Perturbative Non-Equilibrium Thermal Field Theory

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Introduction

Motivation

- the density frontier: ultra-relativistic many-body dynamics

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- the density frontier: ultra-relativistic many-body dynamics
- collider physics:
  - quark gluon plasma/glasma/color glass condensates
- cosmology and the early Universe:
  - baryon asymmetry of the Universe
  - electroweak phase transition
  - reheating/preheating
  - relic densities
- astro-particle physics:
  - compact stars
  - black hole thermodynamics
  - AdS spacetimes
- condensed matter physics

Introduction

Current Approaches

- (semi-classical) Boltzmann transport equations

  [e.g. Kolb and Wolfram, Nucl. Phys. B172 (1980) 224–284]

  - effective resummation of finite-width effects

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- Kadanoff–Baym $\Rightarrow$ quantum Boltzmann equations

  [e.g. Blaizot and Iancu, Phys. Rep. 359 (2002) 355–528; Prokopec et al., Annals Phys. 314 (2004) 208–265]

  - incorporation of off-shell effects
  - truncated gradient expansion in time derivative
  - separation of time scales and quasi-particle approximation
  - varying definitions of physical observables,
    e.g. particle number density
  - underlying perturbation series is ill-defined

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Boundary Conditions

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$\Rightarrow$ interactions switched on at $\tilde{t}_i$

$\Rightarrow$ initial density matrix $\rho(\tilde{t}_i;\tilde{t}_i)$ specified fully in on-shell Fock states

$\Rightarrow$ finite lower bound on time integrals in path-integral action

Canonical Quantisation

Canonical Commutation Relations

- Interaction-picture creation and annihilation operators satisfy:

$$\left[a(\mathbf{p},\tilde{t};\tilde{t}_i), a^\dagger(\mathbf{p}',\tilde{t}';\tilde{t}_i)\right] \;=\; (2\pi)^3 2E(\mathbf{p})\delta^{(3)}(\mathbf{p}-\mathbf{p}')e^{-iE(\mathbf{p})(\tilde{t}-\tilde{t}')}$$

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- Ensemble Expectation Value (EEV) at macroscopic time $t = \tilde{t}_f - \tilde{t}_i$:

$$\langle\Phi(\tilde{t}_f,\mathbf{x};\tilde{t}_i)\Phi(\tilde{t}_f,\mathbf{y};\tilde{t}_i)\rangle_t \;=\; \frac{\mathrm{tr}\,\rho(\tilde{t}_f;\tilde{t}_i)\Phi(\tilde{t}_f,\mathbf{x};\tilde{t}_i)\Phi(\tilde{t}_f,\mathbf{y};\tilde{t}_i)}{\mathrm{tr}\,\rho(\tilde{t}_f;\tilde{t}_i)}$$

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- Most general EEVs permitted:

$$\begin{aligned}\langle a(\mathbf{p},\tilde{t}_f;\tilde{t}_i)a^\dagger(\mathbf{p}',\tilde{t}_f;\tilde{t}_i)\rangle_t \;&=\; (2\pi)^3 2E(\mathbf{p})\delta^{(3)}(\mathbf{p}-\mathbf{p}') \\ &\quad+\; 2E^{1/2}(\mathbf{p})E^{1/2}(\mathbf{p}')f(\mathbf{p},\mathbf{p}',t) \\ \langle a^\dagger(\mathbf{p}',\tilde{t}_f;\tilde{t}_i)a(\mathbf{p},\tilde{t}_f;\tilde{t}_i)\rangle_t \;&=\; 2E^{1/2}(\mathbf{p})E^{1/2}(\mathbf{p}')f(\mathbf{p},\mathbf{p}',t)\end{aligned}$$

Schwinger–Keldysh CTP Formalism

$$\mathcal{Z}[\rho, J_\pm, t] = \mathrm{tr}\left[\bar{\mathrm{T}} e^{-i\int_{\Omega_t} \mathrm{d}^4x\, J_-(x)\Phi_\mathrm{H}(x)}\right] \rho_\mathrm{H}(\tilde{t}_f; \tilde{t}_i) \left[\mathrm{T} e^{i\int_{\Omega_t} \mathrm{d}^4x\, J_+(x)\Phi_\mathrm{H}(x)}\right]$$

$$x_0 \in \left[\tilde{t}_i = -\tfrac{t}{2}, \tilde{t}_f = +\tfrac{t}{2}\right]$$

