Lattice calculation of isospin corrections to $K\ell 2$ and $K\ell 3$ decays

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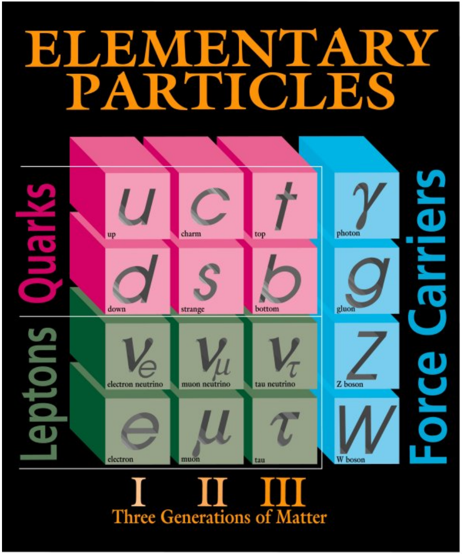
- among the questions left open by the standard model there is the origin of flavour
- the two lightest quarks, the up and the down, have different masses and different electric charges
- nevertheless

$$\frac{m_d - m_u}{\Lambda_{QCD}} \ll 1$$

$$(e_u - e_d)\alpha_{em} \ll 1$$

- for these reasons the group of rotations in this flavour space is a *good* and *very useful* approximate symmetry of the real world



Fermilab 95-759

why isospin breaking?

we do have a lot of precise experimental measurements in the quark flavour sector of the standard model that, combined with CKM unitarity (first row), allow us to **measure** hadronic matrix elements

M.Antonelli et al. Eur.Phys.J.C69 (2010)
G.Colangelo talk at Lattice2012
V.Cirigliano talk

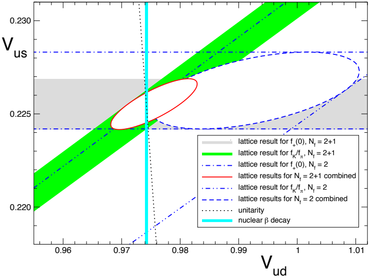
$$\begin{cases} \left|\frac{V_{us}F_K}{V_{ud}F_\pi}\right| = 0.2758(5) \\ \left|V_{us}F_+^{K\pi}(0)\right| = 0.2163(5) \end{cases} \qquad \begin{cases} |V_{ud}|^2 + |V_{us}|^2 = 1 \\ |V_{ud}| = 0.97425(22) \end{cases}$$

where $|V_{ud}|$ comes by combining 20 super-allowed nuclear β-decays and $|V_{ub}|$ has been neglected because smaller than the uncertainty on the other terms, combine to give

$$|V_{us}| = 0.22544(95)$$

$$F_+^{K\pi}(0) = 0.9595(46)$$

$$\frac{F_K}{F_\pi} = 1.1919(57)$$

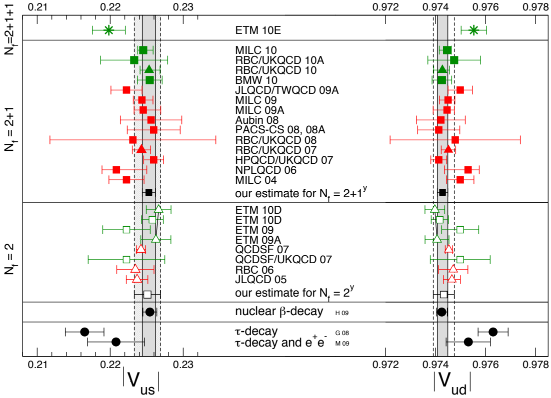


lattice QCD is **still** needed to **postdict** these quantities and, in case, to falsify the standard model

F_K/F_π & $F_+^{K\pi}(0)$ summary from FLAG

concerning theoretical predictions, and lattice QCD in particular, these matrix elements are among the well known quantities

FALG Eur.Phys.J. C71 (2011)
G.Colangelo talk at Lattice2012
J.Laiho and A.Juttner talks



$$F_+^{K\pi}(0) = 0.956(8) \qquad \sim 0.8\% \qquad\qquad \frac{F_K}{F_\pi} = 1.193(5) \qquad \sim 0.5\%$$

to do better we should include effects that we have been neglecting up to now. . .

