} = \mathcal{O}(\delta m^2) \simeq \tilde{S}^{\ell}_{\mu e} = e^{-i\frac{A}{4E}x} \left[-i \sin 2\tilde{\theta}_{12} \sin\left(\frac{\delta\tilde{m}^2 x}{4E} \right) \right]$

$\tilde{S}'_{\tau e} = \mathcal{O}(s_{13}) \simeq \tilde{S}^h_{\tau e} = e^{-i\frac{A}{4E}x} e^{-i\frac{\Delta m^2}{4E}x} \left[-i \sin 2\tilde{\theta}_{13} \sin\left(\frac{\Delta\tilde{m}^2 x}{4E} \right) \right]$

- One can drop the overall phase $e^{-i\frac{A}{4E}x}$ and get:

$$\tilde{S}'_{\mu e} = \left[-i\sin 2\tilde{\theta}_{12}\sin\left(\frac{\delta\tilde{m}^2 x}{4E}\right)\right] + O_2$$

$$\tilde{S}'_{\tau e} = e^{-i\frac{\Delta m^2}{4E}x}\left[-i\sin 2\tilde{\theta}_{13}\sin\left(\frac{\Delta\tilde{m}^2 x}{4E}\right)\right] + O_2$$

which provide all is needed to calculate $P_{e\mu}$ as a quadratic form in $\tilde{S}'_{\mu e}$ and $\tilde{S}'_{e\tau}$. Indeed:

- $$A_{e\mu} = 2\,\mathrm{Re}\left[\tilde{S}'^{*}_{\mu e}\,\tilde{S}'_{\tau e}\right] c_{23}s_{23} = \sin 2\tilde{\theta}_{12}\sin 2\tilde{\theta}_{13}\sin 2\theta_{23}\sin\left(\frac{\Delta\tilde{m}^2 x}{4E}\right)\sin\left(\frac{\delta\tilde{m}^2 x}{4E}\right)\cos\left(\frac{\Delta m^2 x}{4E}\right)$$

$$B_{e\mu} = -2\,\mathrm{Im}\left[\tilde{S}'^{*}_{\mu e}\,\tilde{S}'_{\tau e}\right] c_{23}s_{23} = \sin 2\tilde{\theta}_{12}\sin 2\tilde{\theta}_{13}\sin 2\theta_{23}\sin\left(\frac{\Delta\tilde{m}^2 x}{4E}\right)\sin\left(\frac{\delta\tilde{m}^2 x}{4E}\right)\sin\left(\frac{\Delta m^2 x}{4E}\right)$$

$$C_{e\mu} = |\tilde{S}'_{\mu e}|^2 c_{23}^2 + |\tilde{S}'_{\tau e}|^2 s_{23}^2 = \cos^2\theta_{23}\sin^2 2\tilde{\theta}_{12}\sin^2\left(\frac{\delta\tilde{m}^2 x}{4E}\right) + \sin^2\theta_{23}\sin^2 2\tilde{\theta}_{13}\sin^2\left(\frac{\Delta\tilde{m}^2 x}{4E}\right)$$

$$P_{e\mu} = A_{e\mu}\cos\delta + B_{e\mu}\sin\delta + C_{e\mu}$$

- The solution will be now further reduced by a proper organization of terms, as well as by an expansion in the small parameter:

$$\frac{\delta m^2}{A} = \frac{\delta m^2}{2\sqrt{2}G_F N_e E} \ll 1$$ (for $E \gtrsim 1$ GeV and N_e in the crust/mantle) ← "high-energy" approximation.

In particular, the high-energy approximation (useful for accelerator ν expts) will allow us to express $\delta\tilde{m}^2$, $\Delta\tilde{m}^2$, $\tilde{\theta}_{12}$ and $\tilde{\theta}_{13}$ (the matter parameters) in terms of their vacuum values δm^2, Δm^2, θ_{12} and θ_{13} (together with an expansion in the small parameter s_{13}).

- For $\delta m^2/A \ll 1$:

$$\sin 2\tilde{\theta}_{12} = \sin 2\theta_{12} \Big/ \left[\left(\cos 2\theta_{12} - \frac{A}{\delta m^2}\right)^2 + \sin^2 2\theta_{12}\right]^{\frac{1}{2}}$$

$$= \sin 2\theta_{12} \Big/ \left[\cos^2 2\theta_{12} - \frac{2A}{\delta m^2}\cos 2\theta_{12} + \left(\frac{A}{\delta m^2}\right)^2 + \sin^2 2\theta_{12}\right]^{\frac{1}{2}}$$

$$= \sin 2\theta_{12} \Big/ \left[\left(\frac{A}{\delta m^2}\right)^2\left(1 - \frac{2\delta m^2}{A}\cos 2\theta_{12} + \cdots\right)\right]^{\frac{1}{2}}$$

$$\simeq \sin 2\theta_{12} \Big/ \left[\frac{|A|}{\delta m^2}\left(1 - \frac{\delta m^2}{A}\cos 2\theta_{12}\right)\right] \simeq \sin 2\theta_{12}\,\frac{\delta m^2}{|A|} + O_2$$

$$\delta m^2/\delta\tilde{m}^2 = \sin 2\tilde{\theta}_{12}/\sin 2\theta_{12} = \delta m^2/|A| + O_2 \implies \delta\tilde{m}^2 = |A| + O_2 \quad , \text{ thus:}$$

$$\sin\left(\frac{\delta\tilde{m}^2 x}{4E}\right) \simeq \sin\left(\frac{|A|x}{4E}\right) + O_2$$

- For $s_{13} \ll 1$:

$$\sin 2\tilde{\theta}_{13} = \sin 2\theta_{13} \Big/ \left[\left(\cos 2\theta_{13} - \frac{A}{\Delta m^2}\right)^2 + \sin^2 2\theta_{13}\right]^{1/2}$$

$$\simeq \sin 2\theta_{13} \Big/ \left[\left(1 - \frac{A}{\Delta m^2}\right)^2\right]^{1/2} + O_2 = \sin 2\theta_{13} \Big/ \left|1 - \frac{A}{\Delta m^2}\right| + O_2$$

$$\sin 2\tilde{\theta}_{13} = \left|\frac{\Delta m^2}{\Delta m^2 - A}\right| \sin 2\theta_{13} + O_2$$

$$\Delta\tilde{m}^2 = \Delta m^2 \sin 2\theta_{13}/\sin 2\tilde{\theta}_{13} \simeq \Delta m^2 \left|\frac{\Delta m^2 - A}{\Delta m^2}\right|$$

- We have then:

$$A_{e\mu} \simeq \sin 2\theta_{12}\left(\frac{\delta m^2}{|A|}\right)\sin 2\theta_{13}\left|\frac{\Delta m^2}{\Delta m^2-A}\right|\sin 2\theta_{23}\sin\left(\frac{|A|x}{4E}\right)\sin\left(\Delta m^2\left|\frac{\Delta m^2-A}{\Delta m^2}\right|\frac{x}{4E}\right)\cos\left(\frac{\Delta m^2 x}{4E}\right)$$

$$B_{e\mu} \simeq \sin 2\theta_{12}\left(\frac{\delta m^2}{|A|}\right)\sin 2\theta_{13}\left|\frac{\Delta m^2}{\Delta m^2-A}\right|\sin 2\theta_{23}\sin\left(\frac{|A|x}{4E}\right)\sin\left(\Delta m^2\left|\frac{\Delta m^2-A}{\Delta m^2}\right|\frac{x}{4E}\right)\sin\left(\frac{\Delta m^2 x}{4E}\right)$$

$$C_{e\mu} \simeq \cos^2\theta_{23}\sin^2 2\theta_{12}\left(\frac{\delta m^2}{A}\right)^2\sin^2\left(\frac{Ax}{4E}\right)+\sin^2\theta_{23}\sin^2 2\theta_{13}\left(\frac{\Delta m^2}{\Delta m^2-A}\right)^2\sin^2\left(\frac{|\Delta m^2-A|}{4E}x\right)$$

- Absolute values can be eliminated by inspection of all relevant $\pm$ cases. E.g., by changing sign of (Δm^2-A): $A_{e\mu}$, $B_{e\mu}$, $C_{e\mu}$ do not change. By changing sign of Δm^2: only $A_{e\mu}$ changes. Etc...

  Then we have:

$$A_{e\mu} \simeq \sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}\left(\frac{\delta m^2}{A}\right)\frac{\Delta m^2}{A-\Delta m^2}\sin\left(\frac{Ax}{4E}\right)\sin\left(\frac{A-\Delta m^2}{4E}x\right)\cos\left(\frac{\Delta m^2 x}{4E}\right)$$

$$B_{e\mu} \simeq \sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}\left(\frac{\delta m^2}{A}\right)\frac{\Delta m^2}{A-\Delta m^2}\sin\left(\frac{Ax}{4E}\right)\sin\left(\frac{A-\Delta m^2}{4E}x\right)\sin\left(\frac{\Delta m^2 x}{4E}\right)$$

$$C_{e\mu} \simeq \cos^2\theta_{23}\sin^2 2\theta_{12}\left(\frac{\delta m^2}{A}\right)^2\sin^2\left(\frac{Ax}{4E}\right)+\sin^2\theta_{23}\sin^2 2\theta_{13}\left(\frac{\Delta m^2}{\Delta m^2-A}\right)^2\sin^2\left(\frac{\Delta m^2-A}{4E}x\right)$$

$$P_{e\mu} = A_{e\mu}\cos\delta + B_{e\mu}\sin\delta + C_{e\mu}$$

- Finally, the terms in $P_{e\mu}$ can be organized as:

$$P_{e\mu} = X \sin^2 2\theta_{13} + Y \sin 2\theta_{13} \cos\left(\delta - \frac{\Delta m^2 x}{4E}\right) + Z \quad , \text{ where}$$

$$X = \sin^2\theta_{23} \left(\frac{\Delta m^2}{A - \Delta m^2}\right)^2 \sin^2\left(\frac{A - \Delta m^2}{4E} x\right)$$

$$Y = \sin 2\theta_{12} \sin 2\theta_{23} \left(\frac{\delta m^2}{A}\right)\left(\frac{\Delta m^2}{A - \Delta m^2}\right) \sin\left(\frac{Ax}{4E}\right) \sin\left(\frac{A - \Delta m^2}{4E} x\right)$$

$$Z = \cos^2\theta_{23} \sin^2 2\theta_{12} \left(\frac{\delta m^2}{A}\right)^2 \sin^2\left(\frac{Ax}{4E}\right)$$

as desired.