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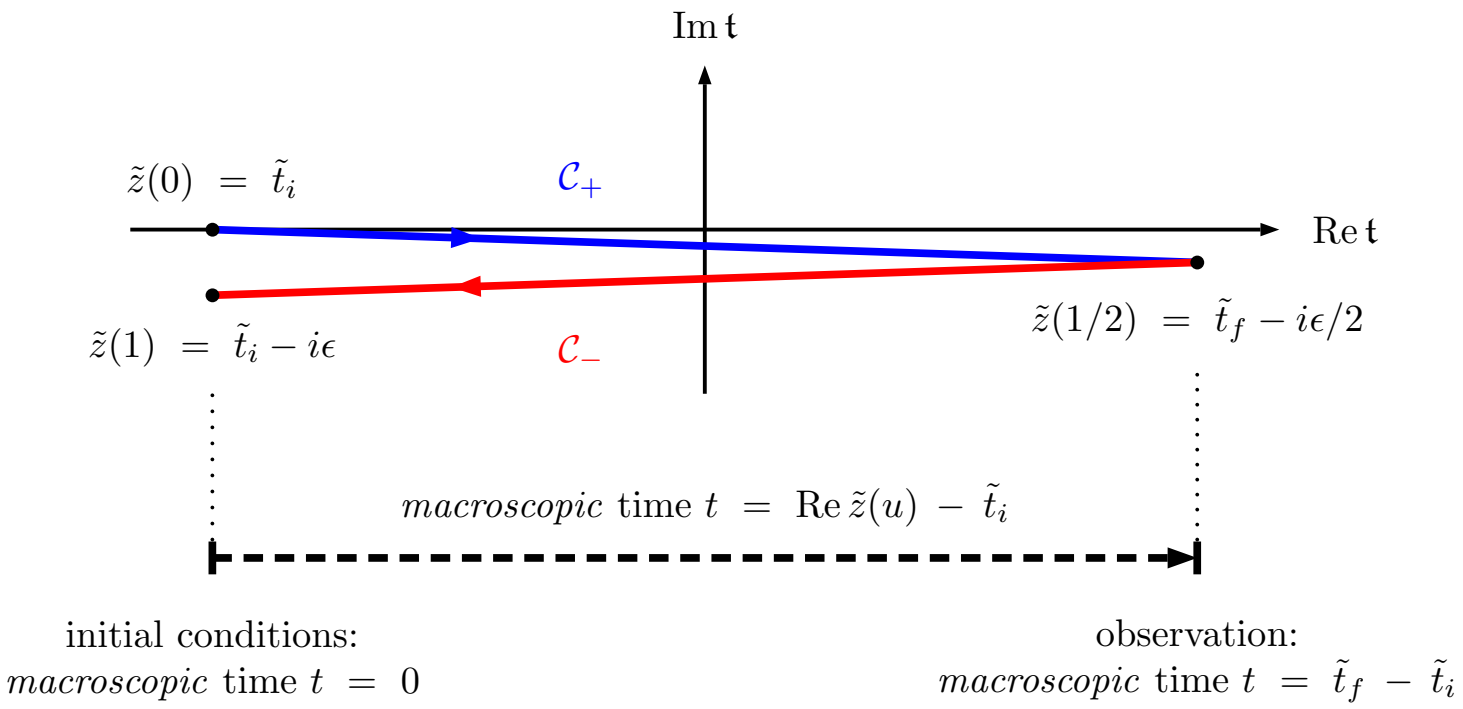


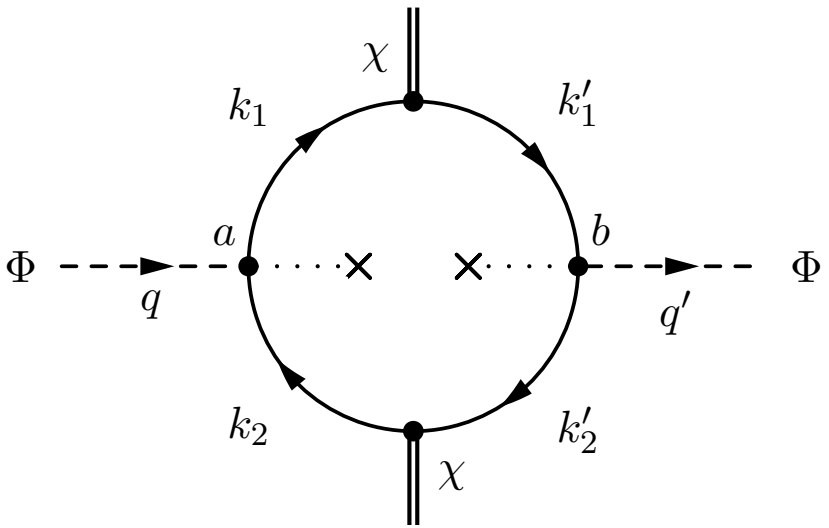
$\Rightarrow$ finite upper and lower bounds on time integrals in path-integral action.

