QUANTUM COSMOLOGY WILL NEED TO BECOME A NUMERICAL SUBJECT

Dr. Edward Anderson
DAMTP CAMBRIDGE

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Halliwell – Hawking Approach

- $\underset{\text{3-metric}}{h_{ij}} = a^2 (\underset{\text{on } S^3}{\Omega_{ij}} + \epsilon_{ij})$ $\qquad \Phi = \sigma^{-1}\left(\phi(t) + \sum_n f_n Q^n\right)$

$$\epsilon_{ij} = \sum_n a_n \Omega_{ij} Q^n + b_n P_{ij}^n + c_n S_{ij}^n + d_n G_{ij}^n$$

various sorts of harmonics (indices suppressed)

- classical → quantum → semiclassical

$$i\hbar \frac{\partial \Psi^{(n)}}{\partial t_{em\,(WKB)}} = H^{(n)}_{\text{2nd order}} \Psi^{(n)}$$

separates out mode by mode.

$$\Psi^{(n)} = \Psi^{(n)}_{\text{scalar}} \cdot \Psi^{(n)}_{\text{vector}} \cdot \Psi^{(n)}_{\text{tensor}}$$

- $H^{(n)}_{\text{2nd order}} = \frac{1}{2} e^{-3\alpha} \times$

$$\left\{ \begin{aligned} & -\left(\frac{a_n^2}{2} + \frac{10(n^2-4)}{n^2-1} b_n^2\right)\partial_\alpha^2 - \left(\frac{15}{2} a_n^2 + \frac{6(n^2-4)}{(n^2-1)} b_n^2\right)\partial_\phi^2 \\ & + \partial_{a_n}^2 - \frac{n^2-1}{n^2-4}\partial_{b_n}^2 - \partial_{f_n}^2 - 2a_n \partial_{a_n}\partial_\alpha - 8b_n \partial_{b_n}\partial_\alpha + 6a_n \partial_{f_n}\partial_\phi \\ & - e^{4\alpha}\left(\frac{1}{3}\left(n^2 - \frac{5}{2}\right)a_n^2 + \frac{n^2-7}{3}\,\frac{n^2-4}{n^2-1} b_n^2 + \frac{2}{3}(n^2-4)a_n b_n - (n^2-1) f_n^2\right) \\ & + e^{6\alpha} m^2 \left(f_n^2 + 6 a_n f_n \phi\right) + e^{6\alpha} m^2 \phi^2 \left(\frac{3}{2} a_n^2 - \frac{6(n^2-4)}{n^2-1} b_n^2\right) \end{aligned} \right\}$$

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$\xrightarrow[\text{magnifies}]{\text{inflation}}$

CMB hot spots 

- E.g. Kiefer et al's approximation scheme

predicts $O\left(\left(\frac{H}{m_{pl}}\right)^2\right)$ suppression of power spectrum for low multipoles.

$\lesssim 10^{-10}$

HALLIWELL-HAWKING MODEL is not Cosmologically ideal.
~ Classical Cosmology is to 2nd order in perts.
On the other hand, this has no emergent t
or whole-universe Interpretation of QM.

Should be able to do better than either scheme:
the best of both worlds.

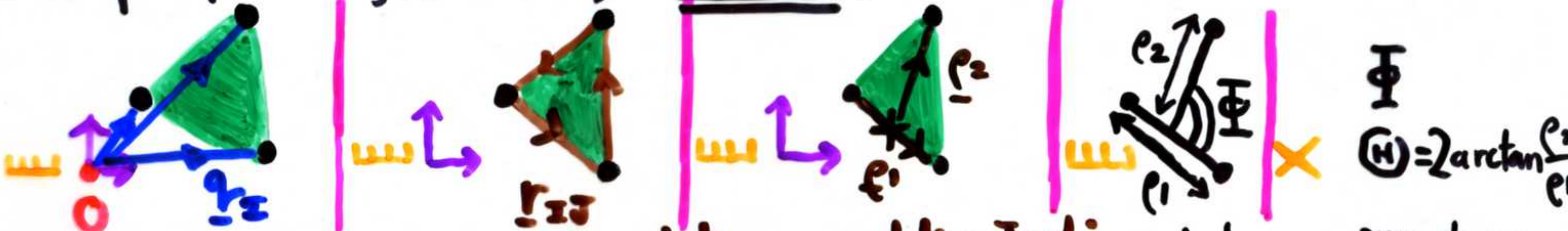
HALLIWELL-HAWKING type eq's are very big;
Use toy models to gain insights.