- In the literature, one can also find the following way to organize terms:

$$P_{e\mu} = x^2 f^2 + 2xy\, fg \cos(\Delta - \delta) + y^2 g^2 \quad , \text{ where}$$

$$x = \sin\theta_{23} \sin 2\theta_{13}$$

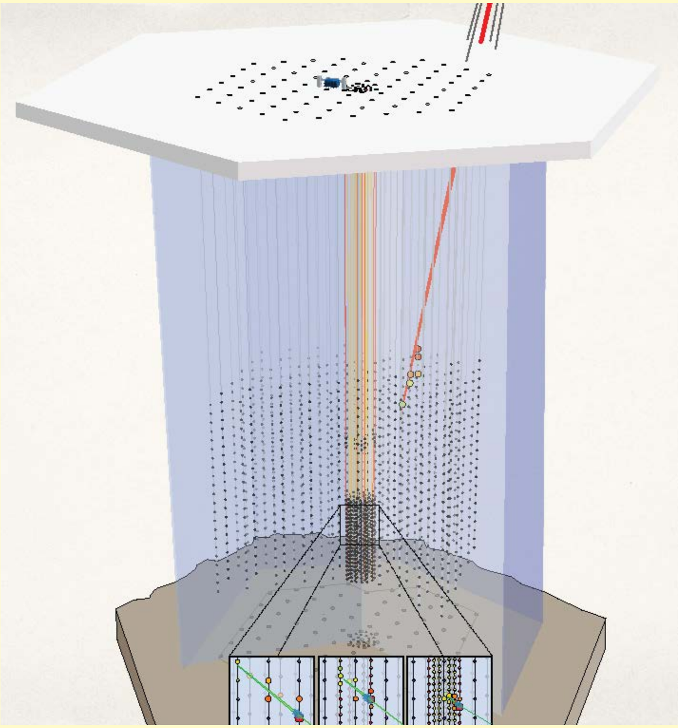
$$y = \frac{\delta m^2}{\Delta m^2} \cos\theta_{23} \sin 2\theta_{12}$$

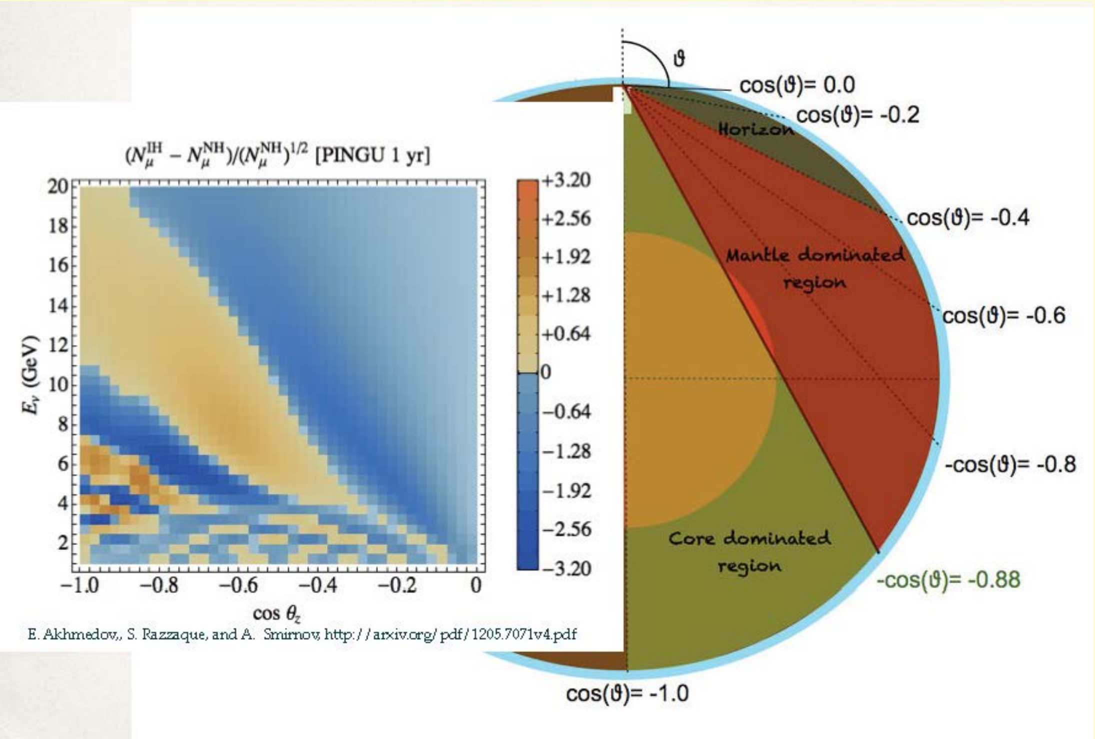
$$\Delta = \Delta m^2 x / 4E$$

$$f = \sin\left(\frac{\Delta m^2 - A}{4E} x\right) \frac{\Delta m^2}{\Delta m^2 - A}$$

$$g = \sin\left(\frac{Ax}{4E}\right) \frac{\Delta m^2}{A}$$

Back to **hierarchy**: the second option (using MSW effect) may be implemented in large-volume detectors (Hyper-Kamiokande, INO...) via atmospheric neutrinos. With very high statistics, they might be sensitive to subleading hierarchy/matter effects beyond the one-dominant-mass scale approximation discussed before.
E.g., recent studies on PINGU in IceCube

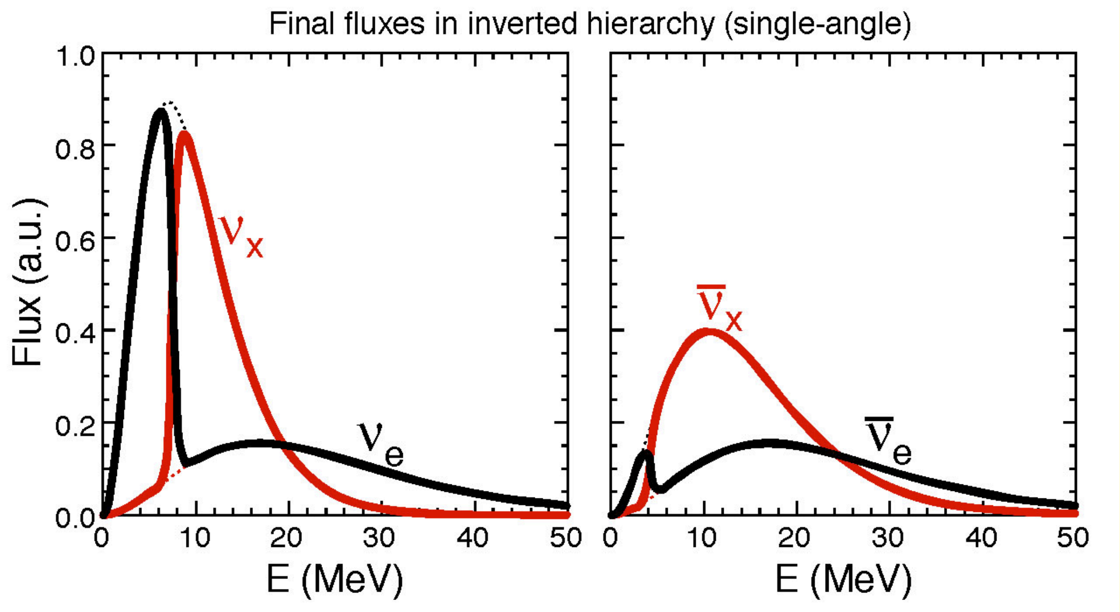




(Akhmedov, Razzaque, Smirnov; slides from E. Resconi)

Again on **hierarchy**: a third option may be provided by neutrino-neutrino forward scattering in **core-collapse SN**. In this case, $\pm\Delta m^2$ compares with $\pm 2\sqrt{2}G_F\, E$ * density (nu + antinu). Maybe the only place to test neutrino-neutrino interactions!

Burst of papers in recent years. Interesting and peculiar nonlinear phenomena arise ("collective effects"), such as spectral split/swap effects in observable spectra (for I.H.)



Very interesting theoretically (coupled, nonlinear flavor histories).
But: rather technical subject, wash-out effects under study.

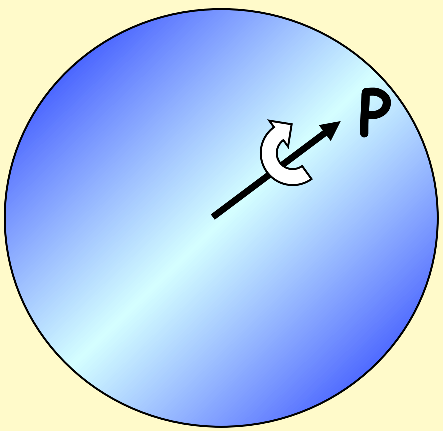
Coupled equations of motion (for 2 flavors, e and $x=\mu,\tau$)

Hamiltonian now depends on neutrino density -> use density matrix.

Liouville equations: $i\partial_t\,\rho=[H,\rho]$ (for each neutrino mode)

Decompose 2x2 (anti)neutrino density matrix over Pauli matrices to get a "**polarization**" (Bloch) 3-vector $\mathbf{P}=(P_1,P_2,P_3)=(P_x,P_y,P_z)$. [Ditto for H.]

Bloch equations: $\partial_t\mathbf{P} = \mathbf{V}\times\mathbf{P}$ (precession-like, $|P|$=const)



Any mode P moves on a Bloch sphere (abstract "flavor space").

"up" direction : ν_e flavor

"down" direct. : ν_x flavor

generic direct. : mixed flavor state

Probability P_{ee} related to $P_3=P_Z$

Coupled equations of motion (cont'd)

The problem is that there are lots of kinematical neutrino modes: continuous distributions over energy and angle(s) -> no less than ∞^2 !

Discretize over energy spectrum (**N_E bins**), and over angular distribution in multi-angle simulations (**N_Θ bins**) -> Get discrete index (indices), **P_i** .

Evolution governed by **$6 \times N_E \times N_\Theta$ coupled Bloch equations** of the form:

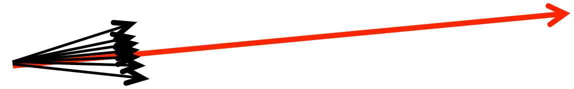
$$\dot{\mathbf{P}}_i = \mathbf{V}_{\text{ector}}[+\omega,\ \lambda,\ \mu,\ \mathbf{P}_j,\ \overline{\mathbf{P}}_j] \times \mathbf{P}_i$$

$$\dot{\overline{\mathbf{P}}}_i = \mathbf{V}_{\text{ector}}[\underset{\text{vacuum}}{\uparrow}-\omega,\ \underset{\text{matter}}{\uparrow}\lambda,\ \underset{\text{self-interaction}}{\uparrow}\mu,\ \underbrace{\mathbf{P}_j,\ \overline{\mathbf{P}}_j}_{\text{ij couplings}}] \times \overline{\mathbf{P}}_i$$

Large, "stiff" set of (strongly) coupled differential equations.

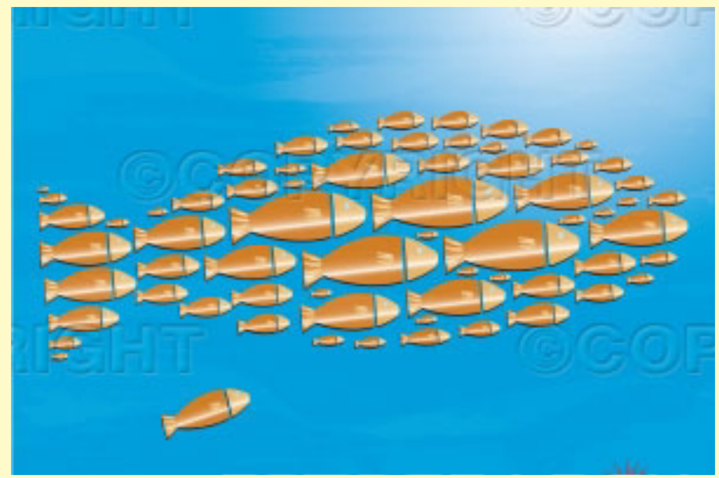
Strong couplings between polarization vectors make the problem difficult, but also make an analytical understanding possible after all !
Key tool of "**near-alignment**" or "**strong polarization**", e.g.:

$\mathbf{P}_i$'s



$$\mathbf{J} = \Sigma_i \, \mathbf{P}_i$$

(global polarization vector)



Neutrino "flavor polarizations" align at high density!
Recent wave of numerical+analytical papers, very surprising "collective" behavior found, significant dependence on hierarchy and on nonzero theta(13) (as well as on new interactions)
Warning: effects may be washed out by many complications in SN...