Non-Homogeneous Free Propagators

Propagator	Double-Momentum Representation
Feynman (Dyson)	$i\Delta^0_{\mathrm{F(D)}}(p,p',\tilde{t}_f;\tilde{t}_i) = \frac{(-)i}{p^2-M^2+(-)i\epsilon}(2\pi)^4\delta^{(4)}(p-p')$ $+2\pi\lvert 2p_0\rvert^{1/2}\delta(p^2-M^2)\tilde{f}(p,p',t)e^{i(p_0-p_0')\tilde{t}_f}2\pi\lvert 2p_0'\rvert^{1/2}\delta(p'^2-M^2)$
$+(-)$ve-freq. Wightman	$i\Delta^0_{>(<)}(p,p',\tilde{t}_f;\tilde{t}_i) = 2\pi\theta(+(-)p_0)\delta(p^2-M^2)(2\pi)^4\delta^{(4)}(p-p')$ $+2\pi\lvert 2p_0\rvert^{1/2}\delta(p^2-M^2)\tilde{f}(p,p',t)e^{i(p_0-p_0')\tilde{t}_f}2\pi\lvert 2p_0'\rvert^{1/2}\delta(p'^2-M^2)$
Retarded (Advanced)	$i\Delta^0_{\mathrm{R(A)}}(p,p') = \frac{i}{(p_0+(-)i\epsilon)^2-\mathbf{p}^2-M^2}(2\pi)^4\delta^{(4)}(p-p')$
Pauli-Jordan	$i\Delta^0(p,p') = 2\pi\varepsilon(p_0)\delta(p^2-M^2)(2\pi)^4\delta^{(4)}(p-p')$
Hadamard	$i\Delta^0_1(p,p',\tilde{t}_f;\tilde{t}_i) = 2\pi\delta(p^2-M^2)(2\pi)^4\delta^{(4)}(p-p')$ $+2\pi\lvert 2p_0\rvert^{1/2}\delta(p^2-M^2)2\tilde{f}(p,p',t)e^{i(p_0-p_0')\tilde{t}_f}2\pi\lvert 2p_0'\rvert^{1/2}\delta(p'^2-M^2)$
Principal-part	$i\Delta^0_{\mathcal{P}}(p,p') = \mathcal{P}\frac{i}{p^2-M^2}(2\pi)^4\delta^{(4)}(p-p')$

Diagrammatics

$$\mathcal{L} \;=\; \tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi \;-\; \tfrac{1}{2}M^2\Phi^2 \;+\; \partial_\mu\chi^\dagger\partial^\mu\chi \;-\; m^2\chi^\dagger\chi \;-\; g\Phi\chi^\dagger\chi$$



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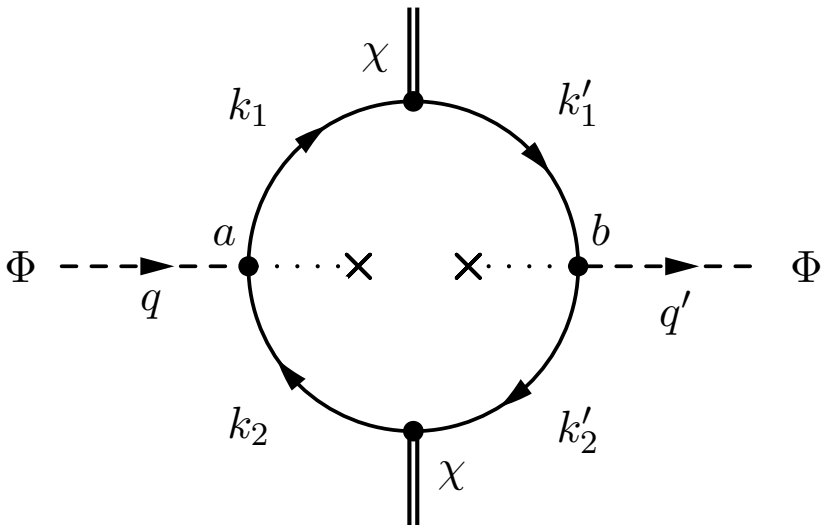

1. time-dependent, energy-non-conserving vertices:

$$\sim\; -ig\frac{t}{2\pi}\mathrm{sinc}\left[\left(\sum_i p_i^0\right)\frac{t}{2}\right]\delta^{(3)}\left(\sum_i \mathbf{p}_i\right)$$