Minisuperspace

Relational Particle Mechanics

Halliwell-Hawking.

minisuperspace is familiar, RPM's :



particle positions | particle relative separations | relative Jacobi vectors | scaled relational coords | pure-shape relational coords

$$\mathcal{E} = \sum_I \frac{p_I^2}{2m_I} + V \longleftrightarrow \mathcal{H}$$

$$\underline{L} = 0 \longleftrightarrow \mathcal{M}_\alpha$$

$\boxed{\text{Nontrivial linear constraints} \quad \text{A}}$

$$\underline{P} = 0$$

clumping :

relative ∠ info

$\boxed{\text{Nontrivial structure} \sim \text{GR inhomogeneity} \quad \text{B}}$

Doable maths :

1d S^{N-2} $C(S^{N-2}) = R^{N-1}$ $\text{Isom}(S^{N-2}) = SO(N-1)$

2d CP^{N-2} $C(CP^{N-2})$ $\text{Isom}(CP^{N-2}) \sim SU(N-1)$.

pure shape | scaled

$\boxed{\text{toy model } 3+1\text{-d GR}}$

Δ : $CP^1 = S^2$, so it has easiest maths for a model with $\frac{A \text{ and}}{B}$

Quantum scheme for Δ RPM

* Q of form G/H, so lies within an example of Isham 8
Kinematical quantization: $G_{can} \circledS V^* = \text{Isom}(Q) \circledS V$ (V finite)

$\stackrel{\Delta}{=} (SU(2) \circledS \mathbb{R}^3) \circledS \mathbb{R}^3$ better thought of as $IHP(\mathbb{C}, 2)$

($\mathbb{R}^3$ vector $\rightarrow$ Pauli matrix map).

* Conformal operator order wave eq (Dynamical quantization)

$$-\hbar^2\left(\partial_\rho{}^2 + \frac{2}{\rho}\partial_\rho - \frac{3}{2\rho^2} + \Delta_{\mathbb{CP}^1 = S^2}\right)\Psi = 2\{E_{Uni} - V(\rho, \Theta, \Phi\}\Psi$$

solution: shape part $Y_{S\,S}(\Theta, \Phi)$ std maths, unusual physics.

scale part : free = Bessel

isotropic HO = assoc. Laguerre.

Semiclassical Approach

- WDE of Δ TISE equivalent.
- Set $\Psi = \underbrace{\phi(h)}_{\exp(iS(h)/\hbar)} |\chi(h, \ell)\rangle$

heavy light

Born-Oppenheimer

slow

fast.

WKB

- D^2 term : $|\chi\rangle\partial_h^2 S$, $\partial_h S \partial_h|\chi\rangle$, $S\partial_h^2|\chi\rangle$ adiabatic terms

WKB

discard in std BO but not QC.

+ generally, 3 types of adiabatic terms $\frac{\omega_h}{\omega_\ell}$, $\frac{\partial_h|\chi\rangle}{\partial_\ell|\chi\rangle}$: $|\chi\rangle$ only, $\frac{\partial_h|\chi\rangle}{\partial_h\psi}$ "mutu

- h-eq.: $\langle\chi|\cdot$ TISE $\sim (\partial_h S)^2 + V = E$ HJE (+ < > + op order)
- ℓ-eq.: $(1-|\chi\rangle\langle\chi|)\cdot$TISE $\sim$ a priori a fluctuation eq.
- $\partial_h S = p_h = dh/dt^{em}$: HJE $\longrightarrow$ eq for t.
- ℓ-eq.: $i\hbar\partial_h S\,\partial_h|\chi\rangle = i\hbar \cancel{\frac{dh}{dt^{em}}\frac{\partial|\chi\rangle}{\partial h}}$: TISE form (+ corrections : next page

Machian Interpretation

- **Leibniz:** there is no time for the Universe as a whole

mathematically implement by Reparametrization inv.

