Heavy quarkonium physics from Euclidean correlators

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Talk at ECT*, based on
[YB, M. Laine, JHEP 1211 (2012) 086]
[YB, A.Rothkopf, Phys. Rev. D **86** (2012) 051503]
[YB, A. Rothkopf, in preparation]

April 3, 2013

Outline

1. Introduction
2. Massive vector current correlator in thermal QCD at NLO
   - Vector current correlators as observable
   - NLO Calculation
   - Results and comparison to the lattice
3. HTL Wilson loop to check the extraction of the static potential
   - HTL and lattice Wilson loop
   - Extraction of the potential from the Wilson loop spectrum
   - Analytic continuation from the Euclidean correlator
4. Conclusion

Outline

Some Euclidean observables for heavy quarkonium

Vector (spatial) current correlator

- Definition:

$$G_{V,ii}(\tau) = \int_x \langle \left(\bar{\Psi}\gamma_{\mu,i}\Psi\right)(\tau,x)\left(\bar{\Psi}\gamma^{\mu}{}_{i}\Psi\right)(\tau,x)\rangle_T$$

- Physics:
  - Transport peak, heavy quark diffusion (large τ)
  - Quarkonium physics, bound state (intermediate τ)
  - High energy scattering (small τ)

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Wilson loop

- Defined as an integral on a square path (r,τ):

$$W_\square = \frac{1}{N_c}\langle \mathcal{P}\mathrm{tr}[e^{-ig\int_\square dx^\mu A_\mu(x)}]\rangle_T$$

- Physics:
  - Heavy bound state physics, heavy quark potential.

How to extract the physics from Euclidean correlators?

Physical quantities we are interested in are defined in Minkowski space. They can be fitted from the correlator's spectrum

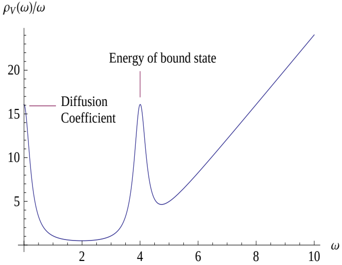
$$\rho(\omega) = Disc[C_E(-i\omega)], \quad \text{or} \quad C_E(\tau) = \int d\omega K(\omega,\tau)\rho(\omega).$$

Perturbation theory:

- Handle Euclidean and Minkowski
- Not accurate for large coupling
- Hard to catch infrared physics

Lattice:

- Contains all the physics
- Bound to Euclidean



- Need analytic continuation to Minkowski space. . .

Can our understanding from perturbation theory help?

Outline

Vector current correlators as observable

Get some intuition from LO perturbation theory

$$G_V(\tau) = G_{00} - G_{ii} = \int_x \langle \left(\bar{\Psi}\gamma_\mu \Psi\right)(\tau, x) \left(\bar{\Psi}\gamma^\mu \Psi\right)(\tau, x)\rangle_T$$

- $G_{00}(\tau)$ is constant (current conservation)
  $\Rightarrow$ one defines the susceptibility $\chi = \beta G_{00}$
  - At LO, $G_{00} = -4N_c \int_p T n'_{\mathrm{F}}(E_p)$
- G_V and G_{ii} have both a constant and a τ-dependent part:
  - At LO, $G_V^{const} = G_{00} - G_{ii}^{const} = -4N_c \int_p \frac{M^2 T n'_{\mathrm{F}}(E_p)}{E_p^2}$
  - $G_V^\tau = -G_{ii}^\tau = -2N_c \int_p \left(2 + \frac{M^2}{E_p^2}\right) D_{Ep,Ep}(\tau)$
  - τ-dependence parametrized by the functions

$$D^{E_{k+1}}_{E_1 \cdots E_k}(\tau) \equiv \frac{e^{(E_1+\cdots+E_k)(\beta-\tau)+(E_{k+1})\tau} + e^{(E_1+\cdots+E_k)\tau+(E_{k+1})(\beta-\tau)}}{[e^{\beta E_1} \pm 1]\cdots[e^{\beta E_n} \pm 1]}$$

