Quantum Effects From Boundaries in de Sitter and anti-de Sitter spaces

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- Quantum vacuum and the Casimir effect
- Quantum effects in de Sitter spacetime
- Quantum effects in anti-de Sitter spacetime
- Applications to Randall-Sundrum-type braneworlds
- Conclusions

Maximally symmetric solutions of Einstein equations

- Minkowski spacetime: $T_{ik} = 0$

$$ds^2 = dt^2 - \sum_{l=1}^{D} (dx^l)^2$$

Cosmological constant

- De Sitter (dS) spacetime: $T_{ik} = \Lambda g_{ik}, \ \Lambda > 0$

$$ds^2 = dt^2 - e^{2t/\alpha} \sum_{l=1}^{D} (dx^l)^2 = \left(\frac{\alpha}{\tau}\right)^2 \left[d\tau^2 - \sum_{l=1}^{D} (dx^l)^2 \right]$$

$\tau = -\alpha e^{-t/\alpha} \leftarrow$ Conformal time $\qquad \alpha^2 = D(D-1)/2\Lambda$

- Anti-de Sitter (AdS) spacetime: $T_{ik} = \Lambda g_{ik}, \ \Lambda < 0$

$$ds^2 = e^{-2\frac{y}{\alpha}} [dt^2 - \sum_{l=1}^{D-1} (dx^l)^2] - dy^2 = \left(\frac{\alpha}{z}\right)^2 \left[d\tau^2 - \sum_{l=1}^{D} (dx^l)^2 \right]$$

$$x^D = z, \ \ z = \alpha e^{y/\alpha}$$

Quantum fields in curved spacetime

- Quantum fields are mainly considered in Minkowski spacetime

- From late 60s $\Rightarrow$ Quantum Field Theory in curved spacetime

  Gravity is treated as a classical field

- Background Fields: Cosmological models, Schwarzshild spacetime, …

- dS and AdS spacetimes have attracted special attention

Why de Sitter space-time?

- De Sitter (dS) space-time is the maximally symmetric solution of the Einstein equations with the positive cosmological constant
- Due to the high symmetry numerous physical problems are exactly solvable on dS background and a
- Better understanding of physical effects in this bulk could serve as a handle to deal with more complicated geometries
- In most inflationary models an approximately dS spacetime is employed to solve a number of problems in standard cosmology
- At the present epoch the Universe is accelerating and can be well approximated by a world with a positive cosmological constant

Quantum vacuum

- Among the most important consequences of quantum field theory is the prediction of non-trivial properties of the vacuum
- Vacuum is a state of quantum field with zero number of quanta

Particle number operator $\longrightarrow$ $\hat{n}|0\rangle = 0$

- Field and particle number operators do not commute

Field operator $\longrightarrow$ $[\hat{\varphi}\,\hat{n}] \neq 0$

- In the vacuum state the field fluctuates:

Vacuum or zero-point fluctuations of a quantum field

Quantum vacuum in external fields

- Properties of the vacuum are manifested in its response to external influences (vacuum polarization in external fields, particle creation from the vacuum)
- Among the most important topics is the investigation of the structure of quantum vacuum in external gravitational fields
- Applications: Physics of the Early Universe, Black Holes, Large Scale Structure of the Universe, CMB Temperature Anisotropies

Casimir effect

- Boundaries can serve as a simple model of external fields

•The presence of boundaries modifies the spectrum of vacuum fluctuations → The vacuum energy is changed → Forces arise acting on boundaries

Casimir effect

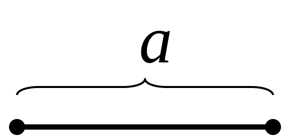
Example: 2D massless scalar field with Dirichlet BC: $\varphi(t,x)=0,\ \ x=0,a$

Boundaries are absent

Momentum $-\infty < p_x < +\infty$

Vacuum energy $E_\infty = \frac{1}{2}\int_{-\infty}^{+\infty} dp_x\omega,\ \ \omega = |\,p_x\,|$