$\rightsquigarrow$ parametrization irrelevance.

- actions $\sim \int ds \sqrt{E-V}$

Jacobi for Mechanics
Misner for minisuperspace
Baierlein-Sharp-Wheeler type for GR

$\Rightarrow$ $\mathcal{H}$ or $\mathcal{E}$ as a primary constraint of quadratic form $\Rightarrow$ $\widehat{\mathcal{H}}$, $\widehat{\mathcal{E}}$ frozen (part of Problem of Time)

- <u>Resolve</u>: <u>emergent time</u> of a <u>Machian</u> version

Mach: "time is to be abstracted from change"

All change to have opportunity to contribute

(gives a generalized local ephemeris time)

Classical: $t^{em(JBB)} = F[h, dh, \ell, d\ell] \approx F[h, dh]$

semiclassical: $t^{em(WKB)} = F[h, dh, |\chi(h,\ell)\rangle] \approx$

9a

$\circledast(1-\ell\partial_\ell)$ $\qquad$ $\pm/\circledast \ln h$

h

$$i\hbar(1-|X\rangle\langle X|)\underline{\clubsuit}|X\rangle = (1-|X\rangle\langle X|)\left\{-\frac{\hbar^2}{2}\left(\underline{\spadesuit}^2+\underline{\spadesuit}+\Delta_\ell\right) + V_\ell^{rec}+J^{rec}\right\}|X\rangle$$

averages visible everywhere

$\boxed{\text{My Full system for semicl QC (+ general than Halliwell-Hawking)}}$

$\frac{d}{dt}\mathrm{em(rec)}$

higher derivs $\qquad$ ordering

ℓ

$$(\circledast \ln h)^2 - \hbar^2\left(\langle\underline{\spadesuit}^2+\underline{\spadesuit}+\Delta_\ell\rangle-\frac{3}{2}\right) - 2i\hbar\left(1+\langle\spadesuit\rangle\right)\circledast\ln h + 2\left(V_h^{rec}+\langle V_\ell^{rec}+J^{rec}\rangle\right)=2E$$

diabatic $\qquad$ back-react

MINISUPERSPACE can be modelled similarly

FLRW ✓

diagonal Bianchi IX ✓ ~ small anisotropy as toy model for small inhomogeneity

HALLIWELL–HAWKING itself will take + work.
c.f. discussion here of Δ for some of the reasons why.

- **Rectified Time**

ℓ-TDSE is simpler for $h^2 \frac{\partial}{\partial t^{em(WKB)}} =: \frac{\partial}{\partial t^{em(rec)}}$

$$i\hbar \frac{\partial}{\partial t^{em(WKB)}} |\chi\rangle = \frac{1}{h^2}\left(\text{std looking } \hat{H}_\ell\right)|\chi\rangle .$$

- **Easiest Regimes to solve**

1) ignore Machian scheme.

$t_0^{em}[h, dh]$ alone from h-eq

then have a ℓ-TDSE to solve

2) maybe treat this as a t-dep pert. problem

$$\text{“} + J(h,\ell)|\chi\rangle \text{”} = \text{“} + J(t_0^{em}, \ell)|\chi\rangle \text{”}$$

Problem: non-balance.

~ energy in/out of ℓ-subsystem whilst h-subsystem's remains fixed

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Types of correction terms in Semicl. Quantum Cosmology

- **diabatic terms** (Massar & Parentani for minisuperspace)

expanding & contracting universes are coupled

QC Klein paradox : initial wave generates

- **Backreaction terms**: e.g include $\langle J \rangle$ in h-eq.

can then balance E, GR has coupling, Machian t.