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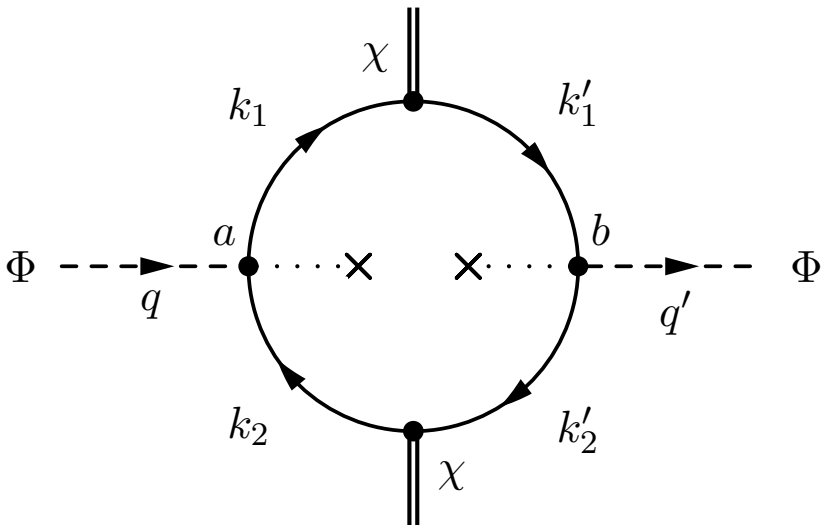

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2. momentum-non-conserving, non-homogeneous free propagators

Master Time Evolution Equations

Partially inverting the CTP Schwinger–Dyson equation:

$$\partial_t f(\mathbf{p}+\tfrac{\mathbf{P}}{2},\mathbf{p}-\tfrac{\mathbf{P}}{2},t) - \iint \frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{d}P_0}{2\pi}\, e^{-iP_0 t}\, 2\,\mathbf{p}\cdot\mathbf{P}\,\theta(p_0)\Delta_<(p+\tfrac{P}{2},p-\tfrac{P}{2},t;0)$$
$$+ \iint \frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{d}P_0}{2\pi}\, e^{-iP_0 t}\,\theta(p_0)\Big(\mathscr{F}(p+\tfrac{P}{2},p-\tfrac{P}{2},t;0) + \mathscr{F}^*(p-\tfrac{P}{2},p+\tfrac{P}{2},t;0)\Big)$$
$$= \iint \frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{d}P_0}{2\pi}\, e^{-iP_0 t}\,\theta(p_0)\Big(\mathscr{C}(p+\tfrac{P}{2},p-\tfrac{P}{2},t;0) + \mathscr{C}^*(p-\tfrac{P}{2},p+\tfrac{P}{2},t;0)\Big)$$

Force and collision terms:

$$\mathscr{F}(p+\tfrac{P}{2},p-\tfrac{P}{2},t;0) \equiv -\int\frac{\mathrm{d}^4 q}{(2\pi)^4}\, i\Pi_{\mathcal{P}}(p+\tfrac{P}{2},q,t;0)\, i\Delta_<(q,p-\tfrac{P}{2},t;0),$$
$$\mathscr{C}(p+\tfrac{P}{2},p-\tfrac{P}{2},t;0) \equiv \frac{1}{2}\int\frac{\mathrm{d}^4 q}{(2\pi)^4}\Big[i\Pi_>(p+\tfrac{P}{2},q,t;0)\, i\Delta_<(q,p-\tfrac{P}{2},t;0)$$
$$- i\Pi_<(p+\tfrac{P}{2},q,t;0)\Big(i\Delta_>(q,p-\tfrac{P}{2},t;0) - 2i\Delta_{\mathcal{P}}(q,p-\tfrac{P}{2},t;0)\Big)\Big]$$

No nested Poisson brackets as in gradient expansion of Kadanoff–Baym equations.

Simple Example

Time-Dependent Width

- $\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}M^2\Phi^2 + \partial_\mu\chi^\dagger\partial^\mu\chi - m^2\chi^\dagger\chi - g\Phi\chi^\dagger\chi$
- $t < 0$: Φ's and χ's in non-interacting equilibria at same temperature
- $t = 0$: interactions switched on