Boundaries are present



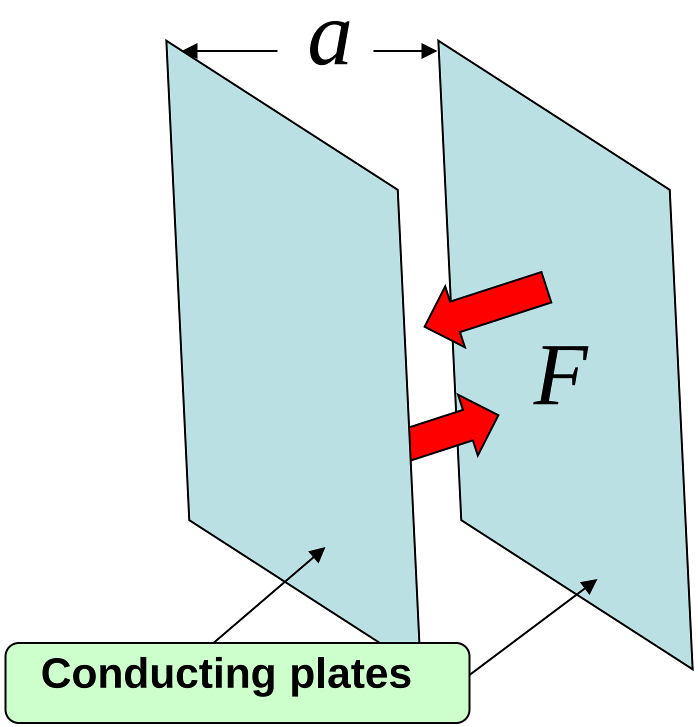
Wave number $p_x = \pi\, n/a,\ \ \ n = 1,2,3,\ldots$

Vacuum energy $E_a = \frac{\pi}{2a}\sum_{n=1}^{\infty} n$

Change in the vacuum energy: $\Delta E(a) = E_a - aE_\infty = -\frac{\pi}{24a}$

Vacuum force: $F = -\frac{\partial \Delta E}{\partial a}$

Ideal metal parallel plates



Two conducting neutral parallel plates in the vacuum attract by the force per unit surface (Casimir,1948)

$$F = \frac{\pi^2 \hbar c}{240 a^4}$$

Casimir effect on curved backgrounds

- An interesting topic in the investigations of the Casimir effect is its dependence on the geometry of the background spacetime

- Relevant information is encoded in the vacuum fluctuations spectrum and analytic solutions can be found for highly symmetric geometries only

- Maximally symmetric curved backgrounds: dS and AdS spacetimes

dS Space: Field and Boundary Geometry

- Field: Scalar field with curvature coupling parameter ξ

$$\left(\nabla_l \nabla^l + m^2 + \xi R\right)\varphi = 0$$

- Bulk geometry: (D+1)-dimensional dS space-time

$$ds^2 = dt^2 - e^{2t/\alpha}\sum_{i=1}^{D}(dz^i)^2 = \alpha^2\eta^{-2}\left[d\eta^2 - \sum_{i=1}^{D}(dz^i)^2\right]$$

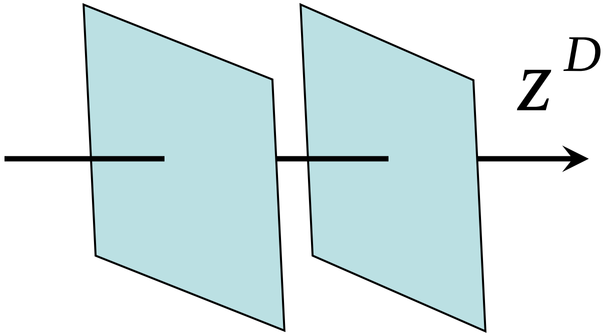
$$\eta = \alpha e^{-t/\alpha},\, R = D(D+1)/\alpha^2,\ \Lambda = D(D-1)/2\alpha^2$$

Cosmological constant

- Boundary geometry: Two infinite parallel plates with Robin BC

$$(1 + \beta_j n^l \nabla_l)\varphi(x) = 0$$

$$z^D = a_1,\ z^D = a_2,\ a_1 < a_2$$



Characteristics of the vacuum state

- Wightman function $W(x, x') = \langle 0|\varphi(x)\varphi(x')|0\rangle$
- Vacuum expectation value (VEV) of the field squared $\langle \varphi^2 \rangle$
- VEV of the energy-momentum tensor $\langle 0|T_{ik}|0\rangle$
- Vacuum forces acting on the boundaries

