About fringe filed in quadrupole and dipole

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Work description and references

The work flow

1. Field expansion, vector potential.
2. Hamiltonian.
3. Calculation of tunes, chromaticities in first order of perturbation theory in action-phase variables.

References

1. Quadrupole fringe: I followed "High Energy Beam Optics" by Klaus G. Steffen
2. Dipole fringe: K. L. Brown and R. Servranckx, “First And Second Order Charged Particle Optics,”AIP Conf. Proc. **127**, 62 (1985).

Quadrupole field expansion up to 5th order

$$B_x(s) = Gy - \frac{G''}{12}(3x^2y + y^3) + O(5)\,, \quad (1)$$

$$B_y(s) = Gx - \frac{G''}{12}(3xy^2 + x^3) + O(5)\,, \quad (2)$$

$$B_s(s) = G'xy - \frac{G'''}{12}(xy^3 + x^3y) + O(6)\,. \quad (3)$$

$$A_x(s) = \frac{G'xy^2}{4} - \frac{G'''}{12}\left(\frac{y^2x^3}{4} + \frac{xy^4}{8}\right) + O(7)\,, \quad (4)$$

$$A_y(s) = -\frac{G'x^2y}{4} + \frac{G'''}{12}\left(\frac{yx^4}{8} + \frac{x^2y^3}{4}\right) + O(7)\,, \quad (5)$$

$$A_s(s) = \frac{G(x^2 - y^2)}{2} - \frac{G''}{12}\frac{(x^4 - y^4)}{4} + O(6)\,. \quad (6)$$

Quadrupole field expansion: Hamiltonian

$$H^{**} = -\sqrt{1 - \left(P_x - \frac{e}{pc}A_x\right)^2 - \left(P_y - \frac{e}{pc}A_y\right)^2} - \frac{e}{pc}A_s\,, \qquad (7)$$

$P_{x,y} = \frac{p_{x,y}}{p}$, $p = p_0(1+\delta)$, p_0 — design momentum, $\frac{eG}{p_0 c} = K_1$.

$$\begin{aligned} H^{**} &= -1 + \frac{P_x^2}{2} + \frac{P_y^2}{2} + \frac{(P_x^2+P_y^2)^2}{8} + \frac{P_xA_x}{B\rho} + \frac{P_yA_y}{B\rho} + \frac{A_s}{B\rho} = \\ &= -1 + \frac{P_x^2}{2} + \frac{P_y^2}{2} + \frac{(P_x^2+P_y^2)^2}{8} + \frac{K_1}{1+\delta}\frac{(x^2-y^2)}{2} + \\ &+ \frac{K_1'}{1+\delta}\frac{(P_xxy^2 - P_yx^2y)}{4} - \frac{K_1''}{1+\delta}\frac{(x^4-y^4)}{48} + \\ &+ \frac{K_3}{1+\delta}\frac{(x^4-6x^2y^2+y^4)}{24} + O(5)\,, \end{aligned} \qquad (8)$$

Quadrupole field expansion: First order perturbation

$$x = \sqrt{2J_x\beta_x}\cos(\psi_x)\,,$$
$$y = \sqrt{2J_y\beta_y}\cos(\psi_y)\,. \tag{9}$$

Hamiltonian averaged over ψ_x and ψ_y

$$\langle H\rangle_{\psi_x,\psi_y} = -1 + \frac{J_x}{\beta_x} + \frac{J_y}{\beta_y} + \frac{K_1'}{4}J_xJ_y(\alpha_y\beta_x - \alpha_x\beta_y) + \frac{K_1''}{32}(J_y^2\beta_y^2 - J_x^2\beta_x^2) +$$
$$+ \frac{3}{16}\frac{J_y^2}{\beta_y^2}(1+\alpha_y^2)^2 + \frac{3}{16}\frac{J_x^2}{\beta_x^2}(1+\alpha_x^2)^2 + \frac{1}{4}\frac{J_xJ_y}{\beta_x\beta_y}(1+\alpha_x^2)(1+\alpha_y^2) +$$
$$+ \frac{K_3}{16}(J_x^2\beta_x^2 - 4J_xJ_y\beta_x\beta_y + J_y^2\beta_y^2)\,. \tag{10}$$