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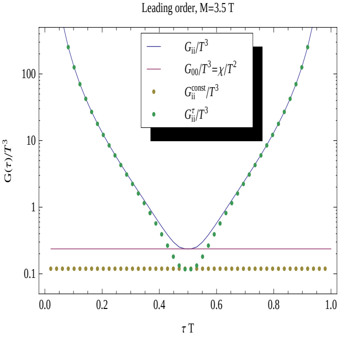
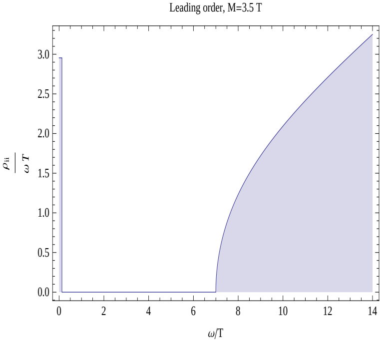
Leading order results

G_{ii}, G_{00} and how they contribution to ρ_{ii}





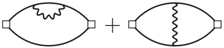
- At $\tau = \beta/2$ the constant part $=$ time dependent part.
- In fact $D_{Ep,Ep}(\beta/2) = -2Tn'_{\mathrm{F}}(Ep)$.
- $D_{Ep,Ep}(\tau) = \frac{K(\tau, 2E_p)}{1+2n_{\mathrm{B}}(E_p)}$, with $K(\tau,\omega) = \frac{\cosh(\omega(\tau-\beta/2)}{\sinh(\omega\beta/2)}$

Diagrams and renormalization

We calculated the NLO vector correlator for "any" quark mass M:

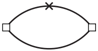
[YB, M.Laine, JHEP 1211 (2012) 086]

The "genuine" 2-loop graphs for $G_V(\omega)$ amount to



$$= 4g^2 C_A C_F \sum\!\!\!\!\!\!\!\int_{K\{P\}} \left\{ -\frac{2(D-2)}{K^2\Delta_P\Delta_{P-Q}} + \frac{4(D-2)}{\Delta_P\Delta_{P-K}\Delta_{P-Q}} \right.$$
$$+ \frac{4(D-2)M^2-(D-2)^2Q^2}{K^2\Delta_P^2\Delta_{P-Q}} + \frac{2(D-2)}{K^2\Delta_{P-K}\Delta_{P-Q}} - \frac{4(D-2)M^2-(D-2)^2Q^2}{\Delta_P^2\Delta_{P-K}\Delta_{P-Q}}$$
$$- \frac{16M^2+2(D-2)^2K\cdot Q-4(D-2)Q^2}{K^2\Delta_P\Delta_{P-K}\Delta_{P-Q}} + \frac{8M^4-2(D-4)M^2Q^2-(D-2)Q^4}{K^2\Delta_P\Delta_{P-K}\Delta_{P-Q}\Delta_{P-K-Q}}$$
$$\left. - \frac{2(D-4)M^2+\frac{1}{2}(D-2)(8-D)Q^2+(D-2)K^2}{\Delta_P\Delta_{P-K}\Delta_{P-Q}\Delta_{P-K-Q}} + \frac{16M^4-4(D-2)Q^2M^2}{K^2\Delta_P^2\Delta_{P-K}\Delta_{P-Q}} + \ldots \right\}$$

The only parameter to renormalize at this order is the mass M:



$$= 4C_A\delta M^2 \sum\!\!\!\!\!\!\!\int_{\{P\}} \left\{ \frac{D-2}{\Delta_P^2} - \frac{2}{\Delta_P\Delta_{P-Q}} + \frac{4M^2-(D-2)Q^2}{\Delta_P^2\Delta_{P-Q}} \right\}$$

Parametrization of the master integrals

- From the Fourier representation,

$$G_V(\tau) = T \sum_{\omega_n} e^{i\omega_n \tau} G_V(\omega_n)$$

- Calculate all master integrals separately
- Write the results into some common basis

$$nF'(E_p),\ D_{E_p,E_p}(\tau),\ D_{\epsilon_k E_p E_{pk}}(\tau),\ D^{\epsilon_k}_{E_p E_{pk}}(\tau), \ldots$$