Evaluation scheme

$$W(x, x') = \sum_{\sigma} \varphi_\sigma(x)\varphi_\sigma^*(x'), \quad \{\varphi_\sigma(x), \varphi_\sigma^*(x)\}$$

Complete set of solutions to the classical field equation

$$\langle \varphi^2 \rangle = \lim_{x' \to x} W(x, x')$$

$$\langle 0|T_{ik}|0\rangle = \lim_{x' \to x} \partial_i \partial'_k W(x, x') + [(\xi - 1/4)g_{ik}\nabla_l \nabla^l - \xi \nabla_i \nabla_k - \xi R_{ik}]\langle \varphi^2 \rangle$$

Vacuum force per unit surface of the plate $= -\left\langle T_D^D \right\rangle_{z^D = a_j}$

Eigenfunctions

- We assume that the field is prepared in the Bunch-Davies vacuum state
- Eigenfunctions in the region between the plates, $a_1 < z^D < a_2$

$$\varphi_\sigma(x) = C_\sigma \eta^{D/2} H_\nu^{(1)}(\eta K) \cos[k_D(z^D - a_1) + \alpha_1(k_D)] e^{i\mathbf{k}\cdot\mathbf{z}}$$

$$\eta = |\tau|,\ K = \sqrt{k^2 + k_D^2},\ \ e^{2i\alpha_1(x)} = \frac{i\beta_1 x - 1}{i\beta_1 x + 1}$$

$$\nu = [D^2/4 - D(D+1)\xi - m^2\alpha^2]^{1/2}$$

- From the BC on the second plate ($a = a_2 - a_1$)

$$(1 - b_1 b_2 y^2)\sin y - (b_1 + b_2) y \cos y = 0,\ b_j = \beta_j/a,\ \ y = k_D a$$

Eigenvalues: $k_D = \lambda_n/a$

Wightman function

- Mode sum: $W(x,x') = \sum_{\sigma} \varphi_\sigma(x)\varphi^*_\sigma(x'), \ \ \sigma = (\mathbf{k}, n)$

- Contains summation over λ_n
- Summation formula

$$\sum_{n=1}^{\infty} \frac{\pi\lambda_n f(\lambda_n)}{\lambda_n + \sin(\lambda_n)\cos[\lambda_n + 2\alpha_1(\lambda_n/a)]} = \int_0^\infty dz f(z) + i\int_0^\infty dz \frac{f(iz) - f(-iz)}{\frac{(b_1z-1)(b_2z-1)}{(b_1z+1)(b_2z+1)}e^{2z} - 1}$$

- Wightman function is decomposed as

$$W(x,x') = W_{dS}(x,x') + W_b(x,x')$$

WF for boundary-free dS spacetime

Boundary-induced part

Renormalization

- For points away from the boundaries the divergences are the same as for the dS spacetime without boundaries

- We have explicitly extracted the part $W_{dS}(x, x')$

- The renormalization of the VEVs is reduced to the renormalization of the part corresponding to the geometry without boundaries

VEV of the energy-momentum tensor: Diagonal components

- Vacuum energy density: $\langle T_0^0 \rangle$
- Parallel stresses = - vacuum effective pressures parallel to the boundaries:

$$\langle T_l^l \rangle, \; l = 1, \ldots, D-1, \quad \langle T_l^l \rangle \neq \langle T_0^0 \rangle$$