Need to solve h & l eq's iteratively for t & $|\chi\rangle$.

- **Higher derivatives**

$(A\partial_t + B\partial_t^2)|\chi\rangle$ corrections

is how Kiefer et al get their cosmologically plausible correction term.

- **Average terms** : the ones I concentrate on for the rest of this talk.

I'd like to do all 4 (& then combos) for Δ & mss, to attain qualitative mastery of the system.

$i\hbar\partial_t\Psi = \hat{H}\Psi$ as an approx to $\left(\frac{\hbar^2}{c^2}\partial_t^2 - \hbar^2\nabla^2 + m^2c^2\right)\Psi = 0$

TDSE

KG eq.

To next order in c^2,
get ∂_t^2 term.

$\longrightarrow$ $\hat{H}^2$ correction term.

For 'ℓ-TDSE' as an approx to WDE,

c^2/G plays role of c above,
and get ∂_t^2 term again

$\longrightarrow$ $\hat{H}_\ell^2$ correction term.

Det 1

Averaged / Expectation terms

- Often argued to be small, incl in QC.
- However, $\langle D^2 \rangle |\chi\rangle$ and $D^2 |\chi\rangle$ are of the same size for Δ free problem since $|\chi\rangle$ are eigenfunctions of this operator.
- Averaged terms turn out to be important in atomic molecular physics and more (see next slide).

Hartree-Fock scheme

Variational Principle
trial wavefunction $\Psi^0 = \frac{1}{\sqrt{n!}} \det |\phi_{a_1}(1) \cdots \phi_{a_n}(n)|$ ① (Slater)
↳ fermions

$H^0 = \sum_{I=1}^{n} h_I \quad h_I = -\frac{\hbar^2}{2m_e}\nabla^2 + V_{\text{fixed nuclear background}}$ BO

Next, include $e^- - e^-$ repulsions: $H^{ee} = \frac{1}{2}\sum_{I \neq J}\sum \frac{e^2}{4\pi\varepsilon_0 r_{IJ}}$

$\langle\Psi|H_0|\Psi\rangle = n\langle\Psi|h_1|\Psi\rangle$ (e^- indistinguishables)

$\langle\Psi|\sum_{I\neq J}\sum \frac{e^2}{4\pi\varepsilon_0 r_{IJ}}|\Psi\rangle = \frac{n(n-1)}{2}\langle\Psi|\frac{e^2}{4\pi\varepsilon_0 r_{12}}|\Psi\rangle \overset{①}{=} \frac{1}{2}\sum_{I\neq J}\sum [\phi_I\phi_J|\phi_I\phi_J]$

$\parallel$ def

$\langle\phi_I^{(1)}\phi_J^{(2)}|\frac{e^2}{4\pi\varepsilon_0 r_{12}}|(\phi_I^{(1)}\phi_J^{(2)} - \phi_J^{(1)}\phi_I^{(2)})\rangle$

Then proceeding variationally,
minimize $E = \sum_{I=1}^{n}\langle\phi_I|h|\phi_I\rangle + \frac{1}{2}\sum_I\sum_J([\phi_I\phi_I|\phi_J\phi_J] - [\phi_I\phi_J|\phi_I\phi_J])$

subject to $\perp$ of the decoupled wavefunctions (incorp by L.M.)

~ a constraint $\sum_I\sum_J \langle\phi_I|\phi_J\rangle - \delta_{IJ} = 0$

Given, to solve $h_1\Phi_I(1) + \sum_{J=1}^{N}(\hat{J}_J(1)\phi_I(1) - \hat{K}_J(1)\phi_I(1)) = \sum_{I=1}^{N}\lambda_{JI}\phi_J(1)$

for $\hat{J}_J(1)\cancel{\phi_I(1)} = \cancel{\phi_I(1)}\langle\phi_J(2)|\frac{e^2}{4\pi\varepsilon_0 r_{IJ}}|\phi_J(2)\rangle$ Coulomb op.