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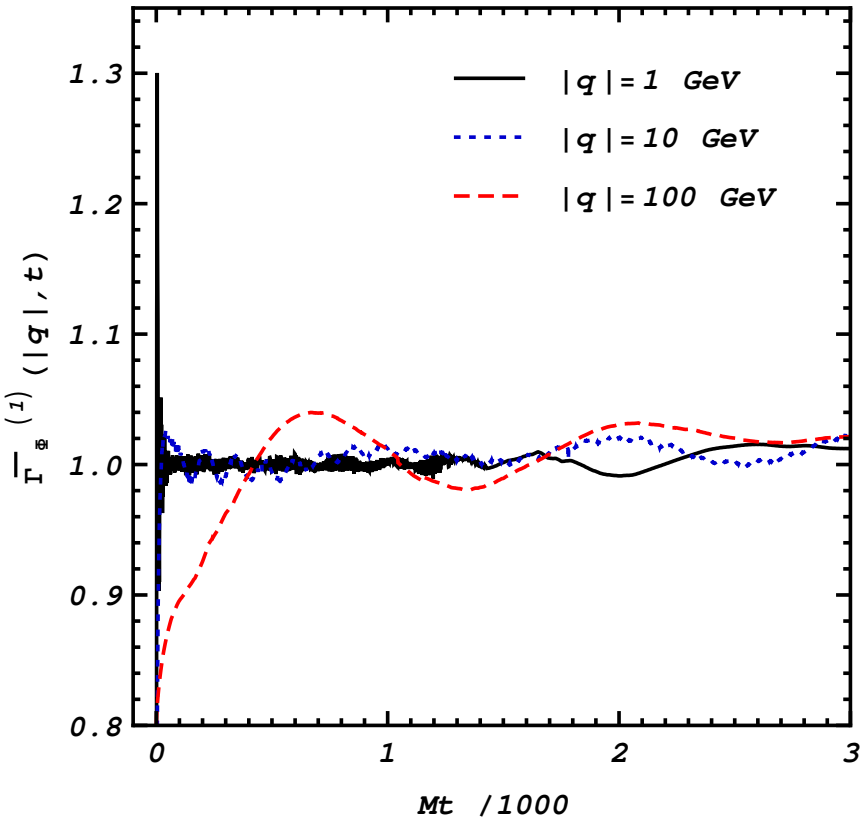

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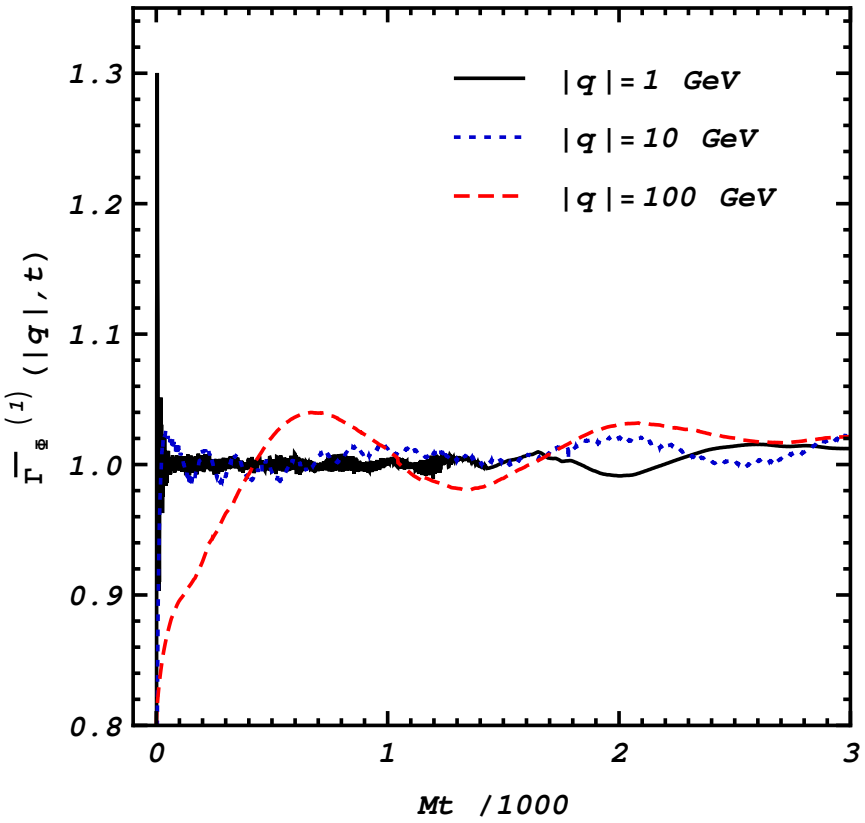
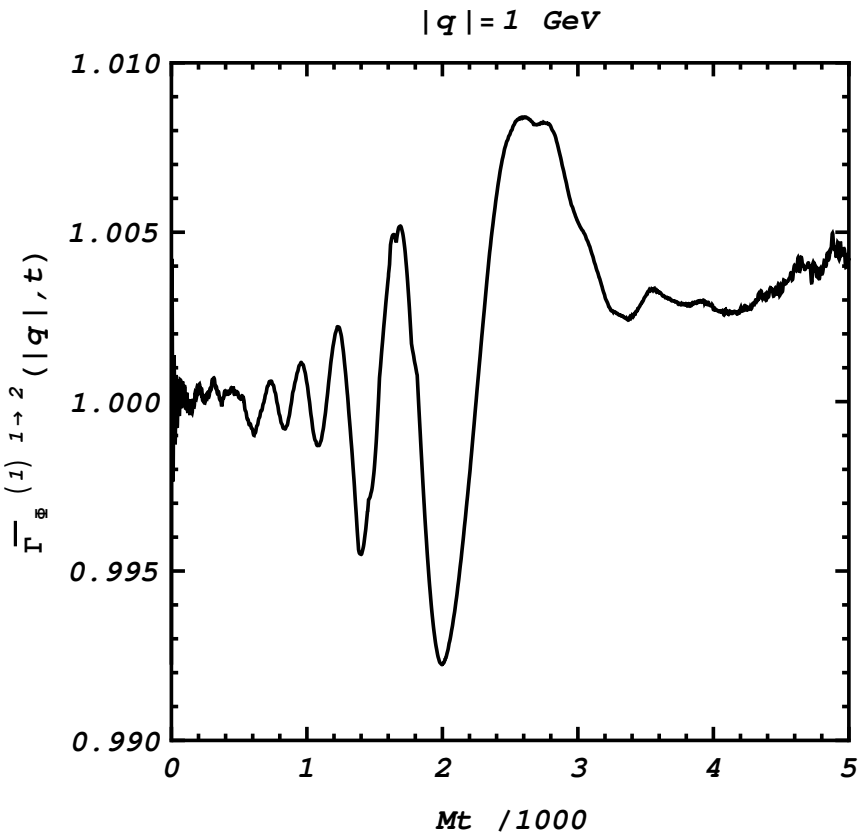