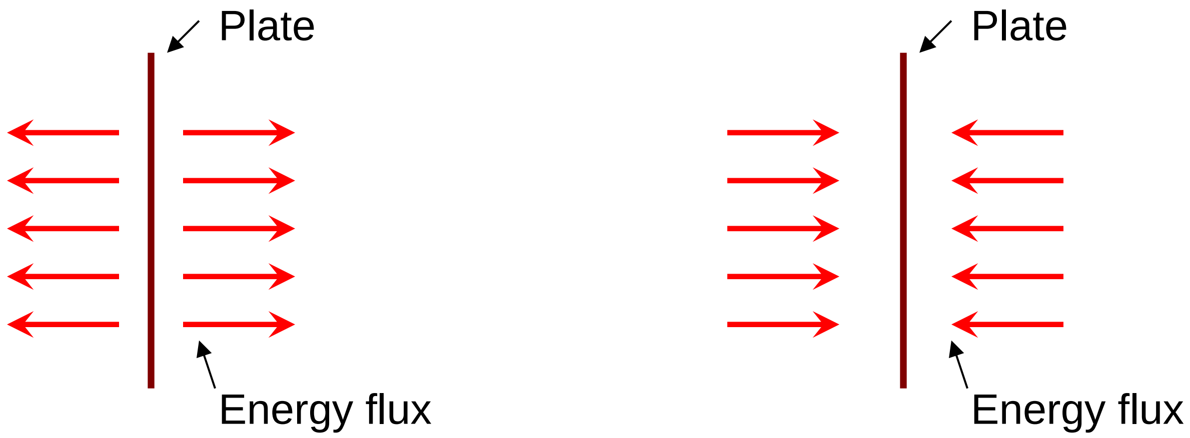
- In Minkowski spacetime $\Rightarrow$ Parallel stresses = Energy density
- Normal stress = - vacuum effective pressure normal to the boundaries:

$$\langle T_D^D \rangle \Leftarrow$$

Determines the vacuum force acting on the boundaries

Off-diagonal component

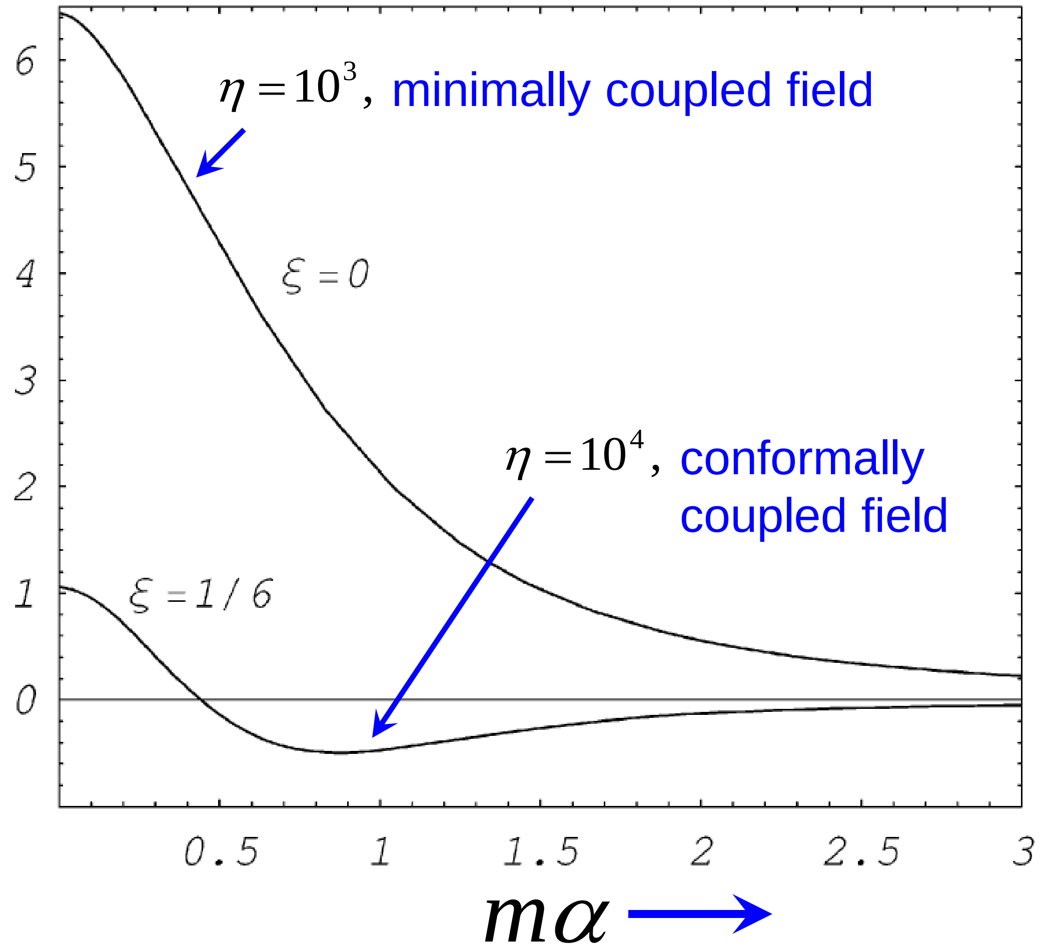
- Nonzero off-diagonal component : $\left\langle T_0^{\ D} \right\rangle$
- Corresponds to energy flux along the direction perpendicular to the plates
- Depending on the values of the coefficients in the boundary conditions and of the field mass, this flux can be positive or negative
- Geometry of a single boundary



