Point Massive Particle in General Relativity

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Notations

$\mathbb{M} \approx \mathbb{R}^4$ - topologically trivial manifold (space-time)

$x^\alpha, \quad \alpha = 0,1,2,3$ - Cartesian coordinates

$g_{\alpha\beta}(x), \quad \operatorname{sign} g_{\alpha\beta} = (+---)$ - Lorentzian signature metric

$q^\alpha(\tau)$ - worldline of a point particle $\qquad \tau$ - parameter along worldline

The action

$$S = \frac{1}{16\pi} \int dx \sqrt{|g|}\, R - M \int d\tau \sqrt{\dot{q}^\alpha \dot{q}^\beta g_{\alpha\beta}}$$

$R(g)$ - scalar curvature

M - mass of a point particle

$\dot{q}^\alpha := \dfrac{dq^\alpha}{d\tau}$ - velocity of a particle

Equations of motion

Einstein's equations: $R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = -\frac{1}{2} T^{\alpha\beta}$

(*)

Geodesic equations: $\left(\ddot{q}^{\alpha} + \Gamma_{\beta\gamma}{}^{\alpha}\Big|_{x=q} \dot{q}^{\beta} \dot{q}^{\gamma} \right) g_{\alpha\delta} = 0$

where $T^{\alpha\beta} = \frac{16\pi M \dot{q}^{\alpha} \dot{q}^{\beta}}{\sqrt{|g|}\, \dot{q}^{0}} \delta(\boldsymbol{x} - \boldsymbol{q})$ - energy-momentum tensor

$\delta(\boldsymbol{x} - \boldsymbol{q}) := \delta(x^1 - q^1)\delta(x^2 - q^2)\delta(x^3 - q^3)$ - three-dimensional δ-function

$R_{\alpha\beta}$ - Ricci tensor

$\Gamma_{\beta\gamma}{}^{\alpha}$ - Christoffel's symbols

The Schwarzschild solution does not satisfy equations of motion (*)

$$ds^2 = \left(1 - \frac{2M}{\rho}\right) dt^2 - \frac{d\rho^2}{\left(1 - \frac{2M}{\rho}\right)} - \rho^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right)$$

Where is the δ-function ?

Point mass in General Relativity

$$S=\frac{1}{16\pi}\int dx\sqrt{|g|}\,R-M\int d\tau\sqrt{\dot{q}^{\alpha}\dot{q}^{\beta}g_{\alpha\beta}}$$

Canonical formulation

$$g_{\alpha\beta}=\begin{pmatrix} N^2+N^{\rho}N_{\rho} & N_{\nu} \\ N_{\mu} & g_{\mu\nu} \end{pmatrix}$$ - ADM parameterization of the metric

$$\alpha=(0,\mu),\qquad \mu=1,2,2$$

N - lapse function

N_{μ} - shift functions

$$N^{\rho}:=\hat{g}^{\rho\sigma}N_{\sigma}\qquad\qquad \hat{g}^{\mu\nu}g_{\nu\rho}=\delta^{\mu}_{\rho}$$

$(g_{\mu\nu},\ p^{\mu\nu}),\ (q^{\mu},\ p_{\mu})$ - canonically conjugate pairs of dynamical variables

N, N^{μ} - Lagrange multipliers

Time gauge: $\tau=q^0=x^0:=t$

Notations: $p:=p^{\mu\nu}g_{\mu\nu},\ \hat{p}^2:=\hat{g}^{\mu\nu}p_{\mu}p_{\nu}$

$\hat{\nabla}_{\mu},\ \hat{\Delta}:=\hat{\nabla}^{\mu}\hat{\nabla}_{\mu}$ - three-dimensional covariant derivative and Laplace-Beltrami operator

