A well-separated pairs decomposition algorithm for kd-trees implemented on multi-core architectures

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October 17, 2013

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Motivation

- A problem
  - Given a set P of n points in R^d:
    - find the two closest points to each other belonging to P (e.g. Eppstein);
    - for each $q \in P$, find its closest neighbour in $P - q$ [6];
    - find all k nearest neighbours of each $q \in P$ [4].

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    - find all k nearest neighbours of each $q \in P$ [4].
- Theoretical limits
  - all solvable in $O(n \log n)$ work **if** an $O(n \log n)$ work algorithm spatial indexing is available.
  - parallel algorithms using p processors and $O(n \log n)$ work theoretically available (Callahan in [4]).

Good for HEP and Physics?

- Multivariate analysis for TMVA in ROOT uses k nearest neighbours search. Repeated analysis might benefit of k-d-tree algorithm with $O(n \log n)$ work with decent scalability.
- Track reconstruction by joining compatible triplets has been approached in CMS using k-d-trees and cellular automata. Even the the simulation of celular automata might demand multidimensional data organisation when the number of dimensions increase.
- N-body computations based on Barnes-Hutt or Fast Multipole Method in general depend on tree methods.
- N-point correlation functions have been tackled in Astronomy by the use of search in k-d-trees [5].

Proposed solution

- K-d-tree based solution built on
  - fair split that will lead to *Well Separated Pair Decomposition*;
  - balanced partitioning giving $O(\log n)$ height independent of input;
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- Implementations in the paper
  - First implementation of a parallel **WSPDP** algorithm.
  - Possibly first implementation of a parallel k-d-tree (Lisp, C/OMP, C++/TBB.)
  - Parallel scalable implementation of Kth-selection algorithm

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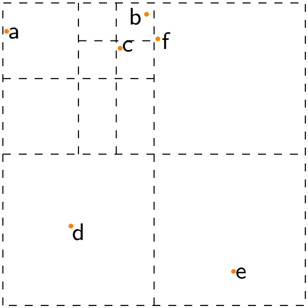

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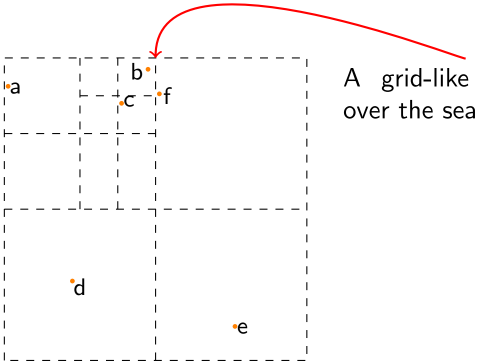

A grid-like structure over the search space

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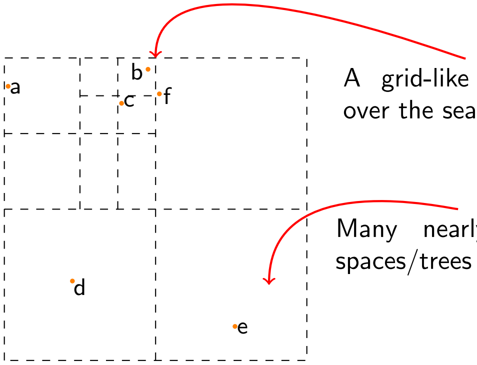

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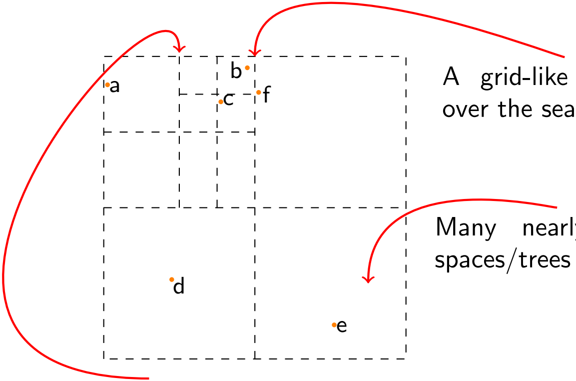