- Sum the masters and simplify
- Remains 2 to 3 integrals to perform numerically

$\Rightarrow$ τ dependence cancels in G_{00}

$\Rightarrow$ G_{00}, G_V UV and IR finite after mass renormalization

- Some terms are IR sensitive, only the sum is finite

Results

$$\frac{T\chi^{\rm NLO}}{4g^2C_AC_F} = \int_p \frac{Tn_{\rm F}{}'(E_p)}{p^2}\int_k\left[\frac{n_{\rm B}(\epsilon_k)}{\epsilon_k}+\frac{n_{\rm F}(E_k)}{E_k}\left(1-\frac{M^2}{k^2}\right)\right]$$

$$\frac{G_{\rm V}^{\rm NLO}\big|_{\rm const.}}{4g^2C_AC_F} = \int_p Tn_{\rm F}{}'(E_p)\int_k\left\{\frac{n_{\rm B}(\epsilon_k)}{\epsilon_k}\left[\frac{1}{p^2}-\frac{3}{E_p^2}\right]\right.$$

$$\left. + \frac{n_{\rm F}(E_k)}{E_k}\left[\frac{1}{p^2}-\frac{3}{E_p^2}-\frac{M^2}{p^2k^2}-\frac{M^2}{E_p^2E_k^2}+\frac{M^2(4E_k^2-M^2)}{2pkE_p^2E_k^2}\ln\left|\frac{p+k}{p-k}\right|\right]\right\}$$

$$\frac{G_{\rm V}^{\rm NLO}\big|_{\tau\text{-dep.}}}{4g^2C_AC_F} = \int_p \frac{D_{2E_p}(\tau)}{4\pi^2}\left[\left(3+\frac{M^2}{E_p^2}\right)\left(1-\frac{p}{2E_p}\ln\frac{E_p+p}{E_p-p}\right)-1+\dots\right]$$

$$+\int_{p,k}\mathbb{P}\left\{\int_z\frac{D_{\epsilon_kE_pE_{pk}}(\tau)}{\epsilon_kE_pE_{pk}(k^2-(E_{pk}-E_p)^2)}\left[-2E_p^2+M^2\frac{k-E_p-E_{pk}}{k+E_p+E_{pk}}+\dots\right]\right.$$

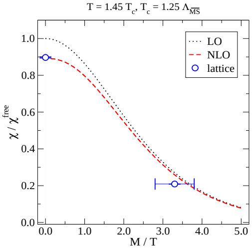
$$\left.+\int_z\frac{D^{\epsilon_k}_{E_pE_{pk}}(\tau)}{\epsilon_kE_pE_{pk}(k^2-(E_{pk}-E_p)^2)}\left[-2E_p^2+\dots\right]+\dots\right.$$

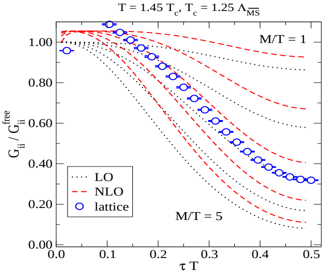
Comparison to the lattice

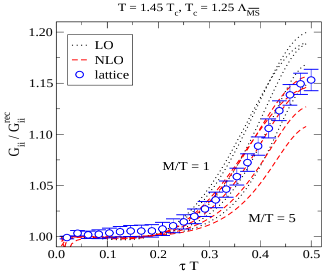
To compare the curves, scale

- G_{ii} and χ to the free, M=0 result
- G_{ii} to the reconstructed correlator

$$G_{ii}^{NLO,rec}(\tau, T) = \int \frac{d\omega}{2\pi}(-\rho_V^{NLO,vac})K(\tau,\omega)$$
$$G_{ii}^{lattice,rec}(\tau, T, T') = \sum_{\tau'(T,T')} G(\tau', T')$$







[Ding, Francis, Kaczmarek, Karsch, Satz, Soeldner, 1204.4945; Ding, Francis, Kaczmarek, Karsch, Laermann, Soeldner, 1012.4963]