Hamiltonian equations of motion

Constraints:

$$H_\perp = \frac{1}{\hat{e}}(p^{\mu\nu} p_{\mu\nu} - p^2) - \hat{e}\hat{R} + \sqrt{M^2 + \hat{p}^2}\ \delta(\mathbf{x} - \mathbf{q}) = 0$$

$$H_\mu = -2\hat{\nabla}^\nu p_{\nu\mu} - p_\mu \delta(\mathbf{x} - \mathbf{q}) = 0$$

$$\hat{e} := \sqrt{|\det g_{\mu\nu}|}$$

Dynamical equations:

$$\dot{g}_{\mu\nu} = \frac{2N}{\hat{e}} p_{\mu\nu} - \frac{N}{\hat{e}} g_{\mu\nu} + \hat{\nabla}_\mu N_\nu + \hat{\nabla}_\nu N_\mu$$

$$\dot{p}^{\mu\nu} = \frac{N}{2\hat{e}} \hat{g}^{\mu\nu} \left(p^{\rho\sigma} p_{\rho\sigma} - \frac{1}{2} p^2 \right) - \frac{2N}{\hat{e}} \left(p^{\mu\rho} p^\nu{}_\rho - \frac{1}{2} p^{\mu\nu} p \right) +$$

$$+\hat{e}(\hat{\Delta} N \hat{g}^{\mu\nu} - \hat{\nabla}^\mu \hat{\nabla}^\nu N) - \hat{e}N \left(\hat{R}^{\mu\nu} - \frac{1}{2} \hat{g}^{\mu\nu} \hat{R} \right) - p^{\mu\rho} \hat{\nabla}_\rho N^\nu - p^{\mu\rho} \hat{\nabla}_\rho N^\nu +$$

$$+\hat{\nabla}_\rho (N^\rho p^{\mu\nu}) - \frac{N p^\mu p^\nu}{2\sqrt{M^2 + \hat{p}^2}} \delta(\mathbf{x} - \mathbf{q})$$

Geodesics:

$$\dot{q}^\mu = -\left.\frac{N}{\sqrt{M^2 + \hat{p}^2}}\right|_{\mathbf{x}=\mathbf{q}} p^\mu - N^\mu \Big|_{\mathbf{x}=\mathbf{q}}$$

$$\dot{p}_\mu = -\partial_\mu \left[N\sqrt{M^2 + \hat{p}^2} - N^\nu p_\nu \right]_{\mathbf{x}=\mathbf{q}}$$

Solution of the equations of motion

Spherical coordinate system t, r, ϑ, φ with particle located at the origin $\boldsymbol{q} = 0$

Staticity: $g_{\alpha\beta} = g_{\alpha\beta}(\boldsymbol{x}), \quad p^{\mu\nu} = 0, \quad p_\mu = 0$

Spherical symmetry: $g_{\mu\nu} = g_{\mu\nu}(r) \quad N = N(r) \quad N_\mu = 0$

Equations of motion:

$$-\hat{e}\hat{R} + 16\pi M \delta(\boldsymbol{x}) = 0$$

$$\hat{e}(\hat{\Delta} N \hat{g}^{\mu\nu} - \hat{\nabla}_\mu \hat{\nabla}_\nu N) - \hat{e}N\left(\hat{R}^{\mu\nu} - \frac{1}{2}\hat{g}^{\mu\nu}\hat{R}\right) = 0$$

$$\partial_\mu N\Big|_{\boldsymbol{x}=0} = 0$$

The main equation

$$\hat{e}\hat{R} = 16\pi M\, \delta(\boldsymbol{x})$$

$$\hat{e} := \sqrt{|\det g_{\mu\nu}|}$$

$\hat{R}$ - three-dimensional scalar curvature

$\delta(\boldsymbol{x}) := \delta(x^1)\delta(x^2)\delta(x^3)$ - three-dimensional δ-function

We are seeking spherically symmetric solution $g_{\mu\nu} = -f^2\delta_{\mu\nu}$ - anzatz

$\delta_{\mu\nu} = \text{diag}\,(+++)$ - Euclidean metric

$$\Delta f - \frac{\partial f^2}{2f} = -4\pi M\,\delta(\boldsymbol{x})$$

$\Delta := \partial_1^2 + \partial_2^2 + \partial_3^2$ - flat space Laplacian

$\partial f^2 := \delta^{\mu\nu}\partial_\mu f\,\partial_\nu f$ - notation

$\mathcal{D}(\mathbb{R}^3)$ - test functions (smooth functions with compact support)

$\mathcal{D}'(\mathbb{R}^3)$ - space of generalized functions (distributions)

Theorem. Function

$$f = 1 + \frac{M}{r} + \frac{M^2}{4r^2} = \left(1 + \frac{M}{2r}\right)^2$$

satisfies equation

$$\Delta f - \frac{\partial f^2}{2f} = -4\pi M \delta(\boldsymbol{x}) \qquad (*)$$

which solution is understood in a generalized sense after integration with a test function.