F_K/F_π & $F_+^{K\pi}(q^2)$ beyond the isospin limit

- it is useful to *divide* the isospin breaking effects into strong and electromagnetic ones,

$$\underbrace{m_u \neq m_d}_{\textbf{QCD}} \qquad\qquad \underbrace{e_u \neq e_d}_{\textbf{QED}}$$

- in the particular and (lucky) case of these observables, the correction to the isospin symmetric limit due to the difference of the up and down quark masses (**QCD**) can be estimated in *chiral perturbation theory*,

$$\begin{cases} F_+^{K\pi}(0) = 0.956(8) \qquad \sim 0.8\% \\ \left(\dfrac{F_+^{K^+\pi^0}(q^2)}{F_+^{K^0\pi^-}(q^2)} - 1 \right)_{QCD} = 0.029(4) \end{cases}$$

A. Kastner, H. Neufeld Eur.Phys.J.C57 (2008)

$$\begin{cases} \dfrac{F_K}{F_\pi} = 1.193(5) \qquad \sim 0.5\% \\ \left(\dfrac{F_{K^+}/F_{\pi^+}}{F_K/F_\pi} - 1 \right)_{QCD} = -0.0022(6) \end{cases}$$

V. Cirigliano, H. Neufeld Phys.Lett. B700 (2011)

- we need first principle lattice QCD calculations to avoid uncertainties coming from the effective theory
- but the home message is: reducing the error on these quantities without taking into account isospin breaking is useless. . .

the collaboration

RM123

JHEP 1204 (2012)

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isospin breaking on the lattice

- the calculation of QED isospin breaking effects on the lattice it has been done for the first time in
Duncan, Eichten, Thacker, Phys. Rev. Lett. 76 (1996)

- the gauge potential is sampled directly, the QED links are obtained by exponentiation and the QCD links are replaced by $U_\mu(x) \to e^{iA_\mu(x)}\, U_\mu(x)$

...
MILC Collaboration, PoS LATTICE2008 (2008) 127
T.Blum et al. Phys. Rev. D82 (2010)
[BMW Collaboration] PoS LATTICE2010 (2010) 121
[T. Ishikawa et al.] Phys. Rev. Lett. 109 (2012)

- because the photons are massless and unconfined this approach may introduce large finite volume effects... we shall come back on QED effects later in this talk

- the calculation of QCD isospin breaking effects on the lattice poses a problem

$$
\begin{aligned}
Z &= \int DUD\psi\; e^{-S_g[U]+S_f[U;m_u,m_d]} \\
&= \int DU\; e^{-S_g[U]} \underbrace{\det(D[U]+m_u)\;\det(D[U]+m_d)}_{\text{must be real and } >0}
\end{aligned}
$$

- furthermore the effect is very small and it can be extremely difficult to see it with limited statistical accuracy

our QCD isospin breaking on the lattice

- our idea is to calculate QCD isospin corrections **at first order** in $\Delta m_{ud} = (m_d - m_u)/2$:

$$
\begin{aligned}
S &= \bar{u}\,(D[U] + m_u)\,u + \bar{d}\,(D[U] + m_d)\,d \\
&= \underbrace{\bar{u}\,(D[U] + m_{ud})\,u + \bar{d}\,(D[U] + m_{ud})\,d}_{S_0} - \overbrace{\frac{m_d - m_u}{2}(\bar{u}u - \bar{d}d)}^{\boldsymbol{\Delta m_{ud}\hat{S}}}
\end{aligned}
$$