... and now, the last topic of this course:

Lesson II - 2nd part (~ 1h):

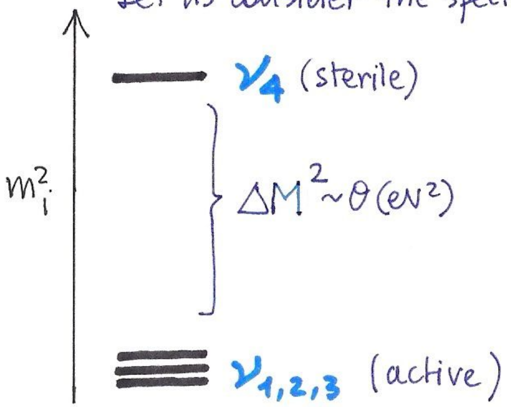
* Oscillations - some open problems:
  - mass hierarchy
  - CP violation and precision frontier
    (+ long exercise on P$e\mu$)
  - **sterile neutrinos**

* Appendix for further reading:
  Notes on statistics and data analysis

Reminder: sterile neutrinos are not forbidden by LEP (N=3 active!) One sterile neutrino at eV scale might lead to further flavor appearance/disappearance phenomena, as previously worked out:

Exercise: One-dominant-mass-scale oscillations for a 4th, sterile neutrino ν_S

Let us consider the spectrum:



Then, the PMNS 3x3 matrix is not unitary, being part of a larger 4×4 mixing matrix:

$$U = \begin{pmatrix} \begin{bmatrix} & U_{PMNS} & \end{bmatrix} & \begin{matrix} U_{e4} \\ U_{\mu 4} \\ U_{\tau 4} \end{matrix} \\ U_{S1} \quad U_{S2} \quad U_{S3} & U_{S4} \end{pmatrix}, \quad \text{with } |U_{S4}|^2 \simeq 1 - \text{epsilon},$$

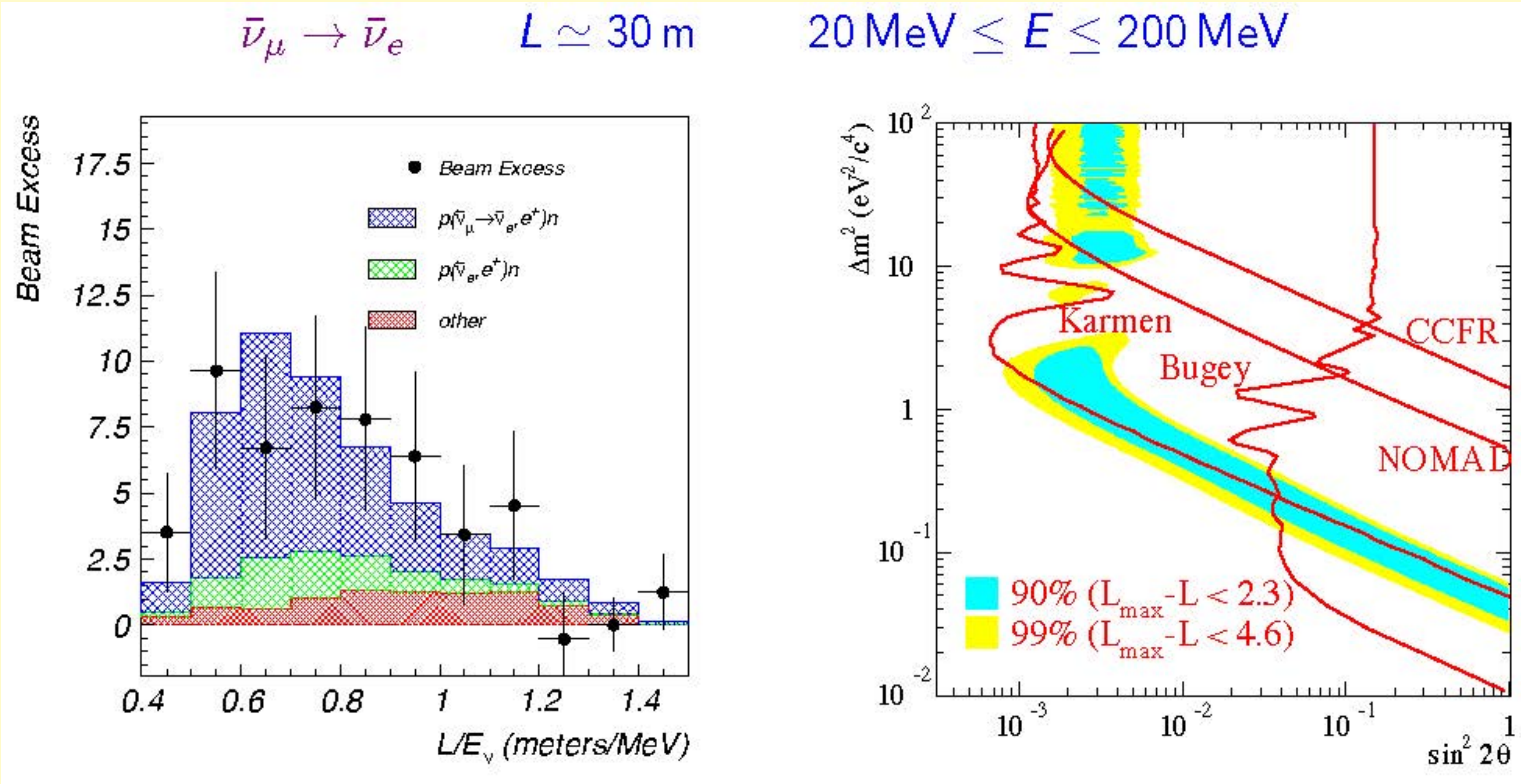
in order not to alter too much the established 3ν phenomenology $\rightarrow |U_{\alpha 4}|^2_{\alpha = e,\mu,\tau} \ll 1$

For experiments sensitive mainly to $\Delta M^2 \sim \mathcal{O}(eV^2)$, one can take the limits $\delta m^2 \rightarrow 0$ and $\Delta m^2 \rightarrow 0$, and apply the same logic as for one dominant mass scale in $3\nu \cong (2\nu) \oplus (1\nu) \Rightarrow 4\nu \simeq (3\nu) \oplus (1\nu)$. Similarly, one gets (for $\alpha, \beta = e \mu \tau$)

Disappearance ($\alpha = \beta$): $1 - P_{\alpha\alpha} \simeq 4 |U_{\alpha 4}|^2 (1 - |U_{\alpha 4}|^2) \sin^2\left(\frac{\Delta M^2 x}{4E}\right)$ ← "singly" suppressed by $|U_{\alpha 4}|^2 \ll 1$

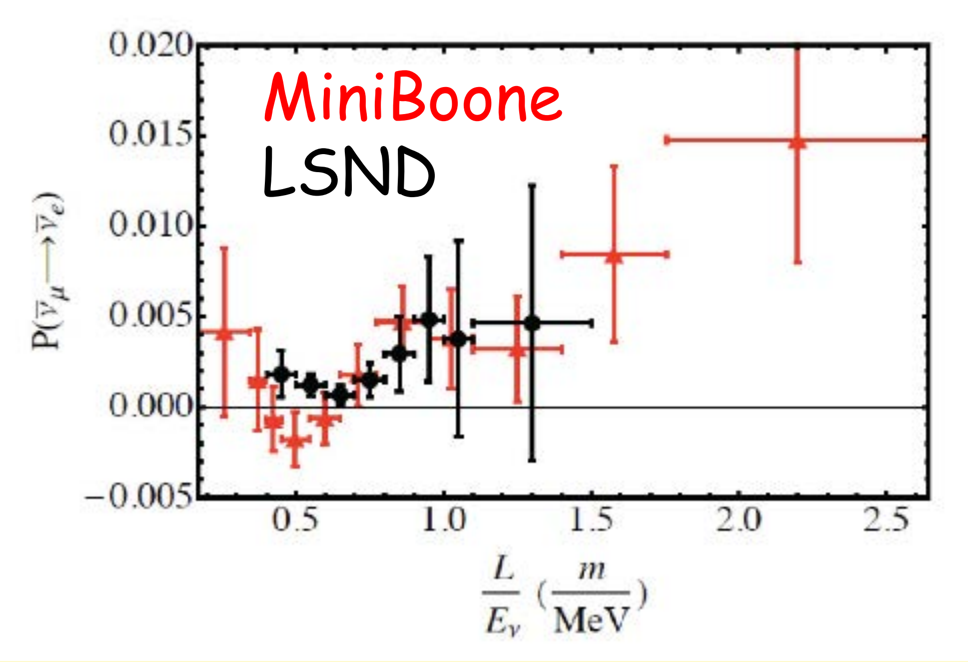
Appearance ($\alpha \neq \beta$): $P_{\alpha\beta} \simeq 4 |U_{\alpha 4}|^2 |U_{\beta 4}|^2 \sin^2\left(\frac{\Delta M^2 x}{4E}\right)$ ← "doubly" suppressed by both $|U_{\alpha 4}| \ll 1$ and $|U_{\beta 4}| \ll 1$

One short-baseline accelerator experiment (LSND) claimed in the 90's flavor oscillation appearance at the O(eV) scale:



Unfortunately, after >20 years, this result has not been either confirmed or ruled out conclusively, even by dedicated appearance experiments (e.g., MiniBoone).

You may or may not see an oscillation pattern here...



... especially if you exclude the two rightmost data points at lowest energy and highest background.

Moreover, if there is a new $\nu_\mu \to \nu_e$ appearance signal, there must be larger $\nu_\mu \to \nu_\mu$ and $\nu_e \to \nu_e$ disappearance signals at the same scale, $\Delta M^2 \sim O(eV^2)$:

Disappearance $(\alpha=\beta)$: $1-P_{\alpha\alpha} \simeq 4|U_{\alpha 4}|^2(1-|U_{\alpha 4}|^2)\sin^2\left(\frac{\Delta M^2 x}{4E}\right)$ ← "singly" suppressed by $|U_{\alpha 4}|^2 \ll 1$

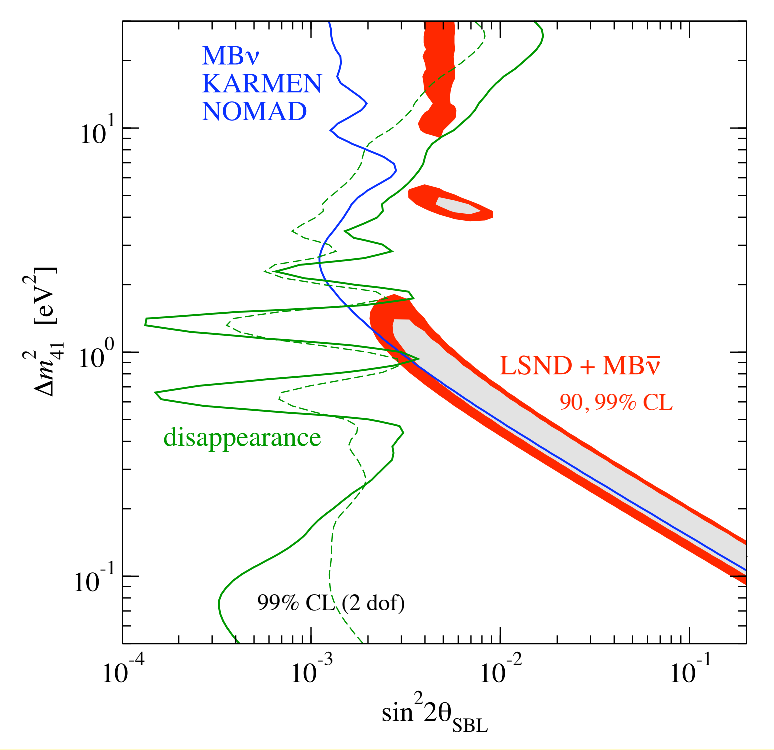
Appearance $(\alpha\neq\beta)$: $P_{\alpha\beta} \simeq 4|U_{\alpha 4}|^2|U_{\beta 4}|^2\sin^2\left(\frac{\Delta M^2 x}{4E}\right)$ ← "doubly" suppressed by both $|U_{\alpha 4}| \ll 1$ and $|U_{\beta 4}| \ll 1$

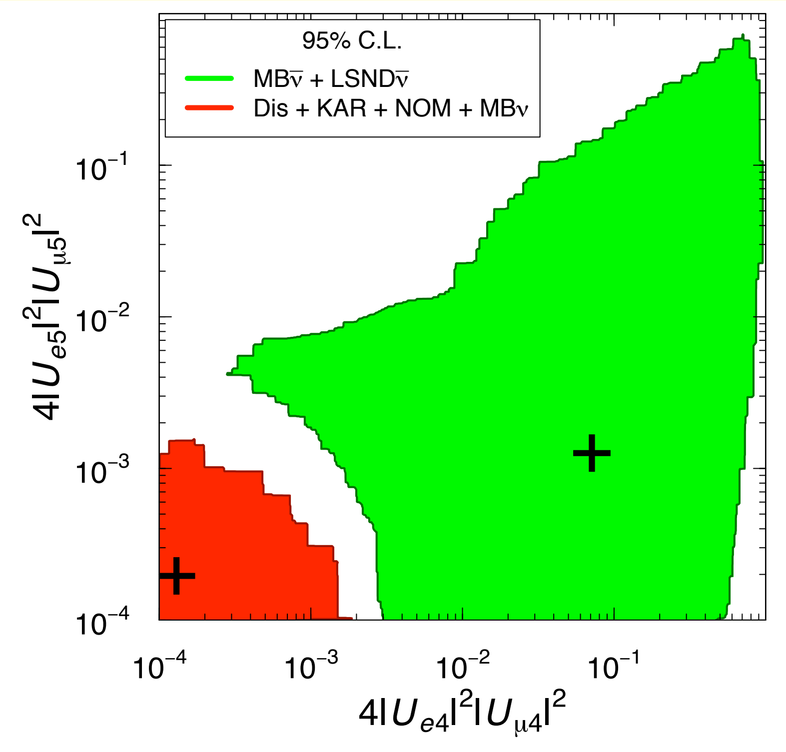
However, no unambiguous disappearance signal has been seen, especially in $\nu_\mu \to \nu_\mu$ mode.