Quadrupole field expansion: First order perturbation

$$\Delta\nu_x = \frac{1}{2\pi}\oint \frac{\partial}{\partial J_x}\langle H\rangle_{\psi_x,\psi_y} ds = \alpha_{xx}J_x + \alpha_{xy}J_y\,, \tag{11}$$

$$\Delta\nu_y = \frac{1}{2\pi}\oint \frac{\partial}{\partial J_y}\langle H\rangle_{\psi_x,\psi_y} ds = \alpha_{yy}J_y + \alpha_{yx}J_x\,, \tag{12}$$

$$\begin{aligned}
\alpha_{xx} &= \frac{3}{16\pi}\oint \frac{(1+\alpha_x^2)^2}{\beta_x^2} ds - \frac{1}{32\pi}\oint K_1''\beta_x^2 ds + \frac{1}{16\pi}\oint K_3\beta_x^2 ds\,,\\
\alpha_{xy} &= \frac{1}{8\pi}\oint \frac{(1+\alpha_x^2)(1+\alpha_y^2)}{\beta_x\beta_y} ds + \frac{1}{8\pi}\oint K_1'(\alpha_y\beta_x - \alpha_x\beta_y) ds - \\
&\quad - \frac{1}{8\pi}\oint K_3\beta_x\beta_y ds\,,\\
\alpha_{yx} &= \alpha_{xy}\,,\\
\alpha_{yy} &= \frac{3}{16\pi}\oint \frac{(1+\alpha_y^2)^2}{\beta_y^2} ds + \frac{1}{32\pi}\oint K_1''\beta_y^2 ds + \frac{1}{16\pi}\oint K_3\beta_y^2 ds\,.
\end{aligned} \tag{13}$$

Quadrupole field expansion: Chromaticity

Hamiltonian expended up to 1st order of δ averaged over $\psi_{x,y}$ (assuming that there is no dispersion)

$$\langle H \rangle_{\psi_x,\psi_y} = -\delta \left[\frac{K_1}{2}(J_x\beta_x - J_y\beta_y) + \frac{K_1'}{4} J_x J_y (\alpha_y\beta_x - \alpha_x\beta_y) + \right. \\ \left. + \frac{K_1''}{32}(J_y^2\beta_y^2 - J_x^2\beta_x^2) + \frac{K_3}{16}(J_x^2\beta_x^2 - 4J_xJ_y\beta_x\beta_y + J_y^2\beta_y^2) \right] . \tag{14}$$

Quadrupole field expansion: Chromaticity

$$\frac{\partial \Delta\nu_x}{\partial \delta} = \frac{1}{2\pi}\oint \frac{\partial^2}{\partial J_x \partial \delta}\langle H\rangle_{\psi_x,\psi_y} ds = \xi_x + \xi_{xx} J_x + \xi_{xy} J_y\,, \tag{15}$$

$$\frac{\partial \Delta\nu_y}{\partial \delta} = \frac{1}{2\pi}\oint \frac{\partial^2}{\partial J_y \partial \delta}\langle H\rangle_{\psi_x,\psi_y} ds = \xi_y + \xi_{yy} J_y + \xi_{yx} J_x\,, \tag{16}$$

$$\xi_x = -\frac{1}{4\pi}\oint K_1 \beta_x ds, \qquad \xi_y = \frac{1}{4\pi}\oint K_1 \beta_y ds, \tag{17}$$

$$\xi_{xx} = \frac{1}{32\pi}\oint K_1'' \beta_x^2 ds - \frac{1}{16\pi}\oint K_3 \beta_x^2 ds\,, \tag{18}$$

$$\xi_{xy} = -\frac{1}{8\pi}\oint K_1'(\alpha_y\beta_x - \alpha_x\beta_y) ds + \frac{1}{8\pi}\oint K_3 \beta_x\beta_y ds\,, \tag{19}$$

$$\xi_{yx} = \xi_{xy}\,, \tag{20}$$

$$\xi_{yy} = -\frac{1}{32\pi}\oint K_1'' \beta_y^2 ds - \frac{1}{16\pi}\oint K_3 \beta_y^2 ds\,. \tag{21}$$

Quadrupole field expansion: Estimations for MQXC.1L5, 150T/m

The following optics of LHC was used "opt_150_0100_0100.madx".

MQXC.1L5 is FF quadrupole: $K_1 = -6.4 \times 10^{-3}\, m^{-2}$, $L = 7.685\, m$, $G = 149\, T/m$, from the IP at $s_0 = 23\, m$. At the IP $\beta_{x,0} = \beta_{y,0} = 0.1\, m$, $\varepsilon_x = \varepsilon_y = 5 \times 10^{-10}\, m \times rad$, action corresponding to amplitude of one sigma is $J_x = \varepsilon_x/2 = 2.5 \times 10^{-10}\, m$.