- the calculation of an observable proceeds as follows

$$
\begin{aligned}
\langle \mathcal{O} \rangle - \boldsymbol{\Delta}\langle \mathcal{O} \rangle &= \frac{\int DU\ e^{-S_g[U] - S_0[U] + \Delta m_{ud}\hat{S}}\ \mathcal{O}}{\int DU\ e^{-S_g[U] - S_0[U] + \Delta m_{ud}\hat{S}}} = \frac{\int DU\ e^{-S_g[U] - S_f^0[U]}\ (1 + \Delta m_{ud}\hat{S})\ \mathcal{O}}{\int DU\ e^{-S_g[U] - S_f^0[U]}\ (1 + \Delta m_{ud}\hat{S})} \\
&= \langle \mathcal{O} \rangle + \boldsymbol{\Delta m_{ud}}\langle \hat{\mathbf{S}}\ \mathcal{O} \rangle - \underbrace{\Delta m_{ud}\langle \hat{S} \rangle}_{=0}
\end{aligned}
$$

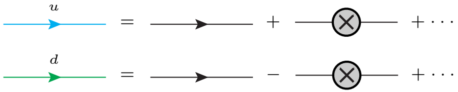
our QCD isospin breaking on the lattice

- to insert $\bar{u}u - \bar{d}d$ within a correlation function amounts (after fermionic Wick contractions) to calculate **the same observables** but with **light propagators squared**

$$\mathcal{S}_u \quad = \frac{1}{D[U]+m_{ud}-\Delta m_{ud}} \quad = \frac{1}{D[U]+m_{ud}} + \frac{\Delta m_{ud}}{(D[U]+m_{ud})^2}$$

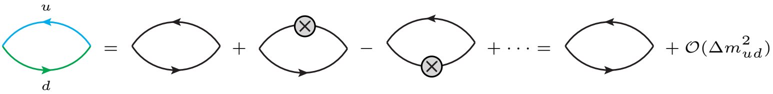
$$\mathcal{S}_d \quad = \frac{1}{D[U]+m_{ud}+\Delta m_{ud}} \quad = \frac{1}{D[u]+m_{ud}} - \frac{\Delta m_{ud}}{(D[U]+m_{ud})^2}$$

- relations that can be represented diagrammatically as

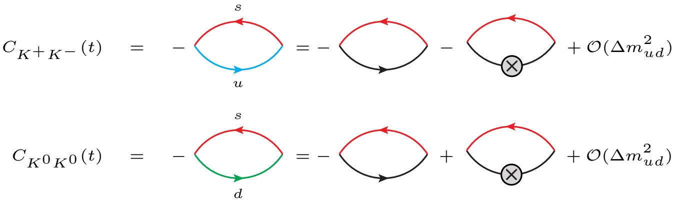


our QCD isospin breaking on the lattice: two point functions

- at first order in Δm_{ud} pion mass and decay constants don't get a correction (here $\pi^{\pm}$ but it works also for π^{0} because $\langle \pi \| \hat{S} \| \pi \rangle = \langle 1, I_3 \| 1, 0 \| 1, I_3 \rangle = 0$)



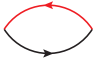
- the kaons do get a correction



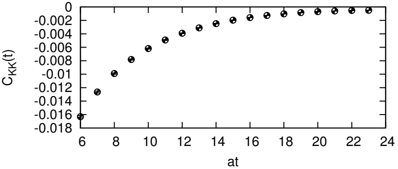
- this means that at first order ($\delta_{.}$ stays for relative variation while $\Delta_{.}$ for absolute variation),

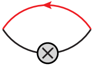
$$\delta_u \left(\frac{F_K}{F_\pi} \right) = \frac{\Delta_u F_K}{F_K} - \frac{\Delta_u F_\pi}{F_\pi} = \frac{F_K - F_{K^+}}{F_K}$$

what do we expect from “corrected” correlation functions?