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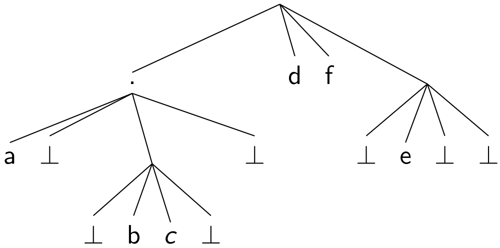
a difficult act to balance parallel work for unequal regions

Quadtrees: a choice for parallel spatial indexing?



- each node partitions a set of points by all *d* attributes, each representing one dimension. partitions for the set in question
- Samet [8]: used by *all* published works on parallel spatial indexing.
- disadvantages:
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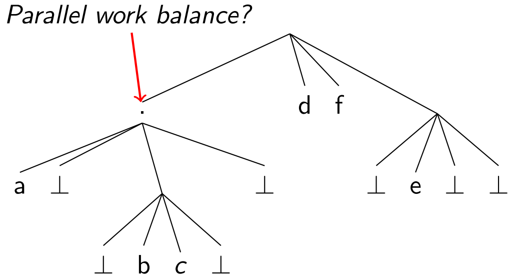

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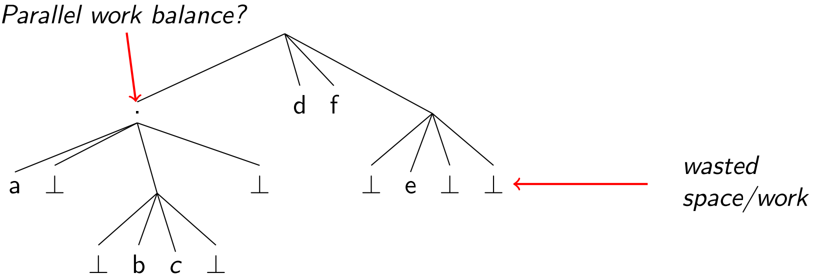

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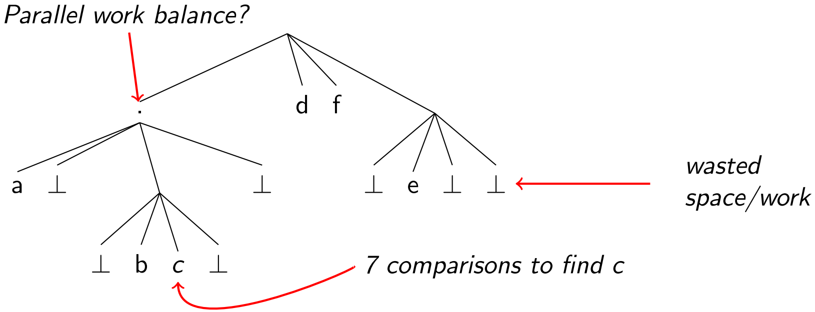

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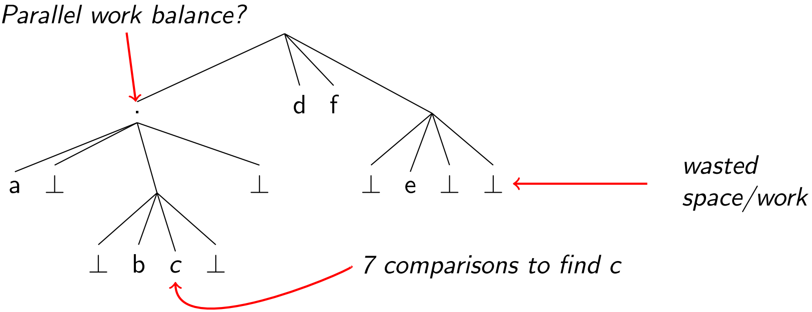