Boundary-free energy density

Renormalized vacuum energy density in D=3 dS space-time

$$\eta\alpha^{D+1}\langle T_0^0\rangle_{\mathrm{dS}}$$



Casimir forces

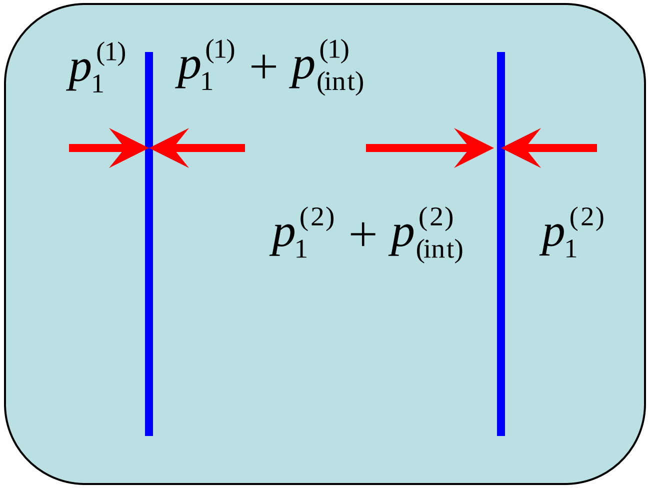
- The vacuum force acting per unit surface of the plate at $z^D = a_j$ is determined by the ${}^{D}_{D}$ component of the vacuum energy-momentum tensor evaluated at this point

- For the region between the plates, the corresponding effective pressures can be written as

$$p^{(j)} = p_1^{(j)} + p_{(\mathrm{int})}^{(j)}, \qquad j = 1, 2$$

Pressure for a single plate

Pressure induced by the the second plate



Casimir forces: Properties

- Depending on the values of the coefficients in the boundary conditions, the effective pressures can be either positive or negative, leading to repulsive or to attractive forces

- For $\beta_1 \neq \beta_2$ the Casimir forces acting on the left and on the right plates are different

- In the limit $\alpha \to \infty$ the corresponding result for the geometry of two parallel plates in Minkowski spacetime is obtained

$$p_{(\mathrm{int})}^{(j)} \approx -\frac{2(4\pi)^{-D/2}}{\Gamma(D/2)} \int_m^\infty du \, \frac{u^2(u^2-m^2)^{D/2-1}}{\frac{(\beta_1 u-1)(\beta_2 u-1)}{(\beta_1 u+1)(\beta_2 u+1)} e^{2au} - 1}$$

Casimir forces: Asymptotics

- Proper distance between the plates is smaller than the curvature radius of the background spacetime: $a/\eta << 1$

$$p_{(\mathrm{int})}^{(j)} \approx -\frac{2(\eta/\alpha)^{D+1}}{(4\pi)^{D/2}\Gamma(D/2)}\int_0^\infty dx \frac{x^D}{c_1(x)c_2(x)e^{2ax}-1}$$

  - ➡In the case of Dirichlet BC on one plate and non-Dirichlet one on the other the vacuum force at small distance is repulsive
  - ➡In all other cases the forces are attractive