Quadrupole field expansion: Estimations for MQXC.1L5, 150T/m

Kinematic term in the quadrupole

$$\alpha_{xx,kin,qd} = \frac{3}{16\pi}\oint\frac{(1+\alpha_x^2)^2}{\beta_x^2}ds \approx \frac{3}{16\pi}\frac{L}{\beta_{x,0}^2} \approx 4.6\times 10^1 m^{-1}, \quad (22)$$

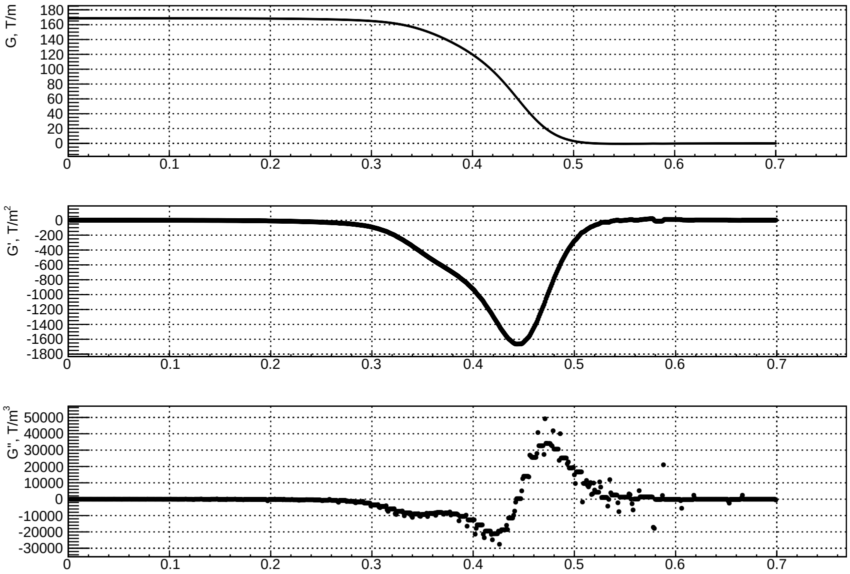
$$\alpha_{yy,kin,qd} = \frac{3}{16\pi}\oint\frac{(1+\alpha_y^2)^2}{\beta_y^2}ds \approx \frac{3}{16\pi}\frac{L}{\beta_{y,0}^2} \approx 4.6\times 10^1 m^{-1}. \quad (23)$$

Kinematic term in the drift from IP to quadrupole

$$\alpha_{xx,kin,dr} = \frac{3}{16\pi}\oint\frac{(1+\alpha_x^2)^2}{\beta_x^2}ds = \frac{3}{16\pi}\frac{s_0}{\beta_{x,0}^2} \approx 1.4\times 10^2 m^{-1}, \quad (24)$$

$$\alpha_{yy,kin,dr} = \frac{3}{16\pi}\oint\frac{(1+\alpha_y^2)^2}{\beta_y^2}ds = \frac{3}{16\pi}\frac{s_0}{\beta_{y,0}^2} \approx 1.4\times 10^2 m^{-1}. \quad (25)$$

Quadrupole field expansion: measurements



Quadrupole field expansion: Estimations for FF quadrupoles

Name	α_{xx}	$\alpha_{xy} = \alpha_{yx}$	α_{yy}
MQXC.3L5	$3.7 \cdot 10^3$	$8.8 \cdot 10^3$	$6.7 \cdot 10^3$
MQXC.B2L5	$6.3 \cdot 10^3$	$-10 \cdot 10^3$	$3.1 \cdot 10^3$
MQXC.A2L5	$4.1 \cdot 10^3$	$7 \cdot 10^2$	$5.6 \cdot 10^2$
MQXC.1L5	$2.9 \cdot 10^3$	$-1.7 \cdot 10^3$	$4 \cdot 10^2$
MQXC.1R5	$4 \cdot 10^2$	$1.7 \cdot 10^3$	$2.9 \cdot 10^3$
MQXC.A2R5	$5 \cdot 10^2$	$7 \cdot 10^2$	$4.1 \cdot 10^3$
MQXC.B2R5	$3.1 \cdot 10^3$	$10 \cdot 10^3$	$6.3 \cdot 10^3$
MQXC.3R5	$6.7 \cdot 10^3$	$-8.6 \cdot 10^3$	$3.7 \cdot 10^3$
Total	$2.8 \cdot 10^4$	$1.6 \cdot 10^3$	$2.8 \cdot 10^4$

Quadrupole field expansion: Estimations for FF quadrupoles

Name	ξ_{xx}	$\xi_{xy} = \xi_{yx}$	ξ_{yy}
MQXC.3L5	$-3.6 \cdot 10^3$	$-8.7 \cdot 10^3$	$-6.5 \cdot 10^3$
MQXC.B2L5	$-6.2 \cdot 10^3$	$10 \cdot 10^3$	$-2.8 \cdot 10^3$
MQXC.A2L5	$-3.7 \cdot 10^3$	$-7 \cdot 10^2$	$-5.6 \cdot 10^2$
MQXC.1L5	$-2.5 \cdot 10^3$	$1.7 \cdot 10^3$	$-4 \cdot 10^2$
MQXC.1R5	$-4 \cdot 10^2$	$-1.7 \cdot 10^3$	$-2.5 \cdot 10^3$
MQXC.A2R5	$-5 \cdot 10^2$	$-7 \cdot 10^2$	$-3.8 \cdot 10^3$
MQXC.B2R5	$-2.8 \cdot 10^3$	$-10 \cdot 10^3$	$-6.2 \cdot 10^3$
MQXC.3R5	$-6.5 \cdot 10^3$	$8.7 \cdot 10^3$	$-3.6 \cdot 10^3$
Total	$-2.6 \cdot 10^4$	$-1.4 \cdot 10^3$	$-3.1 \cdot 10^4$