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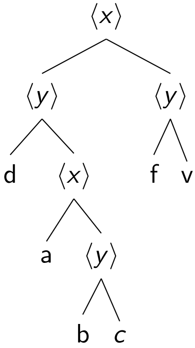

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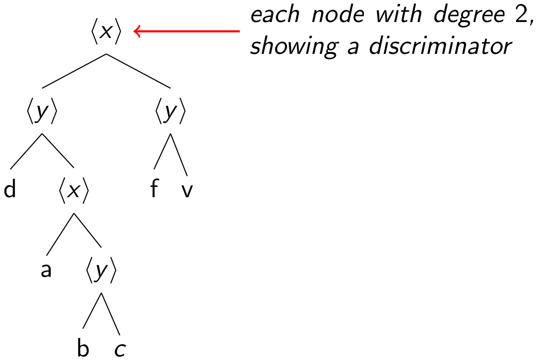

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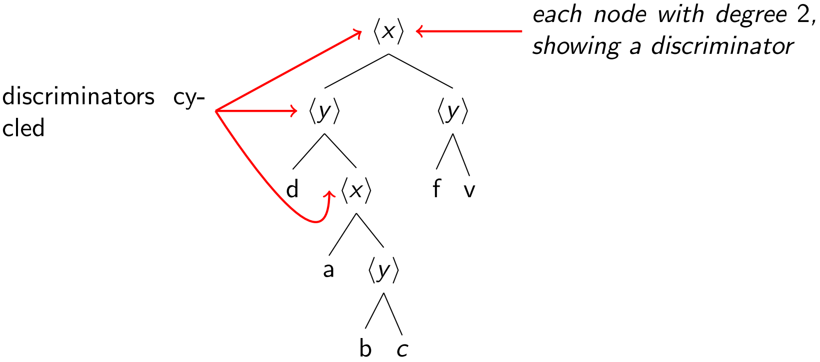

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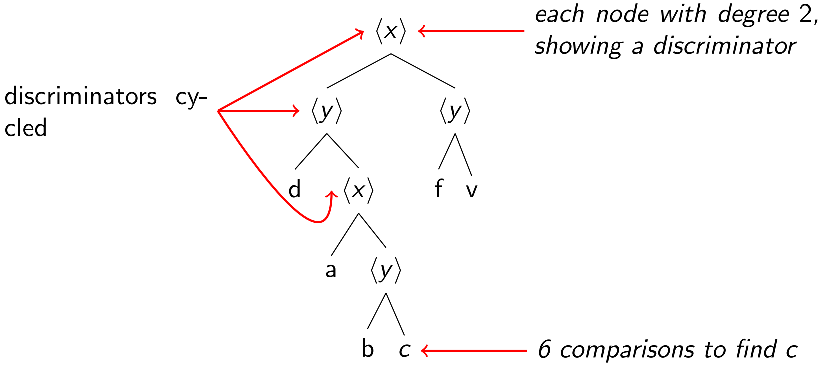

K-d-trees: questions

- K-d-tree by Jon Bentley [2]
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  - discriminators cycle through the k dimension on the path from root to leaf nodes
- Disadvantages
  - cycling through dimensions can still lead to leaf depth unbalancing
  - choice of cut value, a problem to be solved
  - distribution of work on $p > 1$ processors scenario can be complicated due to recursive nature
  - work balancing still a problem

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Fair split based K-d-trees (Callahan)

- Splitting criteria
  - each node defines a range of the space
  - split close to the middle of the range
  - split should guarantee either:
    - balanced distribution of points for children nodes
    - or total of comparisons on the order of the size of the node being split
- Fair split target balanced number of comparisons in sequential split
  - $O(n \log n)$ time for sequential algorithm
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  - split adapts Bentley's "burning the candle from both ends" algorithm [1].
- Problem for parallel algorithm
  - split must "guess" (or brute force search) slabs of splitting
  - Callahan does "hand waving argument" to show that it is possible
  - Har-Peled [6] proposes splitting based on one of
    - radix splitting
    - k-enclosing disk splitting