- At large distances, $a/\eta >> 1$, and for positive values of ν

$$p_{(\mathrm{int})}^{(j)} \approx -\frac{2\alpha^{-D-1}g_\nu^{(j)}}{\pi^{D/2+1}(2a/\eta)^{D-2\nu+2}} \qquad \nu = [D^2/4 - D(D+1)\xi - m^2\alpha^2]^{1/2}$$

- At large distances and for imaginary values of ν

$$p_{(\mathrm{int})}^{(j)} \approx -\frac{4\alpha^{-D-1}|g_\nu^{(j)}|}{\pi^{D/2+1}(2a/\eta)^{D+2}}\cos[2|\nu|\ln(2a/\eta)+\phi_{(j)}]$$

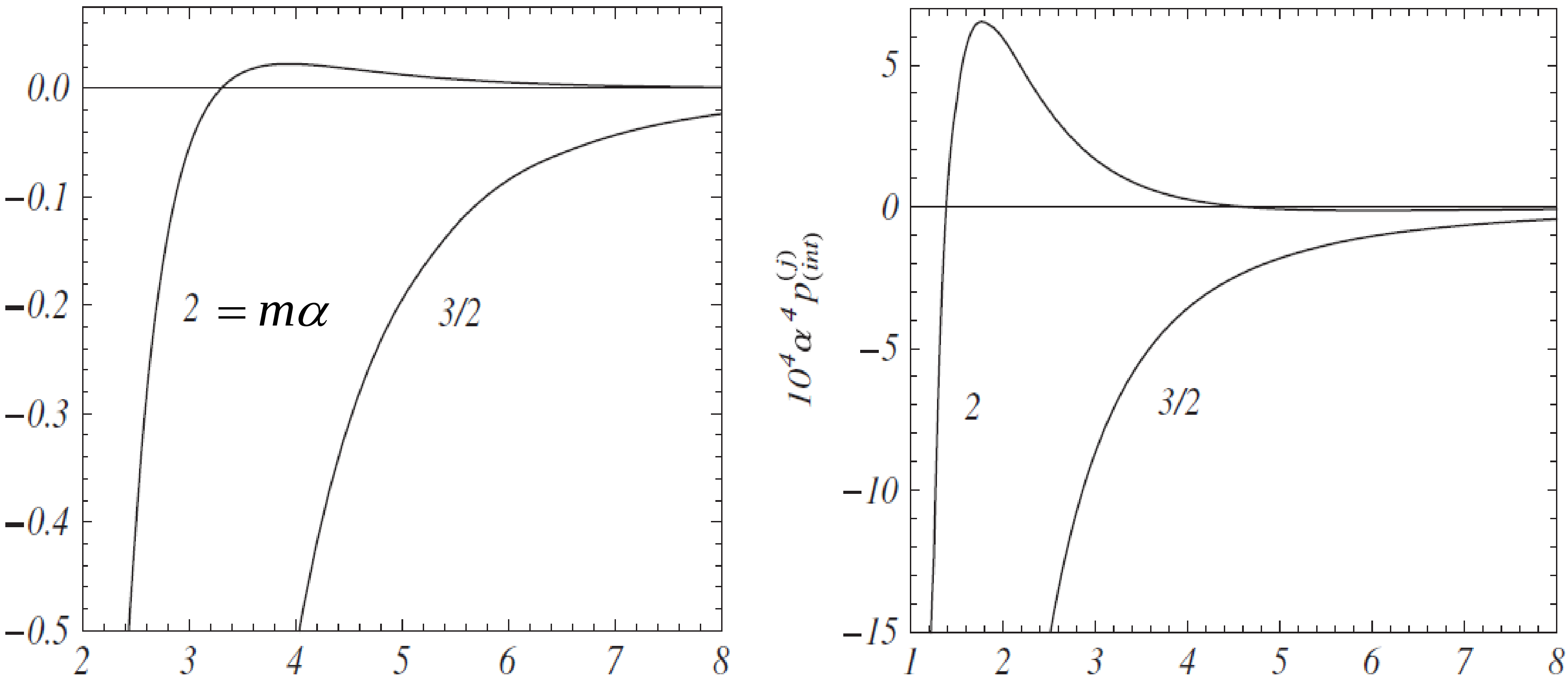
$$p_{(\mathrm{int})}^{(j)} \sim \exp[(D+2)t/\alpha]\cos[2|\nu|t/\alpha+\psi_p]$$