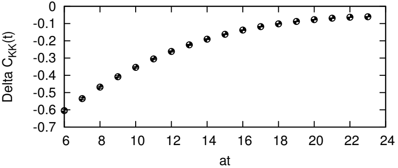
$-$ 

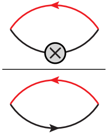
$$= \frac{G_K^2}{2E_K}\; e^{-E_K t}$$



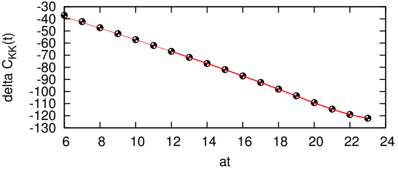


$$= \frac{G_K^2}{2E_K}\; e^{-E_K t}\left[\frac{\Delta(G_K^2/2E_K)}{G_K^2/2E_k} - t\Delta E_K\right]$$

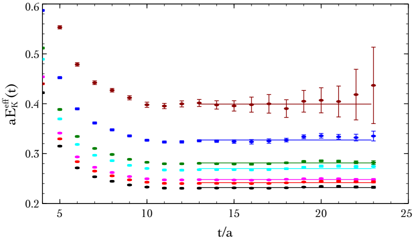


$-$ 

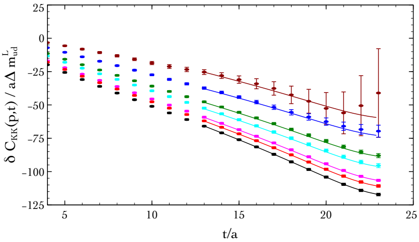
$$= \delta\left(\frac{\mathbf{G_K^2}}{\mathbf{2E_K}}\right) - \mathbf{\Delta E_K t}$$



our QCD isospin breaking on the lattice: kaons two point functions



$$E_K^2(p) = M_K^2 + p^2$$



$$\Delta E_K(p) = \frac{M_K \Delta M_K}{\sqrt{M_K^2 + p^2}}$$

- by considering pseudoscalar-pseudoscalar correlators and by taking into account the finite time extent of the lattice, we fit correlations at different $\vec{p}$ according to,

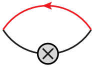
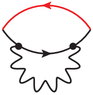
$$\delta C_{KK}(\vec{p}, t) = \delta \left(\frac{G_K^2 e^{-E_K T/2}}{2E_K} \right) + \Delta E_K (t - T/2) \tanh \left[E_K (t - T/2) \right] + \dots$$

- and extract F_K and δF_K according to

$$F_K = (m_s + m_{ud}) \frac{G_K}{M_K^2} \qquad \delta F_K = \frac{\Delta m_{ud}}{m_s + m_{ud}} + \delta G_K - 2\delta M_K$$

extracting $[m_d - m_u]^{QCD}$: QED corrections

- in order to extract $2\Delta m_{ud}^{QCD} = [m_d - m_u]^{QCD}$ we need experimental inputs and we cannot neglect QED corrections
- If we work at first order in the QED coupling constant and Δm_{ud} and neglect terms of $\mathcal{O}(\alpha_{em}\Delta m_{ud})$, *some* of the relevant Feynman diagrams entering kaons two point functions are

$\Delta C_{KK}(t) =$  $- \frac{e_d^2 - e_u^2}{2}$  $- e_s \frac{e_d - e_u}{2}$  $+ \cdots$

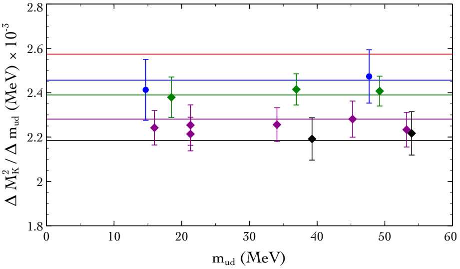
- the electromagnetic corrections to $C_{KK}(t)$ are logarithmically divergent, corresponding to the renormalization of the quark masses, and the separation of QED and QCD effects is ambiguous (**prescription dependent**)
- **in the chiral limit** QED corrections to $M^2_{K^0} - M^2_{K^+}$ and $M^2_{\pi^0} - M^2_{\pi^+}$ are the same (Dashen's theorem)
- **beyond the chiral limit** violations to Dashen's theorem are parametrized in term of small parameters