This fact makes it difficult to reconcile positive LSND results with (mainly) negative results from other epts., not only in models with 1 additional sterile neutrino (3+1) but even with 2 steriles (3+2) →

Appearance/disappearance tension in 3+1 and 3+2

recent fits:



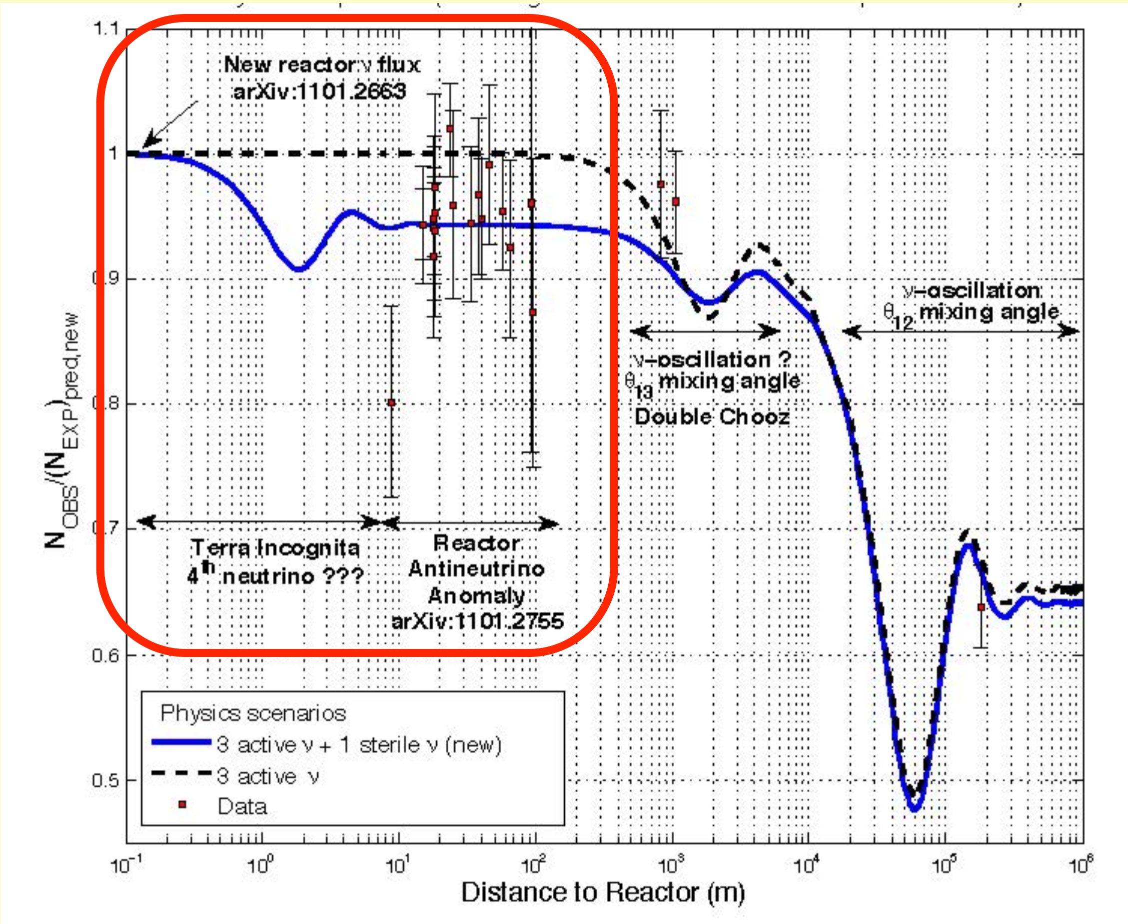


Kopp, Maltoni, Schwetz 2011

Giunti and Laveder 2011

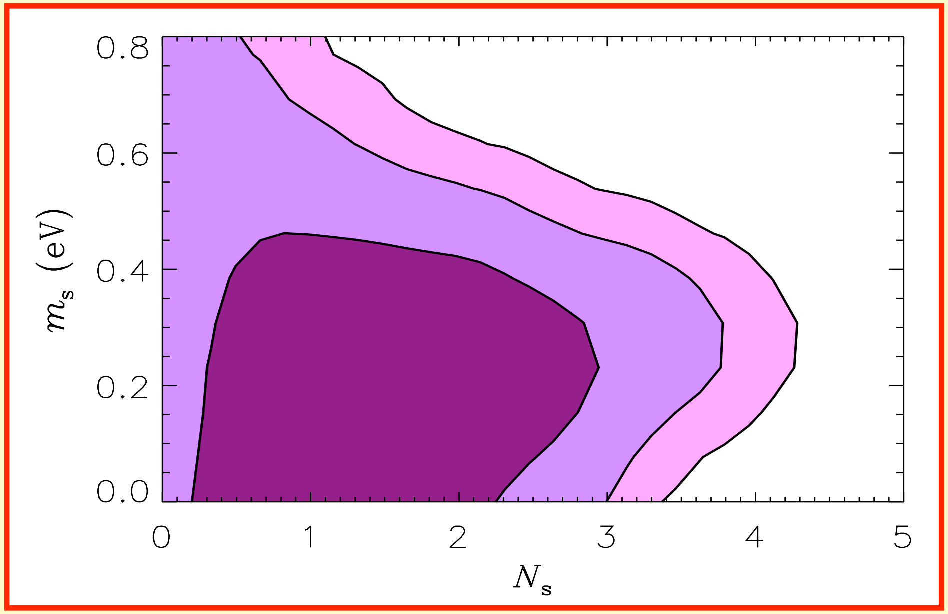
Recently, a possible hint for $\nu_e \rightarrow \nu_e$ disappearance due to sterile ν has been claimed ("reactor neutrino anomaly"), by a reanalysis of new fluxes and old reactor experiments at L<O(100) m.

In addition to the known disappearance due to 3ν oscillations at L>O(100 m), there seems to be an extra deficit at small L:



Mueller et al.

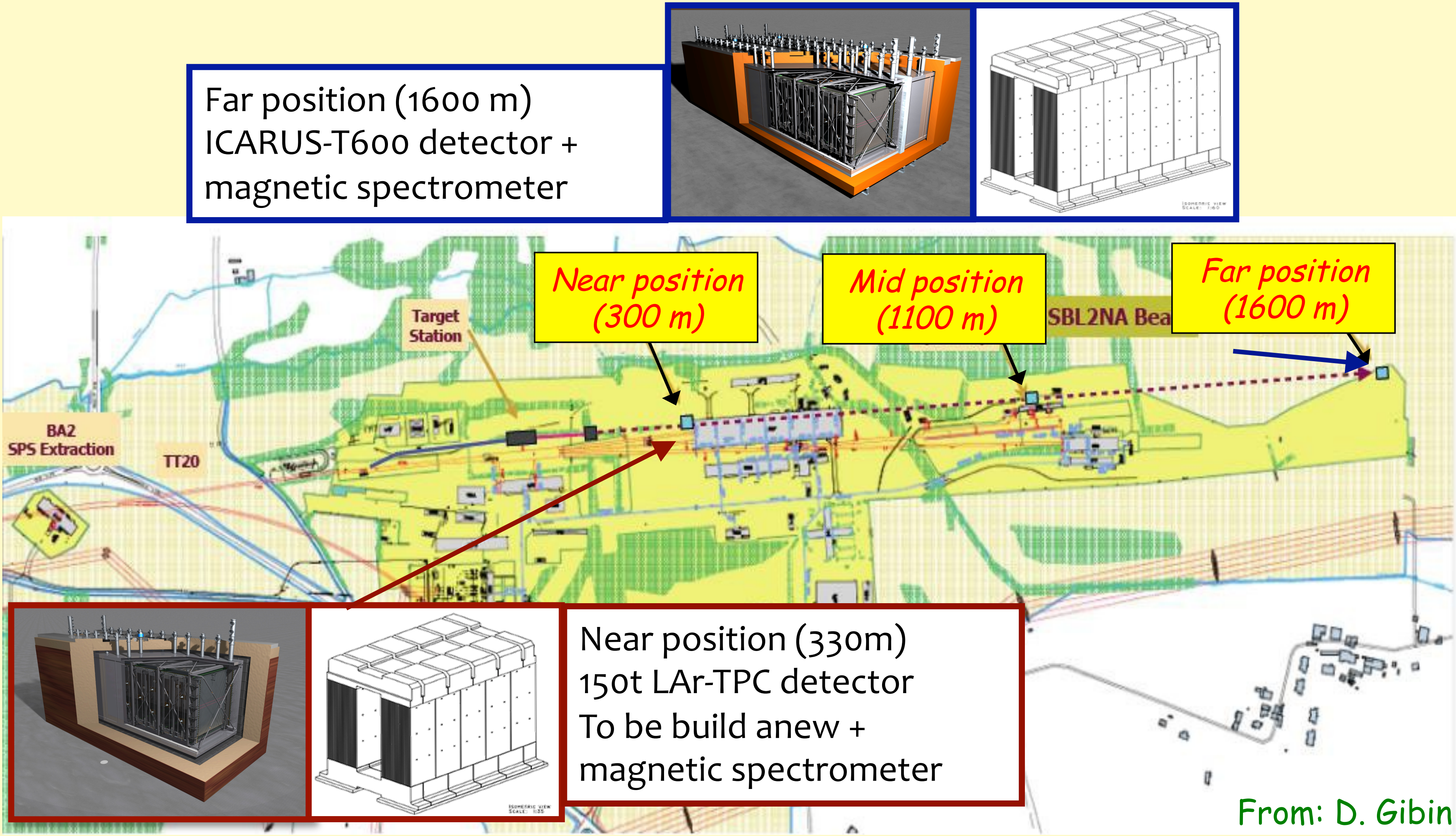
Also, recent data from precision cosmology (which constrain not only the total mass S but also the effective number of ν species N_{eff}) seem to show a preference for $N_{eff} \sim 4$ rather $N_{eff} \sim 3$...



E.g., Hamann et al.

... but with masses somewhat smaller than those required by oscillation mass differences... Situation clearly needs clarification from both cosmology (Planck 2013?) and from new (dis)appearance oscillation experiments.