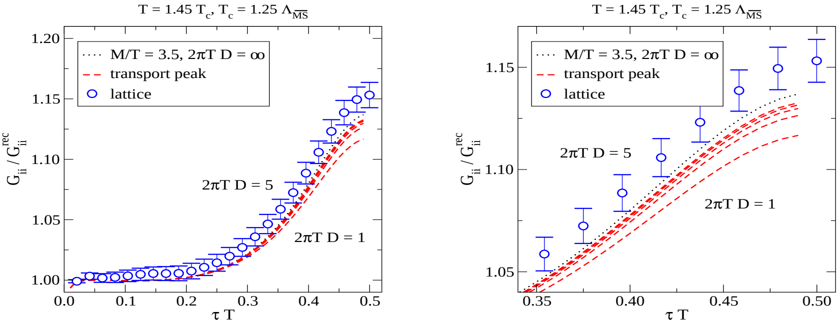
Addition of a transport peak

The spectral function ρ_{ii}/ω has a $\delta(\omega)$ transport peak (LO, NLO)

- In the full result one expects a Lorentzian shape.
- Replace in G_{ii} the constant part by a τ-dependent part from

$$\rho_{ii}^{(\mathrm{L})}(\omega \sim 0) \approx 3D\chi \frac{\omega\eta^2}{\omega^2+\eta^2}$$

- We vary D tuning η to keep the area under $\rho(\omega)/\omega$ constant



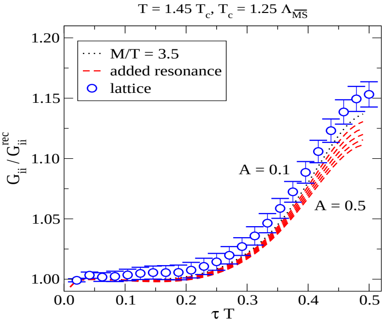
Addition of a bound state peak

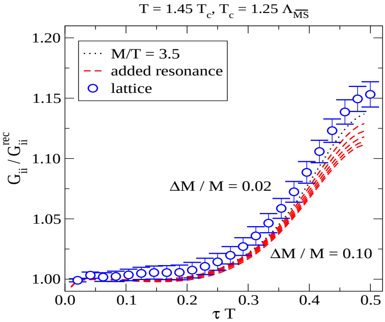
In the temperature range of interest $\rightarrow$ at most one peak.

- Slightly to the left from the free quark-antiquark threshold
- Model this by a skewed Breit-Wigner shape added to NLO

$$\rho_{ii}^{(\mathrm{BW})}(\omega \sim 2M) \;\approx\; \frac{A\,\omega^2\gamma^2}{(\omega - 2M + \Delta M)^2 + \gamma^2}$$

- Set $\Delta M \equiv 2\gamma$, add to the thermal NLO result
- Add to vacuum result with $A \rightarrow 5A$, $\gamma \rightarrow \gamma/5$





Outline

Wilson loop at LO in HTL resummed perturbation theory

LO diagrams [Laine, Philipsen, Romatschke, Tassler, 0611300]

$$W_\square(\tau,r) = 1 + C_F \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{e^{iq_3 r} + e^{-iq_3 r} - 2}{2} \Bigg\{ \frac{\tau}{\mathbf{q}^2 + \Pi_L(0,\mathbf{q})} + \int \frac{dq^0}{\pi} n_B(q^0) \times h(\tau, q^0) \left[\rho_L(q^0,\mathbf{q}) \left(\frac{1}{\mathbf{q}^2} - \frac{1}{(q^0)^2} \right) + \rho_T(q^0,\mathbf{q}) \left(\frac{1}{q_3^2} - \frac{1}{\mathbf{q}^2} \right) \right] \Bigg\},$$

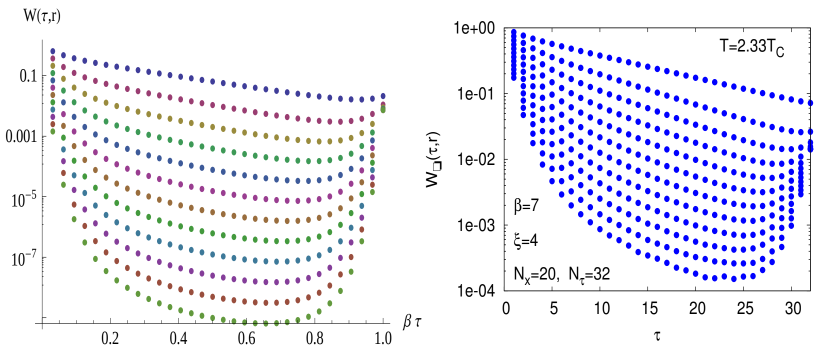
Form this relation we can get:

- $\rightarrow$ The Euclidean correlator, performing the integrals numerically
- $\rightarrow$ The real time correlator $W_\square(\tau = it, r)$
- $\rightarrow$ The potential $\lim_{t\to\infty} \partial_t \log(W_\square(it,r)) = V(r)$
- $\rightarrow$ The spectral function $\rho(r,\omega) = \frac{1}{2\pi} \int dt\, e^{i\omega t} W_\square(it,r)$

I avoid issues of renormalization $\rightarrow$ introduce a lattice-like cutoff Λ

HTL and lattice Wilson loop

Start by comparing the Euclidean correlators:



- HTL Euclidean correlator similar (more power in the UV)
- Resulting HTL potential known
- Use it to test the extraction of the potential

Wilson loop spectral function

- Analytic approximation for $\omega \sim \mathrm{Re}[V(r)]$

$$\rho^{\Lambda}(r,\omega) = \frac{1}{\pi}\frac{|\mathrm{Im}[V(r)]|\cos\delta - (\mathrm{Re}[V(r)]-\omega)\sin\delta}{(\mathrm{Im}[V(r)])^2 + (\mathrm{Re}[V(r)]-\omega)^2}$$

$\Rightarrow$ Skewed Lorenzian with $\delta = \frac{|\mathrm{Im}[V(r)]|}{2T}$

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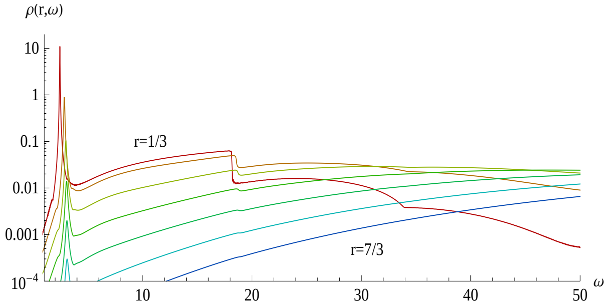
- Numerical results form integration of the real-time correlator:

$\Rightarrow$



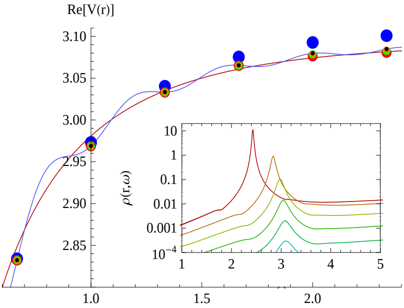
Spectral function fit and extracted value for the potential

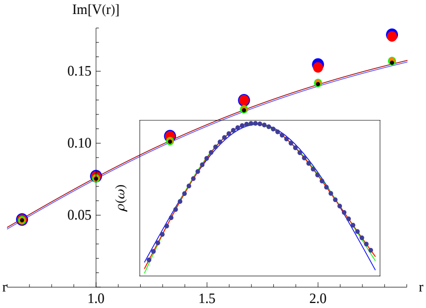
In principle the potential is: $V(r) = \lim_{t\to\infty} \frac{\int d\omega\ \omega\ e^{-i\omega t}\ \rho(r,\omega)}{\int d\omega\ e^{-i\omega t}\ \rho(r,\omega)}$

In the $t \to \infty$ limit, only the first peak $\rho(r,\omega)$ should contribute [YB, Rothkopf, 1208.1899]

$$\rho_\square(r,\omega) = \frac{1}{\pi} e^{\mathrm{Im}[\sigma_\infty(r)]} \frac{|\mathrm{Im}[V(r)]|\cos(\mathrm{Re}[\sigma_\infty(r)]) - (\mathrm{Re}[V(r)]-\omega)\sin(\mathrm{Re}[\sigma_\infty(r)])}{\mathrm{Im}[V](r)^2 + (\mathrm{Re}[V](r)-\omega)^2}$$
$$+c_0(r) + c_1(r) t_{Q\bar{Q}}(\mathrm{Re}[V](r)-\omega) + c_2(r) t_{Q\bar{Q}}^2(\mathrm{Re}[V](r)-\omega)^2 + \cdots$$