Casimir forces between the plates for a *D* = 3 minimally coupled scalar field

$$\xi = 0$$

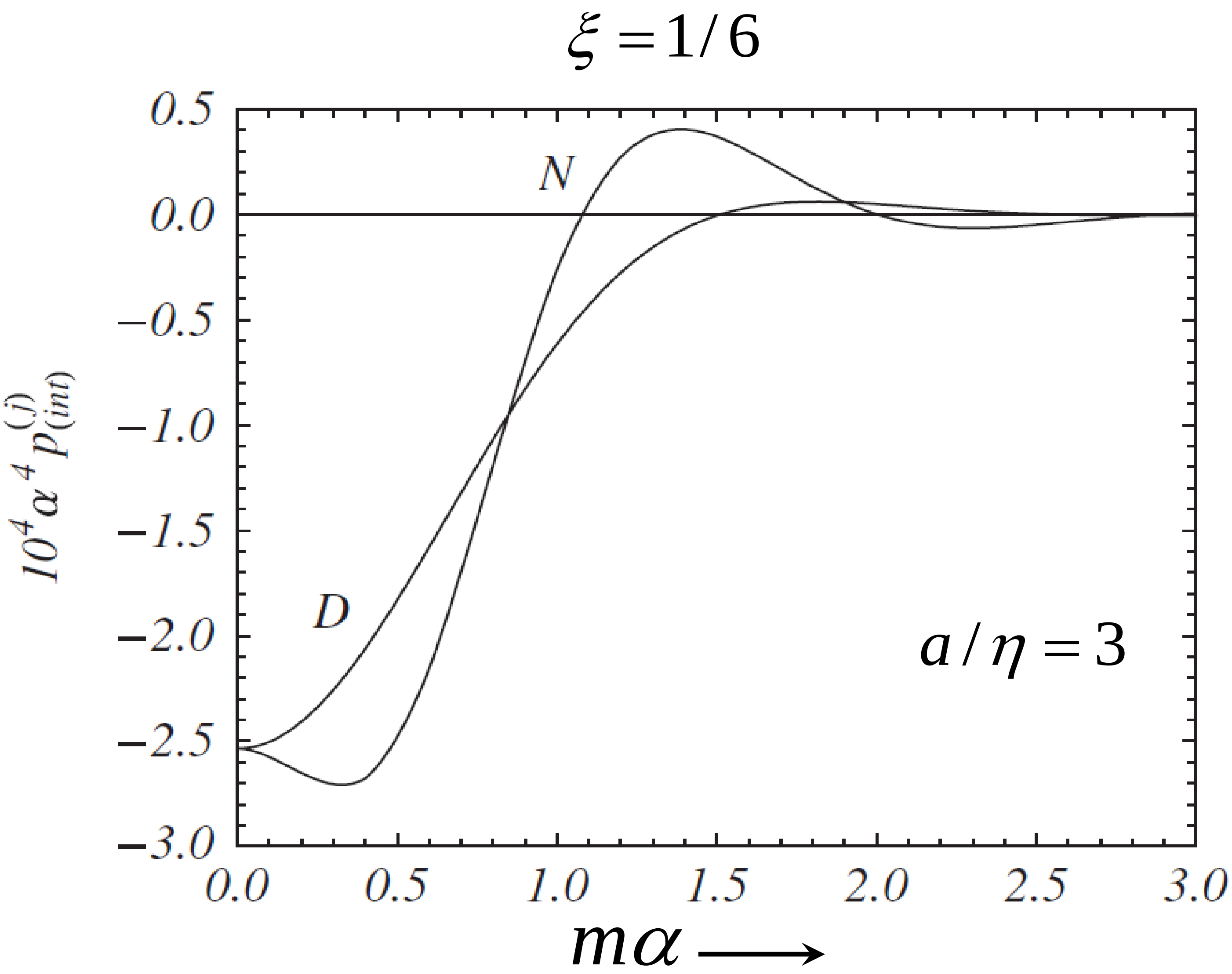
Dirichlet BC

Neumann BC



a/η ⟶

Casimir forces between the plates for a $D = 3$ conformally coupled scalar field



Conclusions

- For the proper distances between the plates larger than the curvature radius of the dS spacetime, $\alpha a/\eta \gtrsim \alpha$, the gravitational field essentially changes the behavior of the Casimir forces compared with the case of the plates in Minkowski spacetime