Proof. $f \in \mathcal{D}'(\mathbb{R}^3)$ - locally integrable function

$(r^\lambda, \varphi) := \int_{\mathbb{R}^3} dx\, r^\lambda \varphi$ - analytic functional $\lambda \in \mathbb{C}$ except poles at $\lambda = -3, -5, -7, \dots$

Gel'fand, Shilov. Vol.1 (1958)

$$\frac{\partial f^2}{2f} = \frac{M^2}{2r^4} \qquad (*) \Longrightarrow \Delta\left(1 + \frac{M}{r} + \frac{M^2}{4r^2}\right) - \frac{M^2}{2r^4} = -4\pi M \delta(\boldsymbol{x})$$

$$\Delta\left(1 + \frac{M}{r}\right) = \Delta \frac{M}{r} = -4\pi M \delta(\boldsymbol{x})$$ - fundamental solution

$$\Delta \frac{M^2}{4r^2} - \frac{M^2}{2r^4} = 0$$

$\Delta r^{\mu+2} = (\mu+2)(\mu+3) r^\mu, \quad \mu \in \mathbb{C}, \ \mathrm{re}\,\mu > 0$ analytic continuation to $\mu = -4$

Theorem. Let three-dimensional metric be $g_{\mu\nu} = -\left(1+\frac{M}{2r}\right)^2 \delta_{\mu\nu}$

Then the lapse function $N = \dfrac{1-\frac{M}{2r}}{1+\frac{M}{2r}}$ satisfy equation

$$\hat{e}(\hat{\Delta} N \hat{g}^{\mu\nu} - \hat{\nabla}^{\mu}\hat{\nabla}^{\nu} N) - \hat{e} N \left(\hat{R}^{\mu\nu} - \frac{1}{2}\hat{g}^{\mu\nu}\hat{R} \right) = 0$$

in a generalized sense.

$$N(r) > 0, \qquad \frac{M}{2} < r < \infty$$

$$N(r) = 0, \qquad r = \frac{M}{2}$$

$$N(r) < 0, \qquad 0 < r < \frac{M}{2}$$

$$N(0) = -1.$$

$N(r)$ - locally integrable function

$N \in \mathcal{D}'(\mathbb{R}^3)$

Geodesic equations

$$\partial_r N = \frac{M}{r^2}\frac{1}{\left(1+\frac{M}{2r}\right)^2} \neq 0$$ - geodesic equation is not fulfilled

$$\left\langle \frac{M}{r^3}\frac{x_\mu}{\left(1+\frac{M}{2r}\right)^2} \right\rangle = 0$$ - after averaging over sphere

Newton's mechanics

$$m\vec{a} = \vec{F} \qquad F_\mu = \partial_\mu \frac{1}{r} = \frac{x_\mu}{r^3}$$ - diverges at $r \to 0$

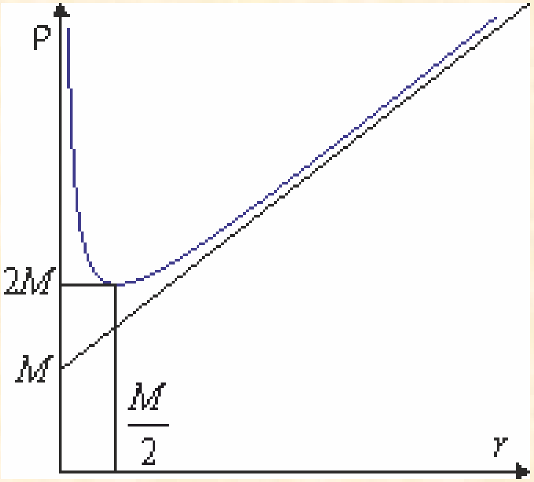
Equations of motions are not fulfilled

$$-\frac{x_\mu}{r^3} = 0$$ - after averaging over sphere

Relation to the Schwarzschild solution

$$ds^2 = \left(1 - \frac{2M}{\rho}\right) dt^2 - \frac{d\rho^2}{1 - \frac{2M}{\rho}} - \rho^2 \left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$