Simple Example

Time Evolution Equations

Truncating the master time evolution equations in a loopwise sense:

$$
\begin{aligned}
\partial_t f_\Phi(|\mathbf{p}|,t) \;=\; & -\frac{g^2}{2}\sum_{\{\alpha\}=\pm 1}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{1}{2E_\Phi(\mathbf{p})}\frac{1}{2E_\chi(\mathbf{k})}\frac{1}{2E_\chi(\mathbf{p}-\mathbf{k})} \\
& \times\; \frac{t}{2\pi}\,\mathrm{sinc}\left[\Big(\alpha E_\Phi(\mathbf{p})-\alpha_1 E_\chi(\mathbf{k})-\alpha_2 E_\chi(\mathbf{p}-\mathbf{k})\Big)\frac{t}{2}\right] \\
& \times\; \left\{\pi+2\mathrm{Si}\left[\Big(\alpha E_\Phi(\mathbf{p})+\alpha_1 E_\chi(\mathbf{k})+\alpha_2 E_\chi(\mathbf{p}-\mathbf{k})\Big)\frac{t}{2}\right]\right\} \\
& \times\; \Big\{\big[\theta(-\alpha)+f_\Phi(|\mathbf{p}|,t)\big]\big[\theta(\alpha_1)\big(1+f_\chi(|\mathbf{k}|,t)\big)+\theta(-\alpha_1)f_\chi^C(|\mathbf{k}|,t)\big] \\
& \qquad \times\; \big[\theta(\alpha_2)\big(1+f_\chi^C(|\mathbf{p}-\mathbf{k}|,t)\big)+\theta(-\alpha_2)f_\chi(|\mathbf{p}-\mathbf{k}|,t)\big] \\
& -\; \big[\theta(\alpha)+f_\Phi(|\mathbf{p}|,t)\big]\big[\theta(\alpha_1)f_\chi(|\mathbf{k}|,t)+\theta(-\alpha_1)\big(1+f_\chi^C(|\mathbf{k}|,t)\big)\big] \\
& \qquad \times\; \big[\theta(\alpha_2)f_\chi^C(|\mathbf{p}-\mathbf{k}|,t)+\theta(-\alpha_2)\big(1+f_\chi(|\mathbf{p}-\mathbf{k}|,t)\big)\big]\Big\}
\end{aligned}
$$