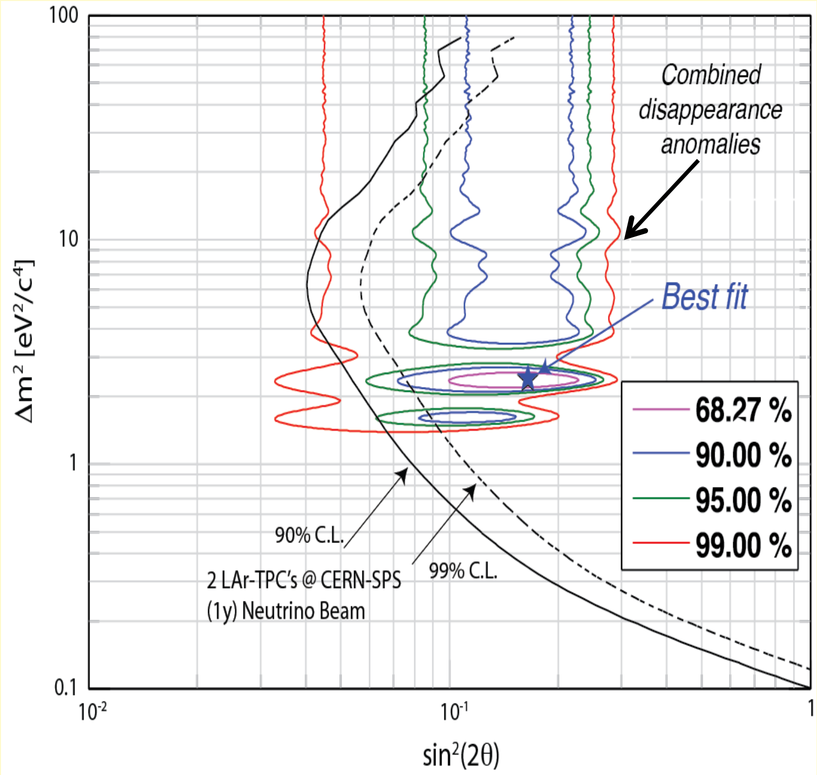
E.g., try to test both disappearance and appearance in one and the same experiment, using near/far comparison and good flavor identification. A recent proposal: ICARUS/NESSiE at CERN:



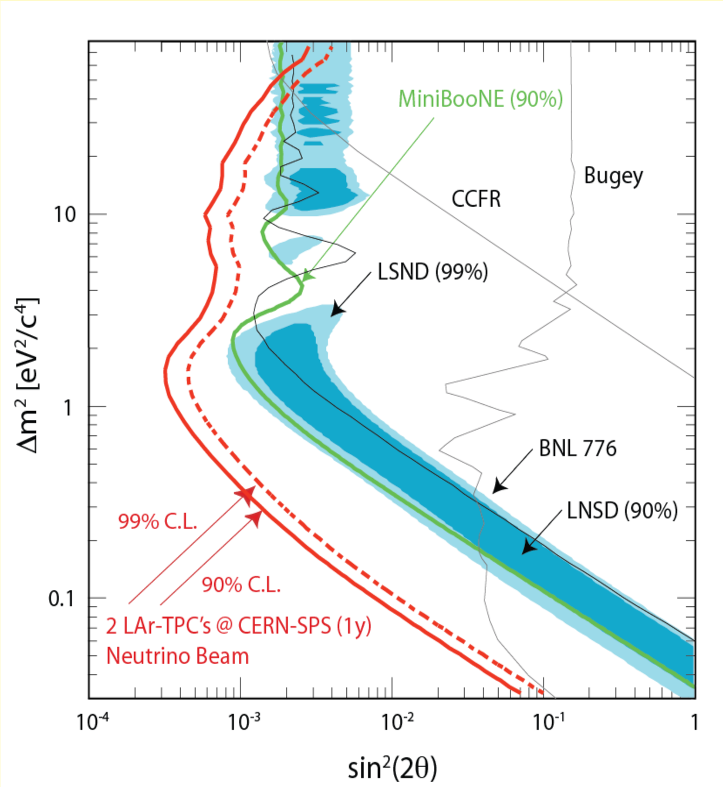
From: D. Gibin

ICARUS/NESSiE estimated potential

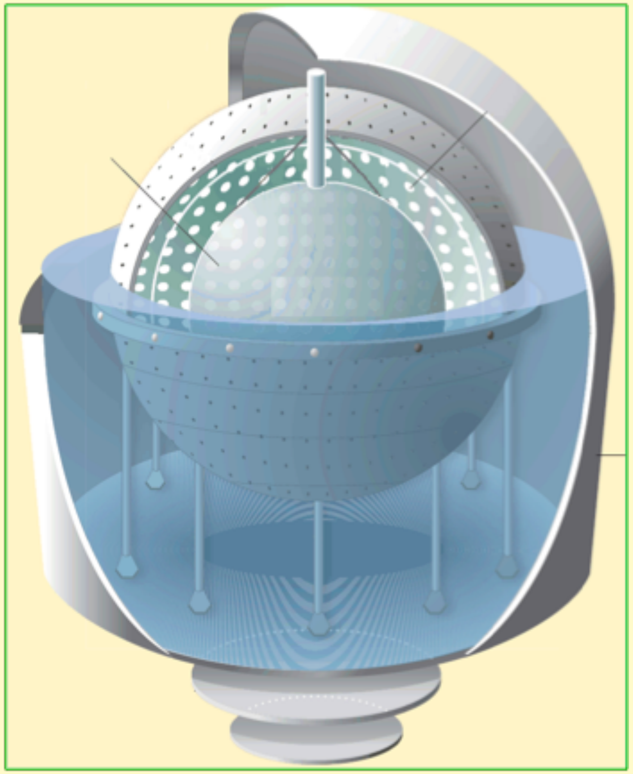
Sensitivity in disappearance:

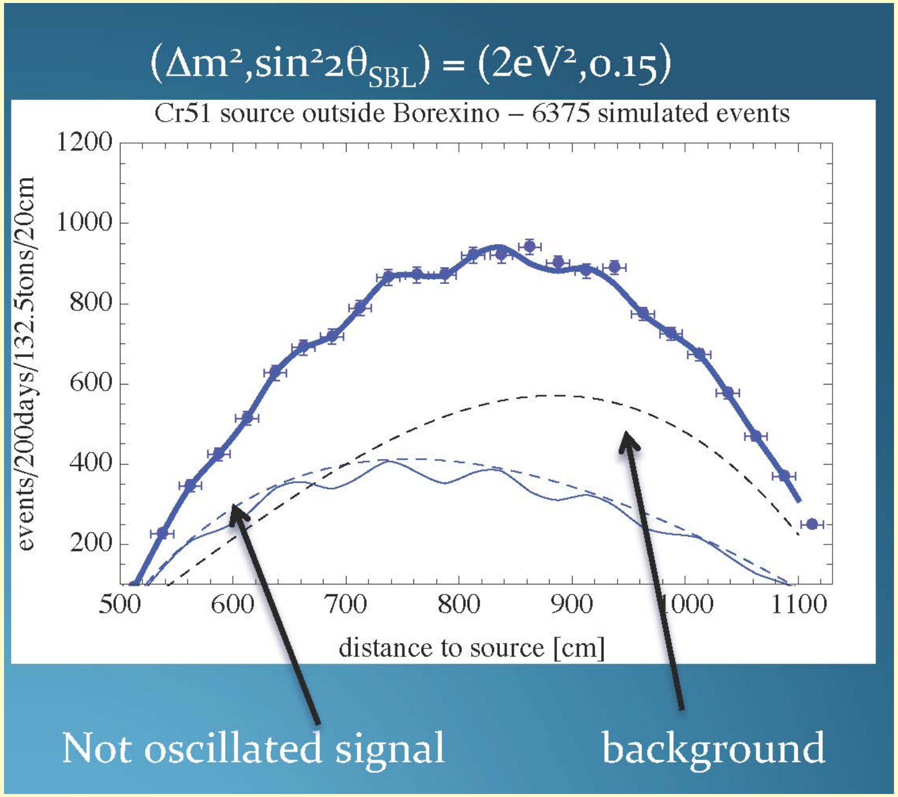


Sensitivity in appearance:



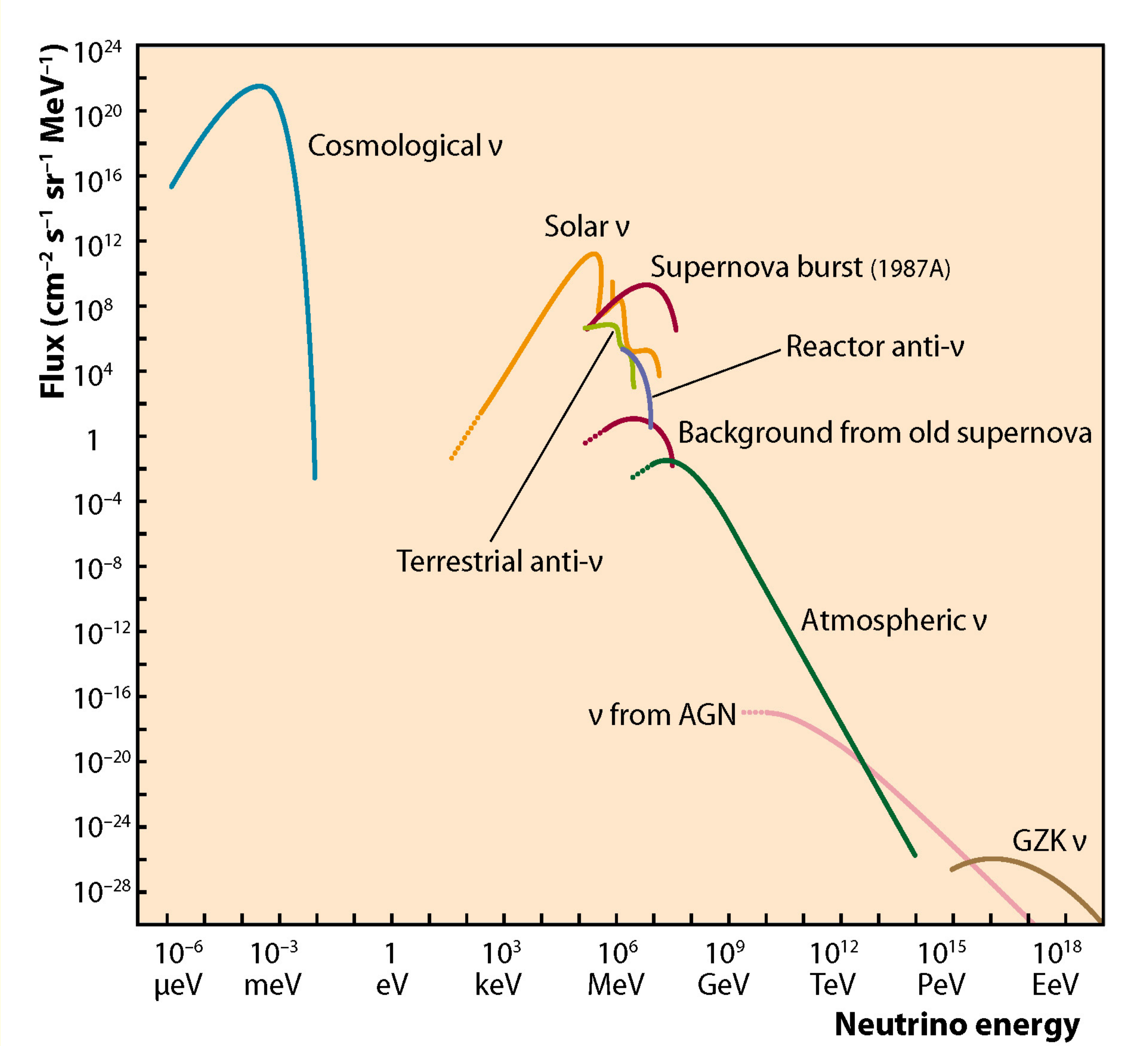
Another idea: test sterile neutrino oscillations with a strong radioactive source inside or just outside Borexino: might observe the oscillation pattern at scale of meters





Many other ideas being discussed. Time will tell !

Finally, we should never forget that there are still vast lands to be explored in the neutrino world...