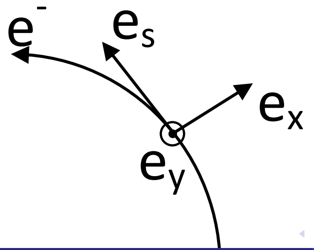
Dipole field expansion: coordinates

Coordinates: $u^i = \{x, s, y\}$.

The basis: $\vec{a_1} = \vec{e_x}$, $\vec{a_2} = (1 + Kx)\vec{e_s}$, $\vec{a_3} = \vec{e_y}$, $K = -\dfrac{eB_0}{p_0 c}$.

Covariant: $B_1 = B_x$, $B_2 = (1 + Kx)B_s$, $B_3 = B_y$.

Contravariant: $B^1 = B_x$, $B^2 = \dfrac{B_s}{(1 + Kx)}$, $B^3 = B_y$.



Dipole field expansion: vector potential

$$
\begin{aligned}
A_1(s) &= B_0'\frac{y^2}{4} - KB_0'\frac{xy^2}{4} + K^2B_0'\frac{x^2y^2}{4} - B_0'''\frac{y^4}{48} + O(5)\,,\\
A_2(s) &= B_0x + KB_0\frac{x^2}{2} - B_0''\frac{xy^2}{4} + (K'B_0' + KB_0'')\frac{x^2y^2}{8} - \\
&\quad - (2KB_0'' + K'B_0')\frac{y^4}{24} + O(5)\,,\\
A_3(s) &= -B_0'\frac{xy}{2} + KB_0'\frac{x^2y}{4} - K^2B_0'\frac{x^3y}{6} + B_0'''\frac{xy^3}{12} + O(5)\,.
\end{aligned}
\tag{26}
$$

Dipole field expansion: field

$$
\begin{aligned}
B_1(s) &= B_x = \frac{1}{1+Kx}\left(\frac{\partial A_3}{\partial s} - \frac{\partial A_2}{\partial y}\right) = (K'B_0' + 2KB_0'')\frac{y^3}{6} + O(4)\,,\\
B_2(s) &= (1+Kx)B_s = (1+Kx)\left(\frac{\partial A_1}{\partial y} - \frac{\partial A_3}{\partial x}\right) = B_0'y - B_0'''\frac{y^3}{6} + O(4)\,,\\
B_3(s) &= B_y = \frac{1}{1+Kx}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial s}\right) =\\
&= B_0 - B_0''\frac{y^2}{2} + (K'B_0' + 2KB_0'')\frac{xy^2}{2} + O(4)\,.
\end{aligned} \tag{27}
$$

Dipole field expansion: Hamiltonian

$$H^{**} = -\frac{e}{p_0 c}\frac{A_2}{1+\delta} - \\ -(1+Kx)\left[1-\left(P_x - \frac{e}{p_0 c}\frac{A_1}{1+\delta}\right)^2 - \left(P_y - \frac{e}{p_0 c}\frac{A_3}{1+\delta}\right)^2\right]^{\frac{1}{2}}, \tag{28}$$

$$H^{**} = -1 + K^2\frac{x^2}{2} + \frac{P_x^2}{2} + \frac{P_y^2}{2} + \frac{(P_x^2+P_y^2)^2}{8} + \\ + Kx\frac{P_x^2+P_y^2}{2} + K'\frac{y^2 P_x}{4} - K'\frac{xyP_y}{2} - K''\frac{xy^2}{4} - \\ -(K'^2 + 8KK'')\frac{y^4}{96} - K'K\frac{x^2 yP_y}{4} + (2K'^2 + KK'')\frac{x^2y^2}{8} + \\ + \delta\left(-Kx - K^2\frac{x^2}{2} - K'\frac{y^2 P_x}{4} + K'\frac{xyP_y}{2} + K''\frac{xy^2}{4}\right) + O(5), \tag{29}$$

Conclusion

- Nonlinear tune shifts from the quadrupole fringe are estimated in the first order of perturbation theory.
- If one considers final focus quadrupole fringe he has to pay attention to the kinematic term in vicinity of the IP.
- Dipole fringe effects are proportional K and its derivatives, most likely very small.