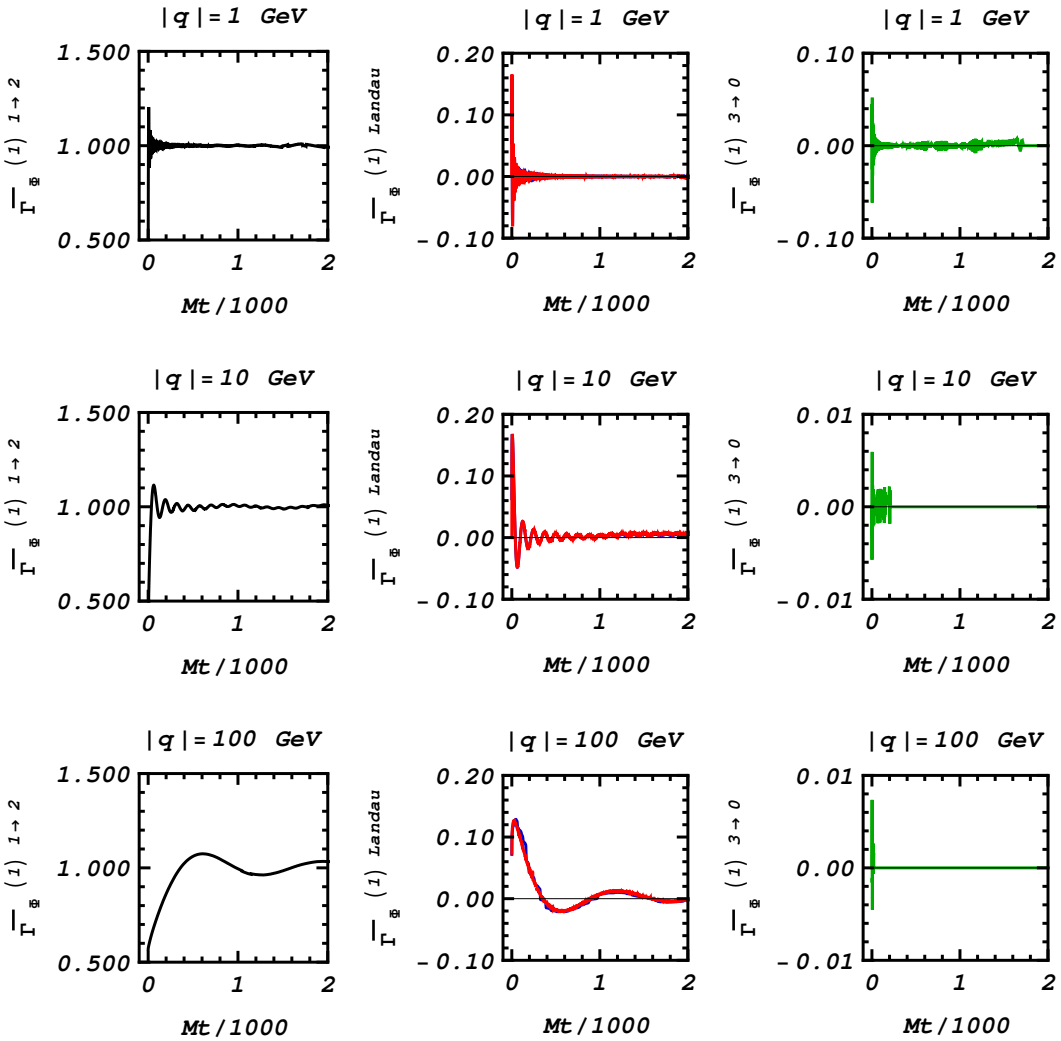
Still valid to all orders in gradient expansion.

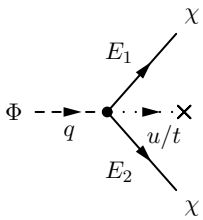
Conclusions

- Obtain master time evolution equations valid to all orders in gradient expansion and to all orders in perturbation theory.
- Loopwise truncation of time evolution equations resum all loop insertions and remain valid to all orders in gradient expansion.
- Underlying non-equilibrium field theory free of pinch singularities.
- Non-homogeneous free propagators and time-dependent vertices break space-time translational invariance from tree-level.
- Early-time dynamics consistently describe energy-violating processes, leading to non-Markovian evolution of memory effects.