$\varepsilon_\gamma = 0.7(5)$ **from FLAG: Eur.Phys.J. C71 (2011) our prescription, for the time being**

$$\left[M^2_{K^0} - M^2_{K^+}\right]^{QCD} = \left[M^2_{K^0} - M^2_{K^+}\right]^{exp} - (1+\varepsilon_\gamma)\left[M^2_{\pi^0} - M^2_{\pi^+}\right]^{exp} \quad = \quad 6.05(63) \times 10^3 \text{ MeV}^2$$

$$\varepsilon_\gamma = 0 \quad \longrightarrow \quad 5.16 \times 10^3 \text{ MeV}^2$$

extracting $[m_d - m_u]^{QCD}$: chiral-continuum extrapolations



$$[m_d - m_u]^{QCD}\,(\overline{MS}, 2GeV) = 2\Delta m_{ud}^{QCD}$$

$$= 2.35(8)(24)\ \text{MeV}$$

chiral perturbation theory formulae can be derived from known results

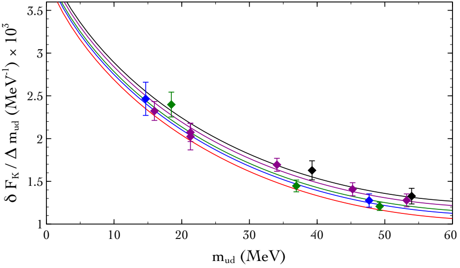
$n_f = 2+1$: **Gasser and Leutwyler Nucl. Phys. B250(1985)**

non unitary $n_f = 2$: **S.Sharpe Phys. Rev. D56(1997)**

$$\frac{\Delta M_K^2}{\Delta m_{ud}} = B_0 \Big\{ 1 + 2(m_{ud}+m_s)\hat{B}_0(2\alpha_8 - \alpha_5) + 4m_{ud}\hat{B}_0(2\alpha_6 - \alpha_4) + \hat{B}_0 m_s \log(2\hat{B}_0 m_s) + \hat{B}_0 \frac{m_s + m_{ud}}{m_s - m_{ud}} \left[m_s \log(2\hat{B}_0 m_s) - m_{ud}\log(2\hat{B}_0 m_{ud}) \right] \Big\}$$

where α_i are low energy constants and $\hat{B}_0 = 2B_0/(4\pi F_0^2)$

calculating δF_K^{QCD}: chiral-continuum extrapolations



$$\left[\frac{F_{K^+}/F_{\pi^+}}{F_K/F_\pi} - 1\right]^{QCD} = -0.0039(3)(2)$$

$$\epsilon_\gamma = 0 \longrightarrow -0.0032(3)$$

to be compared with

$$\left[\frac{F_{K^+}/F_{\pi^+}}{F_K/F_\pi} - 1\right]^{\chi pt} = -0.0022(6)$$

chiral perturbation theory formulae can be derived from known results

$n_f = 2+1$: **Gasser and Leutwyler Nucl. Phys. B250(1985)**
non unitary $n_f = 2$: **S.Sharpe Phys. Rev. D56(1997)**

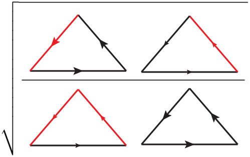
$$\frac{\delta F_K}{\Delta m_{ud}} = \frac{B_0}{2}\left\{\alpha_5 - \hat{B}_0 \frac{m_s + m_{ud}}{m_s - m_{ud}}\left[m_s \log(2\hat{B}_0 m_s) - m_{ud}\log(2\hat{B}_0 m_{ud})\right]\right\}$$

where α_i are low energy constants and $\hat{B}_0 = 2B_0/(4\pi F_0^2)$

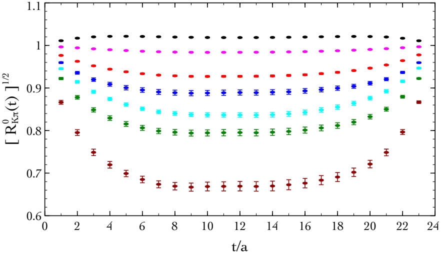
calculating $\delta f_+^{K\pi}(q^2)$

form factors parametrizing semileptonic decays can be calculated with good precision by considering **double ratios** of three point correlation functions