A synoptic view of neutrino fluxes. (from ASPERA roadmap)

Conclusions and Open Problems

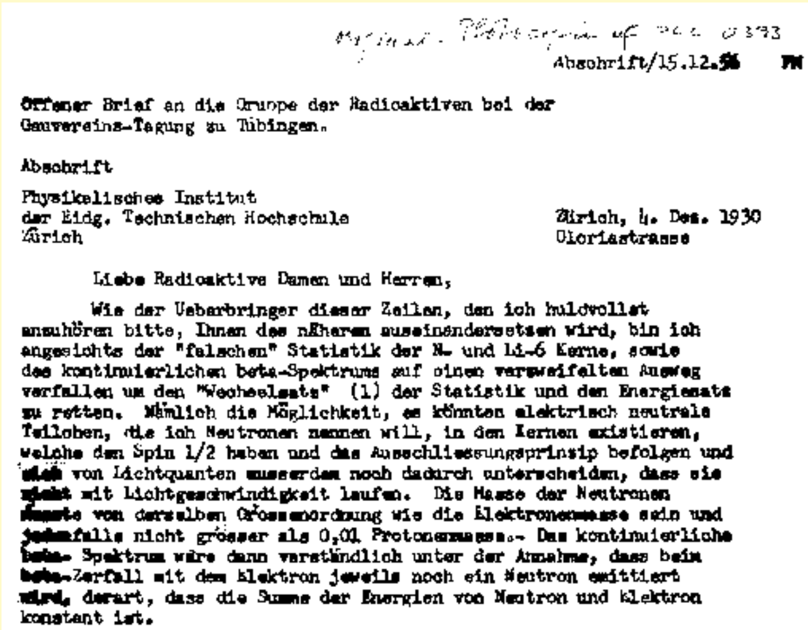
Great progress in recent years …

Neutrino mass & mixing: established fact
Determination of ($\delta m^2, \theta_{12}$) and ($\Delta m^2, \theta_{23}$)
Determination of θ_{13} at reactors (+ accel.)
Observation of (half)-period of oscillations
Direct evidence for solar ν flavor change
Evidence for matter effects in the Sun
Upper bounds on ν masses in (sub)eV range
…………

Further ν_e, ν_τ appearance data at accelerators
Leptonic CP violation
Absolute m_ν from β-decay and cosmology
Test of $0\nu2\beta$ claim and of Dirac/Majorana ν
Matter effects in the Earth, Supernovae…
Normal vs inverted hierarchy
Octant of θ_{23}
Sterile neutrinos in oscillations and cosmology
New neutrino interactions
Deeper theoretical understanding
…………

… and great challenges for the future!

After 83 years… (W. Pauli, Letter from Zurich, 1930)



Abschrift/15.12.56 PM

Offener Brief an die Gruppe der Radioaktiven bei der Gauvereins-Tagung zu Tübingen.

Abschrift

Physikalisches Institut
der Eidg. Technischen Hochschule
Zürich

Zürich, 4. Dez. 1930
Gloriastrasse

Liebe Radioaktive Damen und Herren,

Wie der Ueberbringer dieser Zeilen, den ich huldvollst anzuhören bitte, Ihnen des näheren auseinandersetzen wird, bin ich angesichts der "falschen" Statistik der N- und Li-6 Kerne, sowie des kontinuierlichen beta-Spektrums auf einen verzweifelten Ausweg verfallen um den "Wechselsatz" (1) der Statistik und den Energiesatz zu retten. Nämlich die Möglichkeit, es könnten elektrisch neutrale Teilchen, die ich Neutronen nennen will, in den Kernen existieren, welche den Spin 1/2 haben und das Ausschliessungsprinzip befolgen und sich von Lichtquanten ausserdem noch dadurch unterscheiden, dass sie nicht mit Lichtgeschwindigkeit laufen. Die Masse der Neutronen müsste von derselben Grossenordnung wie die Elektronenmasse sein und jedenfalls nicht grösser als 0,01 Protonenmasse.- Das kontinuierliche beta- Spektrum wäre dann verständlich unter der Annahme, dass beim beta-Zerfall mit dem Elektron jeweils noch ein Neutron emittiert wird, derart, dass die Summe der Energien von Neutron und Elektron konstant ist.

… the neutrino continues to surprise us!

APPENDIX FOR FURTHER READING

Elements of statistics and data analysis

Purpose: become familiar with the treatment of correlated uncertainties, which are often a key ingredient of (neutrino) data analyses.

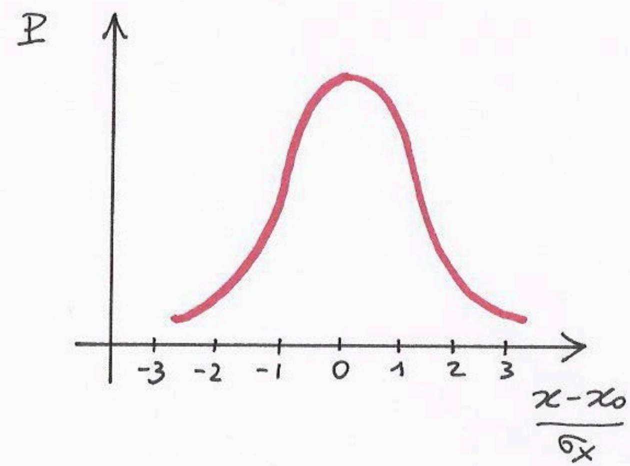
Error distribution for 1 variable

Here we consider only gaussian errors. Also Poisson fluctuations ($\propto \sqrt{N}$) are assumed to be nearly gaussian. [Some comments on small-N cases and on asymmetric errors will be presented at the end.]

Distribution for a single variable:

$$P(x, x_0) = \frac{1}{\sqrt{2\pi} \cdot \sigma_x} e^{-\frac{1}{2}\left(\frac{x-x_0}{\sigma_x}\right)^2},$$

corresponding to quote $x = x_0 \pm \sigma_x$



Area within: $\pm 1\sigma = 68.27\,\%$

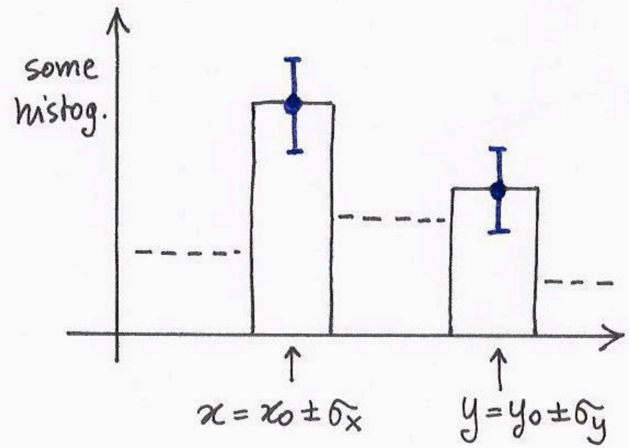
$\pm 2\sigma = 95.45\%$

$\pm 3\sigma = 99.73\%$

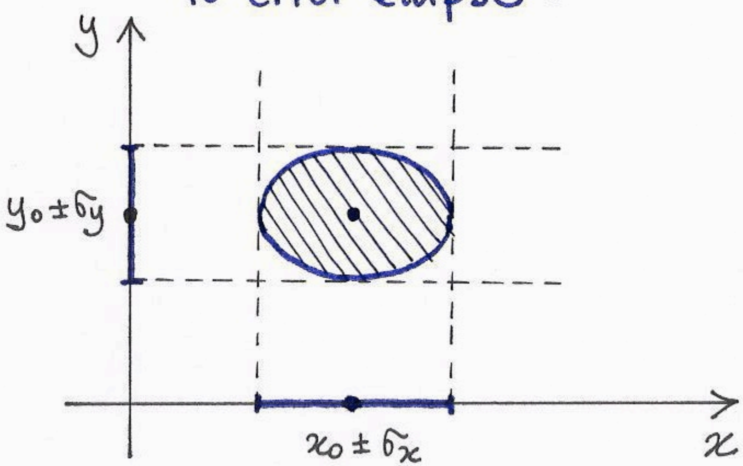
Error distribution for 2 variables (uncorrelated)

Let us consider 2 quantities x & y with errors which have no relation with each other (e.g. statistical errors of two bins):

$$P(x,y;x_0,y_0) = P(x,x_0)\,P(y,y_0)$$



1σ error ellipse:



Ellipse equation:

$$\Delta\chi^2 = 1$$

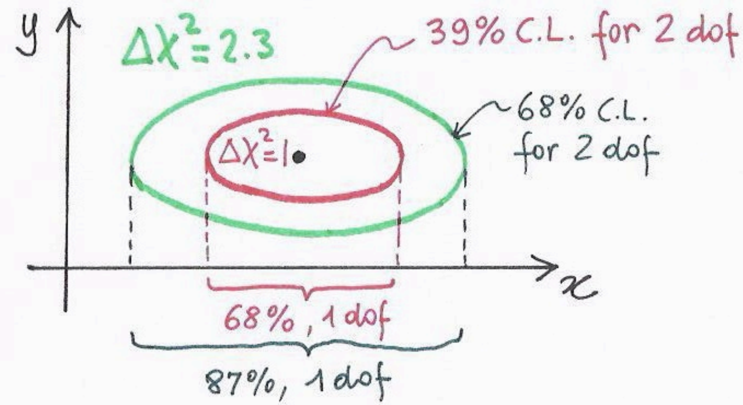
where $\Delta\chi^2 = \left(\frac{x-x_0}{\sigma_x}\right)^2 + \left(\frac{y-y_0}{\sigma_y}\right)^2$

$[\Delta\chi^2 = 0 \text{ at } (x,y) = (x_0,y_0)]$

Note: The probability of finding (x,y) within the 1σ error ellipse is not 68.27% : it is 39.35% !

68% = probability of finding x in $x_0 \pm \sigma_x$, independently on y
68% = " " " y in $y_0 \pm \sigma_y$, " " x
39% = joint probability of finding (x,y) within the 1σ ellipse

If you really want an error ellipse containing 68% joint probab. (68% C.L. for 2 d.o.f.), then you should use $\Delta\chi^2 = 2.3$. Its projections define 87% C.L. for each variable (1 dof).
This is not usually called a "1σ" ellipse.



Confusion may arise if a C.L. is quoted without the corresponding # d.o.f.

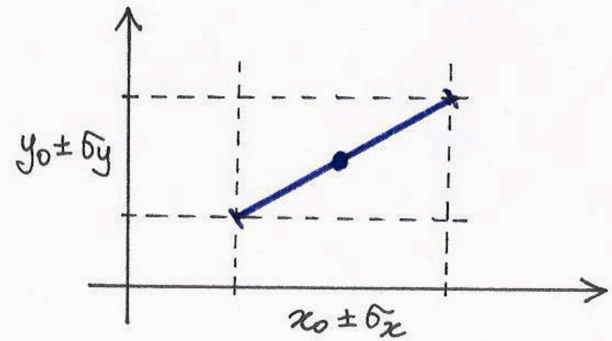
Error distribution for 2 variables (fully correlated)

Consider two variables x and y with errors in one-to-one correspondence, e.g., two bins affected by a common normalization error:

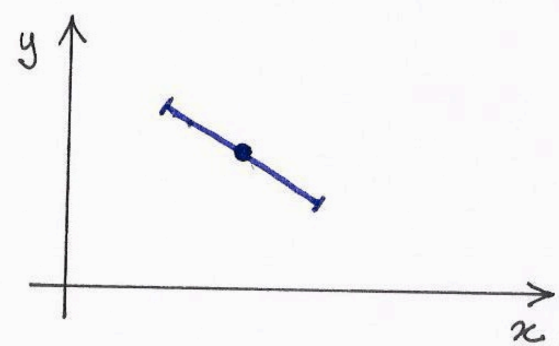
Then the errors go both "up" or "down":

if $x = x_0 + \sigma_x$ then $y = y_0 + \sigma_y$
if $x = x_0 - \sigma_y$ then $y = y_0 - \sigma_y$

The error ellipse is degenerate →
(fully correlated errors)



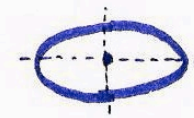
Analogously for full "anticorrelation":
E.g., two bins whose sum is constant;
then, $x = x_0 + \sigma_x$ implies $y = y_0 - \sigma_y$,
and the degenerate ellipse has a negative slope →



Recap: Eqs. for 1σ error ellipses (limit cases)

- No correlation

$$1 = (x-x_0, y-y_0) \underbrace{\begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix}}_{\det \neq 0}{}^{-1} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$



- Full correlation

$$1 = (x-x_0, y-y_0) \underbrace{\begin{pmatrix} \sigma_x^2 & \sigma_x\sigma_y \\ \sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}}_{\det = 0}{}^{-1} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

- Full anticorr.

$$1 = (x-x_0, y-y_0) \underbrace{\begin{pmatrix} \sigma_x^2 & -\sigma_x\sigma_y \\ -\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}}_{\det = 0} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

In general we expect: $1 = (x-x_0, y-y_0) \begin{pmatrix} \text{squared} \\ \text{error} \\ \text{matrix} \end{pmatrix}^{-1} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$

More general 1σ error ellipses

Let us consider two variables (x, y) and two sources of uncertainties:

- statistical (s_x, s_y) with no correlation,
- systematic (c_x, c_y) with full correlation,

namely, $\begin{cases} x = x_0 \pm s_x\,(\text{stat}) \pm c_x\,(\text{syst}) \\ y = y_0 \pm s_y\,(\text{stat}) \pm c_y\,(\text{syst}) \end{cases}$

The errors sum up in quadrature <u>at matrix level</u>:

$$\sigma^2 = \begin{bmatrix} s_x^2 & 0 \\ 0 & s_y^2 \end{bmatrix} + \begin{bmatrix} c_x^2 & c_x c_y \\ c_x c_y & c_y^2 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix} \leftarrow \text{squared error matrix}$$

where

$$\sigma_x^2 = s_x^2 + c_x^2$$

$$\sigma_y^2 = s_y^2 + c_y^2$$

$$\rho = \frac{c_x c_y}{\sigma_x \sigma_y} = \begin{cases} 0 & \text{for } c_x \text{ or } c_y = 0 \quad (\text{no correlation}) \\ 1 & \text{for } s_x = s_y = 0 \quad (\text{full correlation}) \end{cases}$$

In general, the correlation ρ obeys: $0 \le |\rho| \le 1$

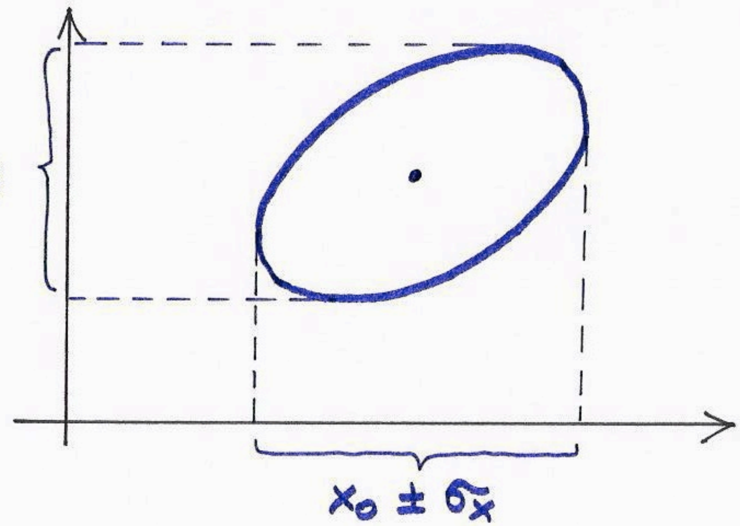
Qualitative shape of 1σ error ellipse:

(s_x, s_y) $\varrho = 0$ $\oplus$ (c_x, c_y) $\varrho = +1$ $=$ (σ_x, σ_y) $0 < \varrho < 1$

Analogously, adding a fully anticorrelated error source:

(s_x, s_y) $\varrho = 0$ $\oplus$ (c_x, c_y) $\varrho = -1$ $=$ (σ_x, σ_y) $-1 < \varrho < 0$

Projections of 1σ error ellipse coincide with $\pm 1\sigma$ ranges for the x and y variables:



Equation of 1σ ellipse:

$$
\begin{aligned}
1 &= \Delta\chi^2 \\
&= (x - x_0,\ y - y_0) \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}^{-1} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \\
&= \frac{1}{1-\rho^2}\left[\left(\frac{x - x_0}{\sigma_x}\right)^2 + \left(\frac{y - y_0}{\sigma_y}\right)^2 - 2\rho\,\frac{(x - x_0)(y - y_0)}{\sigma_x\sigma_y}\right]
\end{aligned}
$$

Probability distribution: $$P = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}\Delta\chi^2}$$

Positively correlated variables and their ratio

Let us consider two variables with positively correlated errors :
$$\begin{cases} x = x_0 \pm \sigma_x \\ y = y_0 \pm \sigma_y \end{cases} \qquad \rho > 0$$

If the correlation is sizable, we expect a significant "cancellation" of errors in the ratio :
$$r = \frac{x}{y} = r_0 \pm \sigma_r \quad \text{with "small } \sigma_r\text{"}$$

The error of the ratio can be evaluated as :

$$\sigma_r^2 = \left(\frac{\partial r}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial r}{\partial y}\right)^2 \sigma_y^2 + 2\rho\left(\frac{\partial r}{\partial x}\right)\left(\frac{\partial r}{\partial y}\right)\sigma_x\sigma_y$$

$$\rightarrow \frac{\sigma_r^2}{r_0^2} = \frac{\sigma_x^2}{x_0^2} + \frac{\sigma_y^2}{y_0^2} - 2\rho\,\frac{\sigma_x}{x_0}\frac{\sigma_y}{y_0} \;;$$ the "$-2\rho\ldots$" term is responsible for error cancellation.

Note that, for $\rho = +1$ (full correlation) : $\frac{\sigma_r^2}{r_0^2} = \left(\frac{\sigma_x}{x_0} - \frac{\sigma_y}{y_0}\right)^2$;
then, if $\frac{\sigma_x}{x_0} = \frac{\sigma_y}{y_0}$ (e.g., for a common normalization uncertainty) the cancellation is complete ($\sigma_r = 0$) as expected.

In general, it is preferable to use correlated variables (x,y) whenever possible, rather than their ratio $r = x/y$.

The main reason is that, if y is distributed as a Gaussian, then $1/y$ is distributed as a Lorentzian ("Breit-Wigner"), with a formally infinite variance. In practice, this may be problematic if σ_y is large and/or one is probing the distribution tails.

Therefore, if we measure $x = x_0 \pm \sigma_x$ and $y = y_0 \pm \sigma_y$, and if we know that $r = r_0 \pm \sigma_r$, it is convenient to keep (x,y) in the fit, together with the correlation

$$\rho = \left(\frac{\sigma_x^2}{x_0^2} + \frac{\sigma_y^2}{y_0^2} - \frac{\sigma_r^2}{r_0^2}\right) \Big/ \left(2\,\frac{\sigma_x}{x_0}\,\frac{\sigma_y}{y_0}\right)$$

(instead of using r directly).

Historically, several "ratios" have been progressively abandoned in neutrino data fits.

Estimates of correlations and covariances

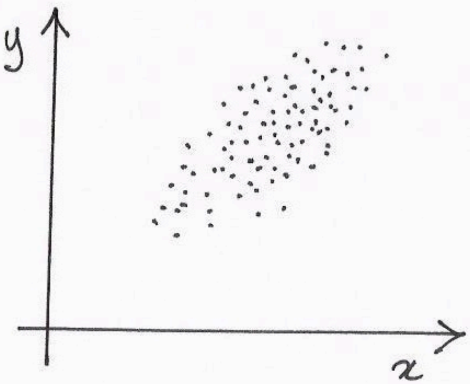
The squared error matrix $\sigma^2 = \begin{bmatrix} \sigma_x^2 & \sigma_{xy}^2 \\ \sigma_{xy}^2 & \sigma_y^2 \end{bmatrix}$

is also called "covariance matrix":

$\sigma_{xy}^2 = \varrho\, \sigma_x \sigma_y =$ covariance of (x, y)

$\sigma_x^2 =$ variance of x

$\sigma_y^2 =$ variance of y



If repeated measurements (or simulations) of the variables x and y are available, →

then:

$$x_0 = \frac{1}{n} \sum_{i=1}^{n} x_i \quad , \quad \sigma_x^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - x_0)^2$$

$$y_0 = \frac{1}{n} \sum_{i=1}^{n} y_i \quad , \quad \sigma_y^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - y_0)^2$$

$$\sigma_{xy}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - x_0)(y_i - y_0) \; ; \quad \varrho = \sigma_{xy}^2 / \sigma_x \sigma_y$$

↓

→ Get: $\left\{ \begin{array}{l} x = x_0 \pm \sigma_x \\ y = y_0 \pm \sigma_y \\ \varrho \end{array} \right.$

However, the ideal situation is to identify and break down all possible error sources to the two main categories of "uncorrelated" errors s and "fully correlated" errors c:

$$\sigma^2 = \begin{pmatrix} \sigma_x^2 & \sigma_{xy}^2 \\ \sigma_{xy}^2 & \sigma_y^2 \end{pmatrix} = \underbrace{\begin{pmatrix} s_x^2 & 0 \\ 0 & s_y^2 \end{pmatrix} + \cdots}_{\text{uncorrelated}} \quad \underbrace{+ \begin{pmatrix} c_x^2 & c_x c_y \\ c_x c_y & c_y^2 \end{pmatrix} + \cdots}_{\text{fully correlated}}$$

[fully anticorrelated errors become fully correlated by changing the sign of one variable].

In this case, one is really "summing in quadrature" all possible, independent (known) error sources.

Generalization to N variables $\{x_i\}_{1 \le i \le N}$

- $\Delta\chi^2 = 1$ error ellipsoid in N-dimensional space is defined by:

  $$1 = \Delta x^T (\sigma^2)^{-1} \Delta x$$

  where Δx is a column vector, $\Delta x = \begin{pmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \\ \vdots \\ x_N - x_N^0 \end{pmatrix}$

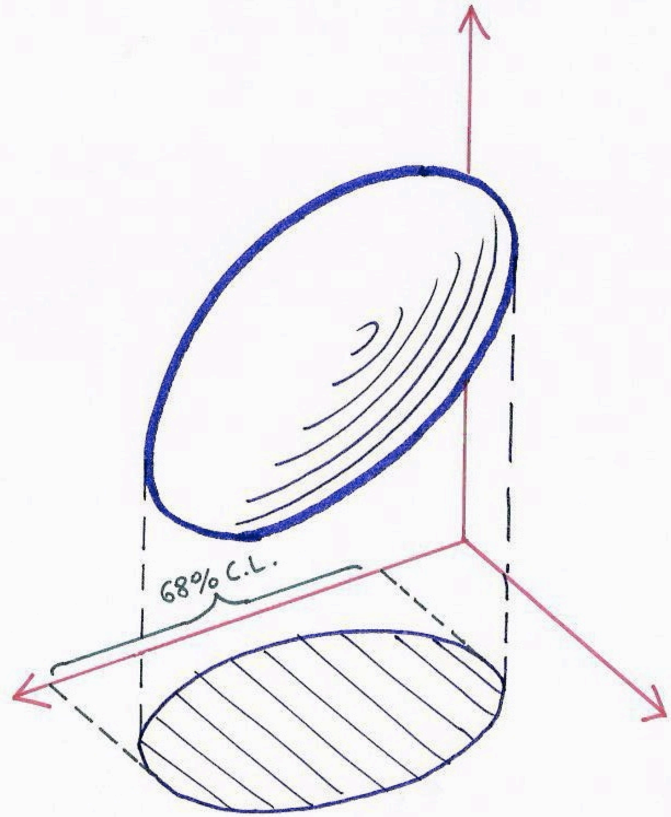
  and $(\sigma^2)^{-1}$ is the inverse of the covariance matrix (symmetrical):

  $$\sigma^2 = \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 & \cdots \\ & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 & \cdots \\ & & \sigma_3^2 & \cdots \\ & & & \ddots \end{pmatrix}$$

- Probability distribution (multivariate Gaussian):

  $$P = \frac{1}{(2\pi)^{\frac{N}{2}} \sqrt{\det \sigma^2}} \, e^{-\frac{1}{2}\Delta\chi^2}$$

- Projection of the $\Delta\chi^2 = 1$ ellipsoid onto one axis x_i gives the 1σ (68% CL) range on the x_i variable ($x_i = x_i^0 \pm \sigma_i$). This holds for any N. Variables $x_j \neq x_i$ are said to be "marginalized" or "projected away"



- However, the joint probability of $(x_1, x_2, \ldots, x_N)$ being inside the ellipsoid decreases with N:

$N=1$	68%	
$N=2$	39%	$< (68\%)^2$
$N=3$	20%	$< (68\%)^3$
$N=4$	9%	$< (68\%)^4$
$\vdots$	$\vdots$	$\vdots$

- Note: n-σ ellipsoids defined by $\Delta\chi^2 = n^2$

Fitting data with a model

In this case we compare theoretical predictions x_i^{theo} with N experimental data x_i^{exp}. Predictions will depend on N_p ($< N$) parameters $\vec{p}$: $x_i^{theo} = x_i^{theo}(\vec{p})$. Recipe:

- Build $\chi^2 = \Delta x^T (\sigma^2)^{-1} \Delta x$, Δx = column vector of $(x_i^{theo} - x_i^{exp})$
  $\sigma^2 = \sigma^2_{theo} + \sigma^2_{exp}$

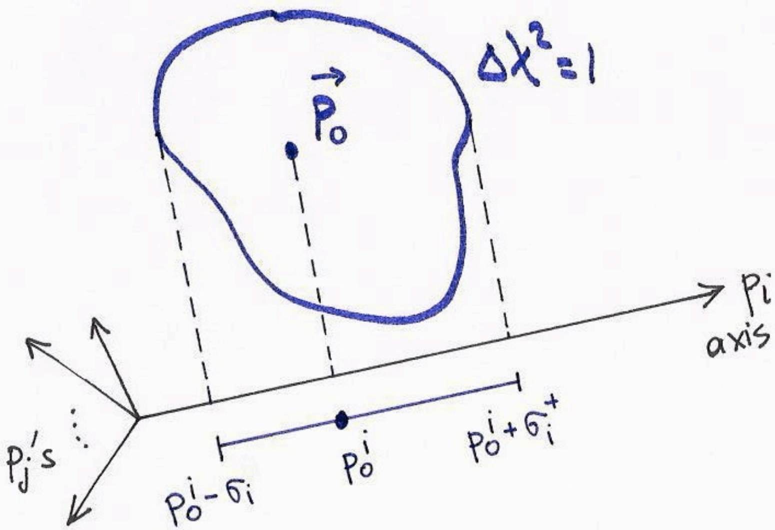
- Find $\chi^2_{min} = \min_{\vec{p}} \chi^2(\vec{p})$ at some $\vec{p} = \vec{p}_0$

- Check that $\chi^2_{min} \sim \underbrace{N - N_p}_{\text{dof for test of hypothesis}} \pm \sqrt{2(N-N_p)}$ ← see, e.g., PDG

- Check not ok : model wrong (χ^2_{min} too high) or "too good" (χ^2_{min} too low). Verify model, underestimated errors....

- Check OK →

Parameter estimation



The Np-dimensional manifold defined by:

$$\Delta\chi^2 = \chi^2(\vec{p}) - \chi^2_{min} = 1$$

represents the "1σ allowed region" of parameters. Projection onto one axis p_i provides $\pm 1\sigma$ ranges:

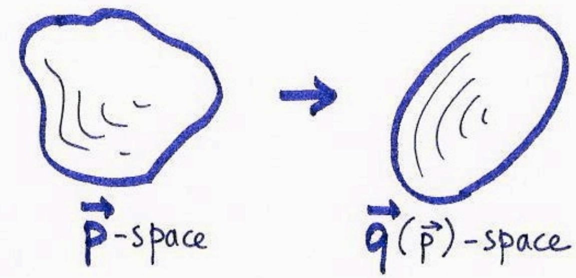
$$p = p_0^i \begin{array}{l} + \sigma_i^+ \\ - \sigma_i^- \end{array} \quad \text{(generally asymmetric)}$$

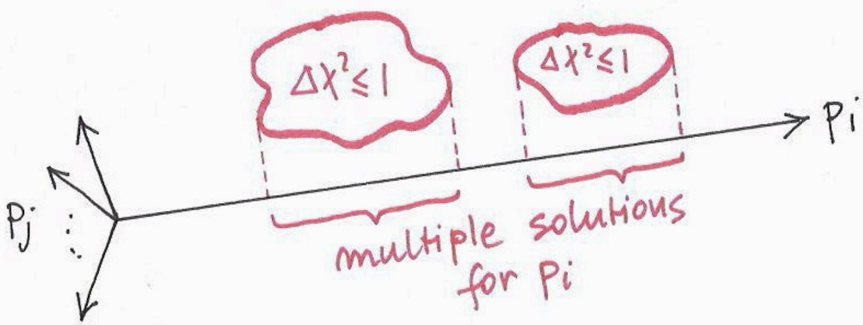
Projection onto p_i ($\equiv$ marginalization over $p_j \neq p_i$) in practice means setting $\Delta\chi^2_i = 1$, where the "reduced" $\Delta\chi^2_i$ is:

$$\Delta\chi^2_i = \min_{p_j \neq p_i} \left(\chi^2(\vec{p}) - \chi^2_{min}\right)$$

Analogously, n-σ ranges are defined by $\Delta\chi^2_i = n^2$

- This procedure can be justified, as far as the allowed region is simply connected. The basic idea is that, through mapping, this volume can be transformed into an ellipsoid ($\rightarrow$ gaussian machinery ...)





- However, if there are several (disconnected) allowed region, the statistical interpretation is problematic. No "consensus" approach in this case.

- In practice, most people keep using the $\Delta\chi^2 = \text{const}$ recipe also for multiple solutions, with some cautionary remarks.

- There is no other way than waiting for new experiments/data to solve the ambiguity ("degeneracy of solutions").

- Sometimes, one is interested not only in $\pm n\sigma$ limits on each variable separately ($\equiv \Delta\chi^2 = n^2$ projections), but on the joint probability of $\vec{p}$ being in a volume defined by $\Delta\chi^2 =$ const.

Relevant tables of $\Delta\chi^2$ level cuts (from PDG):

C.L.	N=1	N=2	N=3
68	1.00	2.30	3.53
90	2.71	4.61	6.25
95	3.84	5.99	7.82
99	6.63	9.21	11.34
99.73	9.00	11.83	14.16

E.g. the joint 95% C.L. region for two variables (p_i, p_j) is defined by $\Delta\chi^2_{ij} = 5.99$, where $\Delta\chi^2_{ij} = \min\limits_{p_k \neq p_{i,j}} \left[\chi^2(\vec{p}) - \chi^2_{min}\right]$.

Analyzing χ^2 contributions.

- The χ^2 is a global quantity. By itself, it does not necessarily "detect" single observables which might be badly fitted
  $\rightarrow$ need to split the total χ^2 into "pieces"

- One possibility is to look at the "residuals" or "pulls" after the fit:

$$(\text{pull})_i = \frac{x_i^{theo}(\vec{p}_0) - x_i^{exp}}{\sigma_i} \qquad \left(\sigma_i = \sqrt{\sigma_{ii}^2} \right)$$

- Then, a large pull (say, $\gtrsim 3\sigma$) signals a potential problem in the data and/or in the model.

The Standard Model fit to LEP data is often shown in terms of such pulls.

- With the previous definitions, however,

$$\chi^2 \neq \Sigma\,(\text{pull})^2$$

since σ^2_{ij} is not diagonal in general.

- It is possible to re-write the same χ^2 in a form $\Sigma\,(\text{pull})^2$ with a somewhat different approach, which also brings some technical advantages.

Previous χ^2 approach : "covariance method"
Alternative " : "pull method"

Covariance method (recap)

- Consider N observables $\{R_n\}_{n=1\ldots N}$

  $\{R_n^{theo}\}$ = theoretical predictions

  $\{R_n^{exp}\}$ = experimental measurements

$$(R_n^{theo} - R_n^{exp}) \pm \underset{\substack{\uparrow \\ \text{uncorrel.} \\ \text{error}}}{u_n} \pm \underbrace{c_n^1 \pm c_n^2 \pm \ldots \pm c_n^K}_{\text{set of } K \text{ systematics produced by independ. sources}}$$

with
$\varrho(u_n, u_m) = 0$ (always uncorrelated)
$\varrho(c_n^k, c_m^k) = 1$ (fully correlated for the same k-th source)
$\varrho(c_n^k, c_m^h) = 0$ ($h \neq k$, uncorrelated from different sources)

- Then: Build $\sigma_{nm}^2 = \delta_{nm} u_n u_m + \sum_{k=1}^{K} c_n^k c_m^k$

  and evaluate $\chi_{cov}^2 = \sum_{n,m=1}^{N} (R_n^{exp} - R_n^{theo}) \left[\sigma_{nm}^2\right]^{-1} (R_m^{exp} - R_m^{theo})$

  as discussed previously

Pull method

- Shift the theoretical predictions linearly in the systematics:

$$R_n^{theor} \to R_n^{theo} + \sum_{k=1}^{K} \xi_k C_n^k$$

where ξ_k = univariate gaussian random variable $\left(\langle \xi_k \rangle = 0,\ \langle \xi_k^2 \rangle = 1\right)$

- Minimize over ξ_k the following sum of squared residuals:

$$\chi^2_{pull} = \min_{\{\xi_k\}} \left[\underbrace{\sum_{n=1}^{N} \left(\frac{R_n^{exp} - \left(R_n^{theo} + \sum_{k=1}^{K} \xi_k C_n^k\right)}{u_n} \right)^2}_{\text{squared residuals}} + \underbrace{\sum_{k=1}^{K} \xi_k^2}_{\text{penalty term}} \right]$$

- At minimum $\left(\xi_k \stackrel{def}{=} \bar{\xi}_k\right)$: $\bar{R}_n^{theo} = R_n^{theo} + \sum_{k=1}^{K} \bar{\xi}_k C_n^k$ ← "shifted" predictions

and $$\chi^2_{pull} = \sum_{n=1}^{N} \left(\frac{R_n^{exp} - \bar{R}_n^{theo}}{u_n} \right)^2 + \sum_{k=1}^{K} \bar{\xi}_k^2$$

$$= \sum_{n=1}^{N} \begin{pmatrix} \text{pull of} \\ \text{observable} \end{pmatrix}_n^2 + \sum_{k=1}^{K} \begin{pmatrix} \text{pull of} \\ \text{systematic} \end{pmatrix}_k^2$$ ← "diagonal" form

- It turns out that: $\chi^2_{pull} \equiv \chi^2_{covariance}$ (some algebra needed)

  so the methods are numerically equivalent.

- The pull approach may be more convenient for large N. The inversion of a large $N \times N$ covariance matrix may be unstable, especially if systematics dominate. The pull method leads to k equations (linear) in the ξ_k's, which is solved by a $k \times k$ matrix inversion, with $k \ll N$ usually.

- In addition, relatively large ξ_k's may signal systematic "offsets" required to match data and theory.

- Several ν data analyses are now performed in terms of χ^2_{pull}.

See hep-ph/0206162 for details

Comment on low-statistic bins

- The fit to a histogram may become problematic if one (or more) bin contains a low number of events N^{exp} (or even none, $N^{exp}=0$).

  In this case, the Gaussian approximation

$$\chi^2 \ni \frac{(N^{exp}-N^{theo})^2}{N^{exp}} \qquad \underline{\text{fails}}$$

- In this case, the PDG suggests an alternative form, which embeds more properly the Poisson nature of the fluctuations:

$$\chi^2 \ni 2\left(N^{theo}-N^{exp}+N^{exp}\ln\frac{N^{exp}}{N^{theo}}\right)$$

$$\left[\text{or } 2N^{theo} \text{ if } N^{exp}=0\right]$$

- Additional systematics can still be embedded in the pull approach via:

$$N^{theo} \to N^{theo} + \sum_k \xi_k c_n^k$$

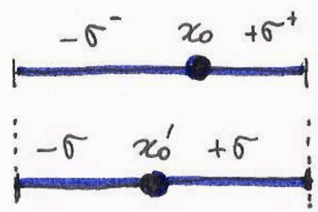
- Gaussian limit recovered at large N.
  Hint: expand logarithm at second order in $(N^{theo}-N^{exp})/N^{exp}$.

Comment on asymmetric errors

- If a variable x is affected by asymmetric errors : $\quad x = x_0 {}^{+\sigma^+}_{-\sigma^-}$
  there is no "consensus recipe" to write a χ^2 contribution.

- Sometimes the range is conservatively symmetrized to the largest error : $\quad x = x_0 \pm \sigma_{max} \quad , \quad \sigma_{max} = \max(\sigma^+, \sigma^-)$

- A better recipe has been argued in physics/0403086 :
  shift $x_0 \to x_0'$ so that the new $\pm 1\sigma$ range reproduces the old one.



$$x_0' + \sigma \equiv x_0 + \sigma^+$$
$$x_0' - \sigma \equiv x_0 - \sigma^-$$

Then the χ^2 contribution is : $\left(\frac{x - x_0'}{\sigma}\right)^2$.