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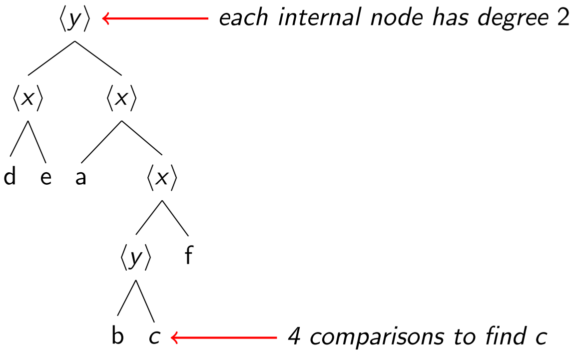

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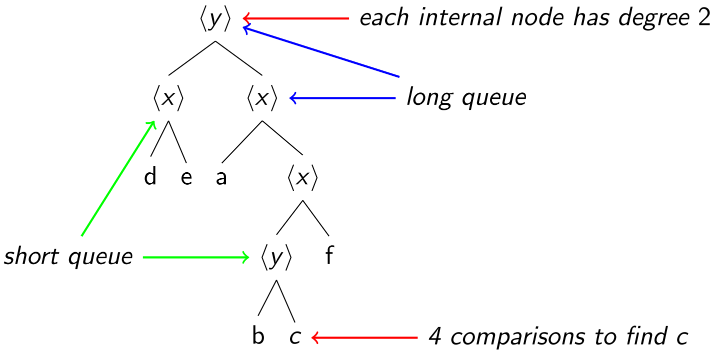

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- asynchronously parallel algorithm: full use of all processors as they become available.

Splitting for balanced k-d-tree

- Split close to the median.
  - Based on Blum, Floyd, Pratt, Tarjan algorithm of kth-selection
  - Parallel map blocks of 5 elements to its median
  - Recursion until block of up to 5 central elements is found.
  - Easy elimination of recursion.
  - Guaranteed time in $O(n)$.
  - Median of final blocks always greater than $\frac{3n}{10}$ and less than $\frac{7n}{10}$ elements of initial set.
  - Additional element from final block used for a two pivots split.
- Option for split at the middle of the range based on A. Moore [7].

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- Short queue split:
  - Each process available takes one node to split with same algorithm as long queue.

Performance

- Based on gcc (-fopenmp) 4.6, Ubuntu 13.04, Intel xeon 5660.
- Sets with up to 2^{16} points processed asynchronously with one processor.
- Speed-up and efficiency shown in the table only for the 1572864 set.

points in 3-d	1 proc	4 procs	8 procs	12 procs
65536	0.856s	0.868s	0.857s	0.854s
252144	3.206s	1.337s	1.235s	1.391s
524288	6.551s	2.545s	1.338s	1.512s
786432	9.724s	3.991s	2.781s	1.862s
1572864	18.43s	7.515s	4.532s	3.623s
S_p	1	2.45	4.14	5.08
E_p	1	0.61	0.51	0.42

- Speed-up $S_{12} = \frac{T_{12}}{T_1}$
- Efficiency $E_{12} = \frac{S_{12}}{12}$

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  - *Careful! Categorization pattern can be very costly!*
- Is *WSPDP* too heavy?
  - Yes, when number of dimensions increase [6].
  - Sequential approximation algorithm available. Parallel?

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  - Extensive testing/tuning for algorithm with an initial phase of sampling to distribute points when running on cluster.
  - MIC computation could be more feasible due to lower communication costs.

Conclusion

- The simple structure of k-d-trees offers promising alternatives regarding:
  - Restricting the height of resulting trees.
  - balancing of work load in parallel implementation.
- Challenges that need to be worked:
  - Irregular parallelism of present algorithm not the best regular SIMD as found in GPUs.
  - Improving memory management in parallel execution might result in huge gains in computing time and efficiency.
  - Scheduling of synchronised steps affects time and efficiency.

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