Parameters: $e^{\mathrm{Im}[\sigma_\infty(r)]}$, $\mathrm{Im}[V(r)]$, $\mathrm{Re}[V(r)]$; $\mathrm{Re}[\sigma_\infty(r)]$; $c_0(r)$; $c_1(r)$; $c_2(r)$

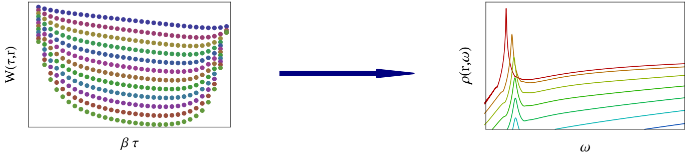




The lowest peak contains all the information for the potential!

MEM to get the spectral function

Can we get the spectral function?



- From $N_\tau = 32$ points, usual MEM (with 32 basis functions) fails given the large ω range needed
- Too few basis function to reconstruct the Euclidean correlator
- One might remove points at small/large τ to get the peak
- In any case not well controlled

Extended MEM to get the spectral function

From $N_\tau = 32$ points, extended MEM
(with 80 basis functions, $\beta = 20$, $\omega \in [-10, 15]$, $r = 1$):

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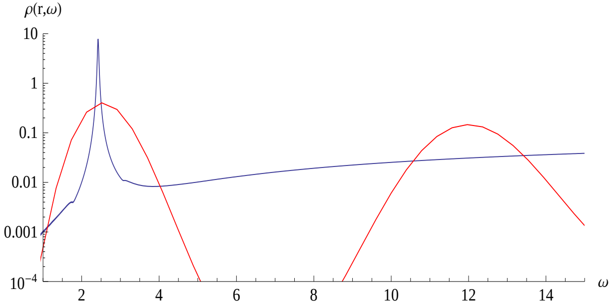

- Position of the peak $\equiv \mathrm{Re}[V](r) \sim$ OK
- Euclidean correlator reconstructed
- Low resolution, Numerically expensive, unstable results

Bayesian curve fitting to get the spectral function

- New entropy functionnal
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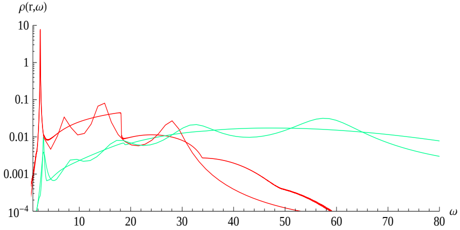
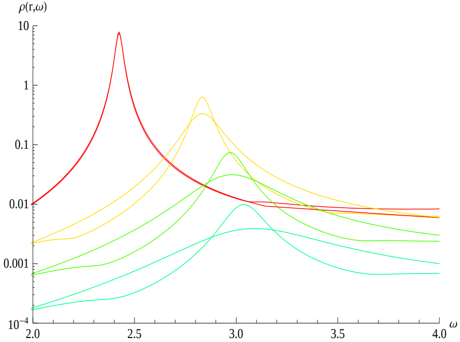
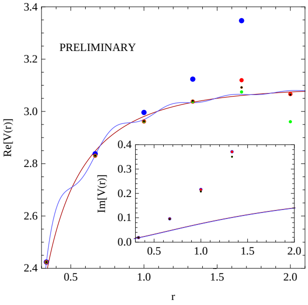
Bayesian curve fitting to get the spectral function

- New entropy functionnal
- Basis: 800 bars
- $N_\tau = 32$
- Great result at small r!
- $\mathrm{Re}[V(r)]$ correct.
- $\mathrm{Im}[V(r)]$ OK at small r.







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Conclusion

G_V:

- Encodes a lot of information
- Hard to disentangle the different types of physics
- Require very accurate lattice data

Next:

- ρ at NLO for any $M \leftarrow$ physics
- Oher channels G_S, G_{PS}, G_A, behave differently

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$W_\square$:

- We are now confident that we can get $\mathrm{Re}[V(r)]$
- The previous measurements of $\mathrm{Im}[V(r)]$ were MEM artifacts

Next:

- What should be $N_\tau,\ \delta$ to get the correct Im[V(r)]?