- the Schwarzschild metric in Schwarzschild coordinates, white and black holes, no point particle



$$\rho = r\left(1 + \frac{M}{2r}\right)^2$$

- change of radial coordinate

$$ds^2 = \left(\frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}}\right)^2 dt^2 - \left(1 + \frac{M}{2r}\right)^4 \left[dr^2 + (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\right]$$

- the Schwarzschild metric in isotropic coordinates, point particle, no black hole

Schwarzschild metric in isotropic coordinates

$$ds^2 = \left(\frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}} \right)^2 dt^2 - \left(1 + \frac{M}{2r} \right)^4 \left[dr^2 + (d\vartheta^2 + \sin^2\vartheta \, d\varphi^2) \right]$$

$$-\infty < t < \infty, \quad 0 < r < \infty, \quad 0 < \vartheta < \pi, \quad 0 < \varphi < 2\pi$$

$\mathbb{M} \approx \mathbb{R}^4$ - topologically trivial space-time

Asymptotic flatness at $r \to \infty$

All componentrs are smooth for $0 < r < \infty$

$$\det g_{\alpha\beta} = -\left(1 - \frac{M}{2r} \right)^2 \left(1 + \frac{M}{2r} \right)^{10} r^4 \sin^2\vartheta$$

Metric is degenerate on the sphere $r_* = \frac{M}{2} \quad \Leftrightarrow \quad \rho_s = 2M$

↑ Schwarzschild radius (horizon)

Asymptotic at large distances: $g_{00} \simeq 1 - \frac{2M}{r} \quad r \to \infty$

$\Longrightarrow$ Newton's law

Geodesics (extremals) $x^{\alpha}(\tau)$

Geodesic equations:

$$\ddot{x}^{\alpha} + \Gamma_{\beta\gamma}{}^{\alpha}\dot{x}^{\beta}\dot{x}^{\gamma} = 0$$

$$\ddot{t} = -\frac{2}{\left(1+\frac{M}{2r}\right)\left(1-\frac{M}{2r}\right)}\frac{M}{r^2}\dot{t}\dot{r}$$

$$\ddot{r} = -\frac{\left(1-\frac{M}{2r}\right)}{\left(1+\frac{M}{2r}\right)^7}\frac{M}{r^2}\dot{t}^2 + \frac{1}{\left(1+\frac{M}{2r}\right)}\frac{M}{r^2}\dot{r}^2 + \frac{\left(1-\frac{M}{2r}\right)}{\left(1+\frac{M}{2r}\right)}r(\dot{\vartheta}^2 + \sin^2\vartheta\,\dot{\varphi}^2)$$

$$\ddot{\vartheta} = -\frac{\left(1-\frac{M}{2r}\right)}{\left(1+\frac{M}{2r}\right)}\frac{2}{r}\dot{r}\dot{\vartheta} + \sin\vartheta\cos\vartheta\,\dot{\varphi}^2$$

$$\ddot{\varphi} = -\frac{\left(1-\frac{M}{2r}\right)}{\left(1+\frac{M}{2r}\right)}\frac{2}{r}\dot{r}\dot{\varphi} - 2\frac{\cos\vartheta}{\sin\vartheta}\,\dot{\vartheta}\dot{\varphi}$$

Integrals of motion

$$C_0 = \dot{x}^\alpha \dot{x}^\beta g_{\alpha\beta} = \left(\frac{1-\frac{M}{2r}}{1+\frac{M}{2r}} \right)^2 \dot{t}^2 - \left(1+\frac{M}{2r} \right)^4 \left[\dot{r}^2 + r^2 (\dot{\vartheta}^2 + \sin^2\vartheta \dot{\varphi}^2) \right]$$

$$C_1 = K_1^\alpha x^\beta g_{\alpha\beta} = \left(\frac{1-\frac{M}{2r}}{1+\frac{M}{2r}} \right) \dot{t}$$

$$C_2 = K_2^\alpha \dot{x}^\beta g_{\alpha\beta} = -\left(1+\frac{M}{2r} \right)^4 r^2 \sin^2\vartheta \dot{\varphi}$$

$K_1 = \partial_0 \qquad K_2 = \partial_\varphi$ - Killing vector fields

Angular momentum is conserved ⟹ Motion in the plane $\vartheta = \frac{\pi}{2}$

Freedom in a choice of canonical parameter: $C_1 = 1$ ⟹ At large distances $\tau = t$