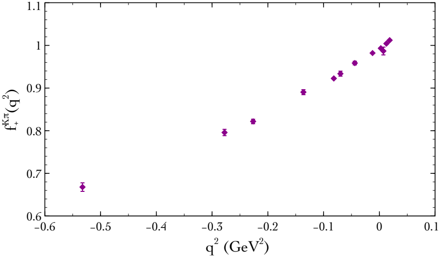
$$\frac{\langle \pi | V^\mu_{su} | K \rangle}{2\sqrt{E_\pi E_K}} =$$



$$=$$



and

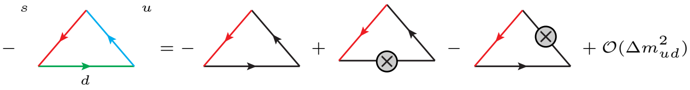


$$\langle \pi | V^0_{su} | K \rangle = (E_K + E_\pi) f_+^{K\pi} + (E_K - E_\pi) f_-^{K\pi}$$

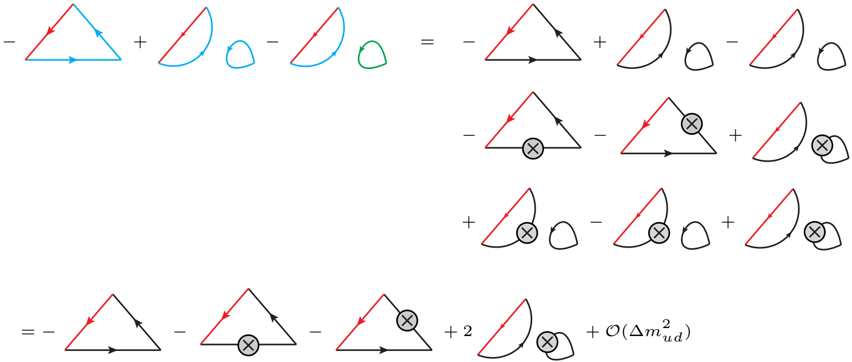
$$\langle \pi | \vec{V}_{su} | K \rangle = (\vec{p}_i + \vec{p}_f) f_+^{K\pi} + (\vec{p}_i - \vec{p}_f) f_-^{K\pi}$$

calculating $\delta f_+^{K\pi}(q^2)$

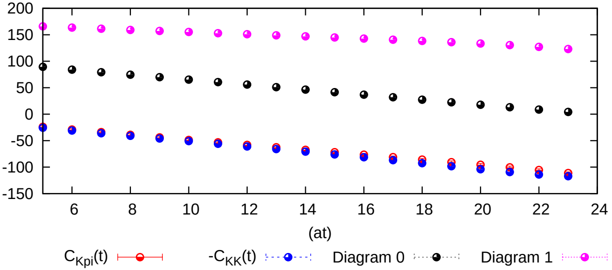
the diagrammatic expansion in the $K^0 \to \pi^- \ell \nu$ is



and is different, because of the **disconnected diagrams**, from the $K^+ \to \pi^0 \ell \nu$ case



calculating $\delta f_+^{K\pi}(q^2)$

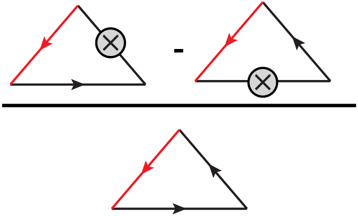


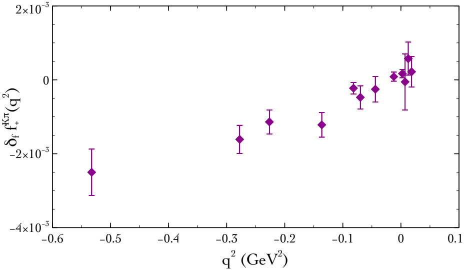
what do we expect from corrected three point correlation functions?