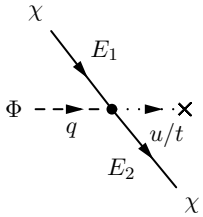
Backup Slides

Simple Example: Evanescent Processes

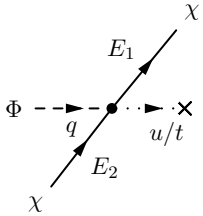


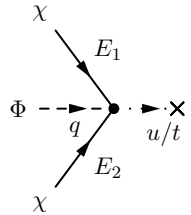


$1 \to 2$ decay (left)



$2 \to 1$ Landau damping (center)

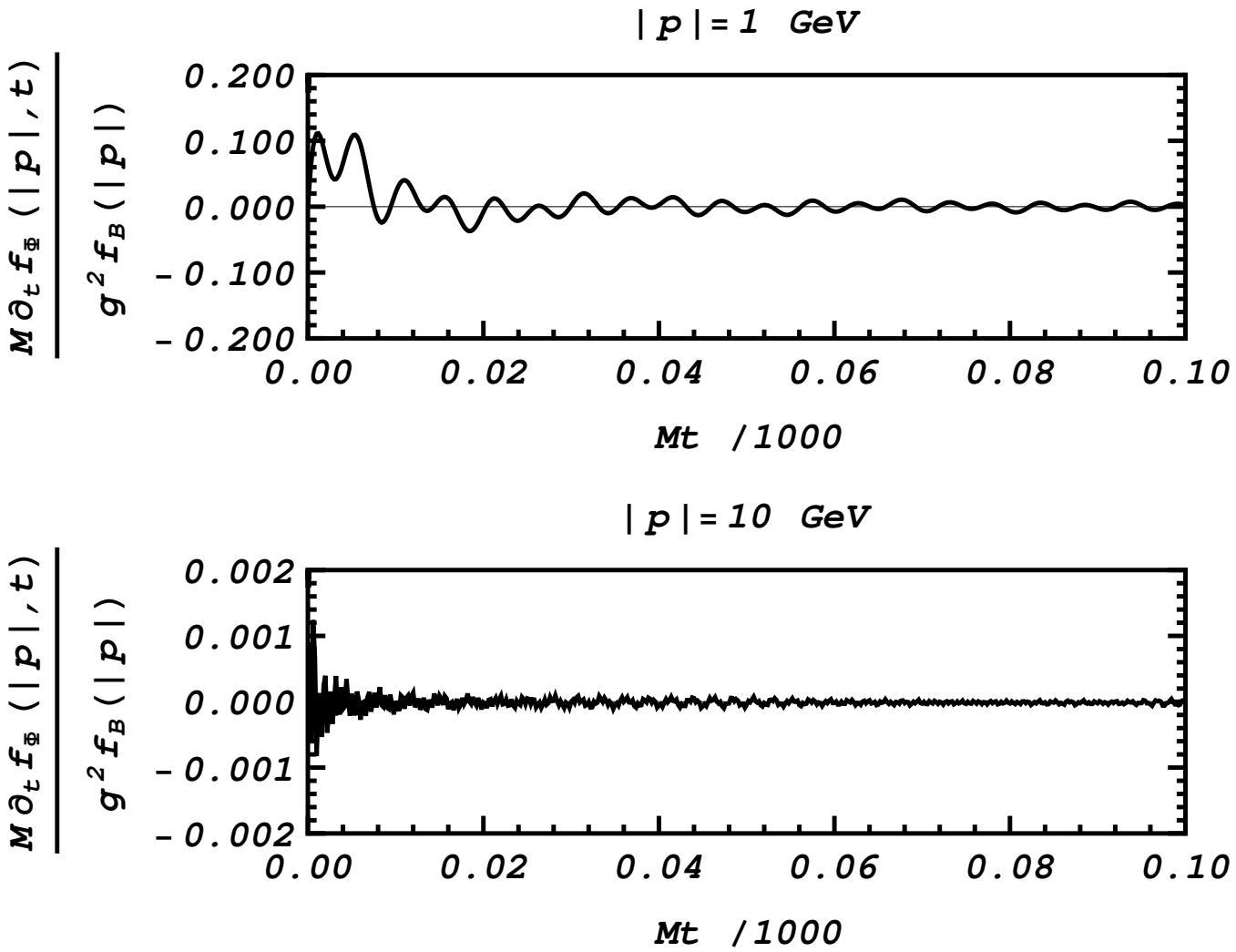




$3 \to 0$ total annihilation (right)

Backup Slides

Not Just a Complicated Zero



Backup Slides

Pinch Singularities: $\delta^2(\sum p_0)$

- early times: $\delta^2(\sum p_0) \;\rightarrow\; \delta_t^2(\sum p_0)$
- intermediate times:
  - pinch singularities grow: $t\delta(\sum p_0)$
  - equilibration occurs: $f(t) - f_{eq} \;=\; \delta f(t) \;=\; \delta f(0)e^{-\Gamma t}$
- late times: $f \;\rightarrow\; f_{eq}$ and pinch singularities cancel

$\Leftarrow$ finite time domain

$\Leftarrow$ f's in free propagators evaluated at time of observation