Newtonian limit

For $\tau = t$ at large distances and small velocities $\frac{M}{r} \sim \varepsilon, \quad \dot{r} \sim \varepsilon, \quad \varepsilon \ll 1$

Geodesic equations: $\ddot{r} = -\frac{M}{r^2} + r(\dot{\vartheta}^2 + \sin^2\vartheta\,\dot{\varphi}^2)$

$$\ddot{\vartheta} = -\frac{2\dot{r}\dot{\vartheta}}{r} + \sin\vartheta\,\cos\vartheta\,\dot{\varphi}^2$$

$$\ddot{\varphi} = -\frac{2\dot{r}\dot{\varphi}}{r} - 2\frac{\cos\vartheta}{\sin\vartheta}\dot{\vartheta}\dot{\varphi}$$

Action for Newton's particle

$$S = \int dt\,L := \int dt\left(\frac{v^2}{2} + \frac{M}{r}\right) = \int dt\left(\frac{\dot{r}^2 + r^2(\dot{\vartheta}^2 + \sin^2\vartheta\,\dot{\varphi}^2)}{2} + \frac{M}{r}\right)$$

$$E = \frac{v^2}{2} - \frac{M}{r} = \frac{\dot{r}^2 + r^2(\dot{\vartheta}^2 + \sin^2\vartheta\dot{\varphi}^2)}{2} = \text{const}$$ - energy ⟸ C_0

$$L_z = -\frac{\partial L}{\partial\dot{\varphi}} = -r^2\sin^2\vartheta\,\dot{\varphi} = \text{const}$$ - angular momentum ⟸ C_2

Circular geodesics

$$r = \text{const}, \quad \tau = \text{t}$$

$$\dot{\varphi}^2 = \frac{1}{\left(1+\frac{M}{2r}\right)^6}\frac{M}{r^3} = \text{const} \quad \Rightarrow \quad \dot{\varphi}^2 = \frac{M}{r^3} = \text{const}$$ - Newton's limit

$$\lim_{r\to 0}\dot{\varphi}^2 = 0$$

Lightlike radial geodesics

$$C_0 = 0 \qquad \dot{\varphi} = 0 \qquad \pm\tau + \text{const} = r + \frac{M^2}{4r}$$

The points $r = 0$ and $r = \infty$ are complete

$$\ddot{r} = -\frac{1}{\left(1-\frac{M^2}{4r^2}\right)^3}\frac{M^2}{2r^3}$$

Attraction $r_* < r < \infty$ Repulsion $0 < r < r_*$

The Einstein-Rosen Bridge

Coordinate transformation $\rho \to u:$ $\qquad \frac{1}{2}u^2 = \rho - 2M$

$$ds^2 = \frac{u^2}{u^2 + 4M}dt^2 - (u^2 + 4M)du^2 - \frac{1}{4}(u^2 + 4M)(d\vartheta^2 + \sin^2 d\varphi^2)$$

Two copies of external Schwarzschild solution are mapped onto $u > 0$ and $u < 0$

The metric is degenerate at $u = 0$

Transformation to isotropic coordinates: $\qquad u = \sqrt{2r}\left(1 - \frac{M}{2r}\right)$

$$u > 0 \quad \Leftrightarrow \quad \frac{M}{2} < r < \infty$$

$$u < 0 \quad \Leftrightarrow \quad 0 < r < \frac{M}{2}$$

The Schwarzschild metric in isotropic coordinates
is globally isometric to Einstein-Rosen bridge

Conclusion

This is the gravitational field around massive point particle of mass M

$$ds^2 = \left(\frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}} \right)^2 dt^2 - \left(1 + \frac{M}{2r} \right)^4 \left[dr^2 + (d\vartheta^2 + \sin^2\vartheta \, d\varphi^2) \right]$$

$$-\infty < t < \infty, \quad 0 < r < \infty, \quad 0 < \vartheta < \pi, \quad 0 < \varphi < 2\pi$$

$\mathbb{M} \approx \mathbb{R}^4$ - topologically trivial and asymptotically flat

The space-time is geodesically complete at $r \to 0, \ \infty$

and incomplete at $r \to \frac{M}{2}$

$r > \frac{M}{2}$ - attraction $\qquad 0 < r < \frac{M}{2}$ - repulsion