$$C^{\mu}_{K\pi}(t) = Z^{\mu}_{K\pi}\ e^{-E_K t}\ e^{-E_\pi (T-t)}$$

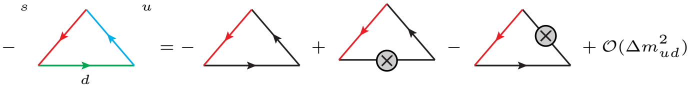
$$\Delta C^{\mu}_{K\pi}(t) = \left(\Delta Z^{\mu}_{K\pi} - Z^{\mu}_{K\pi}\Delta E_K t\right)\ e^{-E_K t}\ e^{-E_\pi (T-t)}$$

$$\delta C^{\mu}_{K\pi}(t) = \left(\delta \mathbf{Z}^{\mu}_{\mathbf{K}\pi} - \mathbf{\Delta E_K t}\right)$$





$$\left[\frac{f_+^{K^0\pi^-}(0) - f_+^{K\pi}(0)}{f_+^{K\pi}(0)}\right]^{QCD} = 0.85(18)(1) \times 10^{-4}$$



- in this work we have **not calculated disconnected diagrams**
- we can only show results for the $K^0 \to \pi^- \ell\nu$ case (above)
- this is a quantity that cannot be measured directly and the missing contribution, according to χpt, is expected to be much bigger
- the results given here make us confident on the possibility of completing the calculation by including disconnected diagrams

outlooks

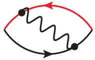
- first results obtained by applying our method look very promising
- the method is general and can be applied to many observables, even at second order: we plan to apply it to $M_{\pi^+} - M_{\pi^0}$
- we shall also refine our results in the case of nucleon masses and form factors
- first small steps toward the calculation of other observables that are relevant for phenomenological applications (long distance effects, etc.)
- U-spin corrections?

$$\langle \text{surface of the unitarity triangle} \rangle \propto (m_s^2 - m_d^2)\,(m_b^2 - m_d^2)\,(m_b^2 - m_s^2)$$

- we are computing QED effects by ourself . . .

non-compact QED on the lattice: our approach

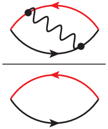
we include QED interactions perturbatively and calculate the photon propagator stochastically

$-$ 

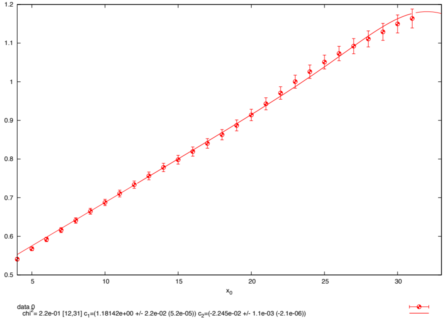
$$= -\frac{e_s e_u}{2} \sum_{x,y} D_{\mu\nu}(x-y)\; T\langle 0|\, \bar{s}(t)\gamma_5 u(t)\; V_s^{\mu}(x)\; V_u^{\nu}(y)\; \bar{u}(0)\gamma_5 s(0)\,|0\rangle$$

$$= -\frac{e_s e_u}{2} \sum_{x,y} \sum_{B} B_{\mu}(y) C_{\nu}(x)\; T\langle 0|\, \bar{s}(t)\gamma_5 u(t)\; V_s^{\mu}(x)\; V_u^{\nu}(y)\; \bar{u}(0)\gamma_5 s(0)\,|0\rangle$$

does it works?



$$\sim \frac{\partial}{\partial e^2} \left(\frac{G_K^2}{M_K} e^{-M_K} \right) =$$



well, from the numerical point of view it seems to work. the physics will come soon!!

QED corrections to hadronic matrix elements: non factorable contributions

- weak charged currents are, by definition, **not** invariant under QED gauge transformations
- gauge invariant observables are the decay rates
- electromagnetic interactions include long distance contributions that cannot be easily *separated* from factorable contributions because of QED gauge dependence and infrared divergences
- in the case of $K\ell 2$ decays, one cannot define a "decay constant" at $O(\alpha_{em})$