$\hat{K}_J(1)\phi_I(1) = \phi_J(1)\langle\phi_J(2)|\frac{e^2}{4\pi\varepsilon_0 r_{IJ}}|\phi_I(2)\rangle$ Exchange op.

~ is a Coupled integro-differential eq] (*)

Requires numerics (or heavy approximation
~ Roothaan finite-matrix eq.)

Detailed features of previous slide do not matter
– we are doing QC, not Mol Phys.

(*) remains true.

Differences (all available)

(but not all the differences we need)

- QC is not particularly fermionic.
- need t-dep HF ~ Dirac 1930
  and + comprehensibly in Blaziot & Ripka 1986
- ultimately (Halliwell-Hawking)
  we need it for field theory
  ~ but it has appeared in Condensed Matter
  and in other contexts in Cosmology.
- adiabatic t-dep HF in Baranger & Vénéroni 1978

Variational Principles for QM

$E[\psi^*,\psi] = \frac{\langle\psi|\hat{H}|\psi\rangle}{\|\psi\|^2}$ Ritz $J[\psi^*,\psi] = \langle\psi|\hat{H}|\psi\rangle - \underset{\uparrow}{E}\|\psi\|^2$

Lagrange multiplier

HF, insert $\hat{H}_\Delta$

$\int_{R(3,2)} \mathcal{D}R(3,2)$

$\underbrace{R(3,2)}_{\mathbb{R}^3}$

e.g. Dirac or Blaizot-Ripka

t-dep is fine: $S[\psi,\psi^*] = \langle\psi|(i\hbar\partial_t - \hat{H} - \Lambda)|\psi\rangle$

For semicl QCos ℓ-eq:

$S[\chi^*,\chi] = \int dS^2\, \chi^* \left\{ i\hbar \underline{\Phi} - \hat{N}_\ell - \frac{1}{2}\left(i\hbar\langle\underline{\Phi}\rangle - \langle\hat{N}_\ell\rangle - (\Lambda - \langle\Lambda\rangle) \right) \right\} \chi$

$S(3,2) = S^2$

t-indep part.

insert also $-\hbar^2\left((\underline{\Phi}^2 + \underline{\Phi})/2 - \langle\underline{\Phi}^2 + \underline{\Phi}\rangle/4\right)$ to include that

Variational Principles

h-part:

$\rightarrow$ coeff of $\partial_h S$ $\rightarrow$ coeff of 1 in h-eq.

$$S_h = \int dt^{em(rec)} \left(-A \pm \sqrt{A^2 - B}\right)^2 / h^2 \; =: S_h^{eval.}$$

is + convenient than

$$S_h' = \int dt^{rec} (T - V - \langle O \rangle) \text{ form.}$$

Then, combined,

$$S[\Psi, \Psi^*; h] = S_{semi}[\chi^*, \chi] + \mu \left(S_h - S_h^{eval}\right)$$

μ: Lagrange Multiplier

$\langle\ \rangle$ here doesn't enter t-eq since $\mu = 0$ post-variationally

Still assumes we know t.

Have not as yet got a VP for chroniferous procedure itself.

CONCLUSION

- QC eq's have many terms.
- Perturbation & Variation methods appropriate
- Both are nonstd.
- VP's, HF type procedure. Some provided.
- Next step: trial wavefunctions
  ~ mode decoupling of Halliwell-Hawking model
  \+ its $\Psi^{(n)} = \Psi^{(n)}_{S} \Psi^{(n)}_{V} \Psi^{(n)}_{T}$ decomposition.
- Δ model – qualitatively useful so far, isn't so useful now. (do have $\Psi_{pert} = \Psi(S_p)\Psi(S(6))\Psi(S$ $\Sigma)$
- Anything better from microphysics?
- Eventual need to investigate cvgce of the method.