- In particular, the forces may become repulsive at large separations between the plates

- A remarkable feature of the influence of the gravitational field is the oscillatory behavior of the Casimir forces at large distances, which appears in the case of imaginary ν

- In this case, the values of the plate distance yielding zero Casimir force correspond to equilibrium positions. Among them, the positions with negative derivative of the force with respect to the distance are locally stable

Conclusions

- In dS spacetime, the decay of the VEVs at large separations between the plates is power law (monotonical or oscillating), independently of the field mass

- This is quite remarkable and clearly in contrast with the corresponding features of the same problem in a Minkowski bulk. The interaction forces between two parallel plates in Minkowski spacetime at large distances decay as $1/a^{D+1}$ for massless fields, and these forces are exponentially suppressed for massive fields by a factor of e^{-2ma}

Why anti-de Sitter?

- Questions of principal nature related to the quantization of fields propagating on curved backgrounds

- AdS spacetime generically arises as a ground state in extended supergravity and in string theories

- AdS/Conformal Field Theory correspondence: Relates string theories or supergravity in the bulk of AdS with a conformal field theory living on its boundary

- Braneworld models: Provide a solution to the hierarchy problem between the gravitational and electroweak scales. Naturally appear in string/M-theory context and provide a novel setting for discussing phenomenological and cosmological issues related to extra dimensions

Quantum effects in AdS bulk

- *Field:* Scalar field with an arbitrary curvature coupling parameter

$$(\nabla^i \nabla_i + m^2 + \zeta R)\varphi(x) = 0$$

- *Bulk geometry:* (D+1)-dimensional spacetime with the line element

$$ds^2 = e^{-2ky}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2$$

1/k AdS curvature radius

- *Boundaries (Branes):* Located at *y*=*a* and *y*=*b*
- *Boundary conditions*:

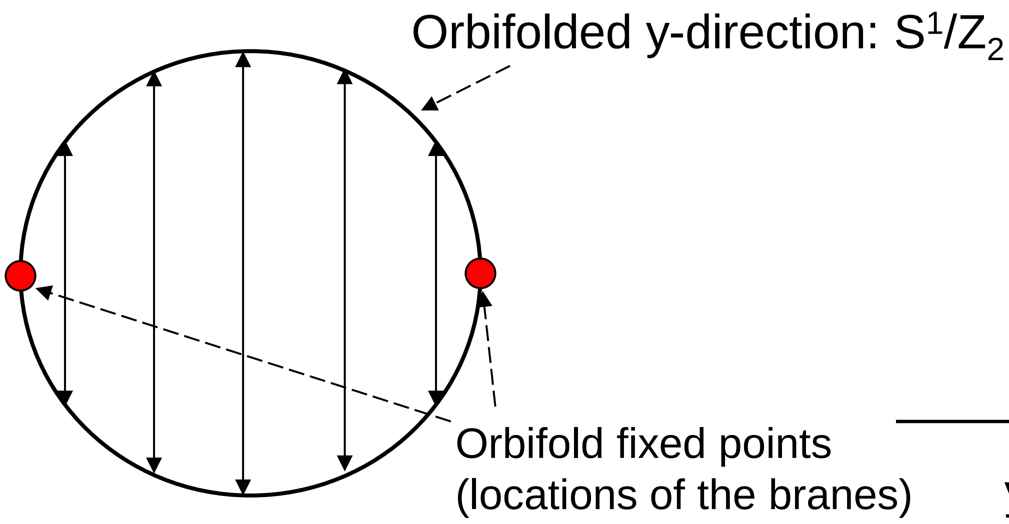
$$(\tilde{A}_y + \tilde{B}_y \partial_y)\varphi(x) = 0, \quad y = a, b$$

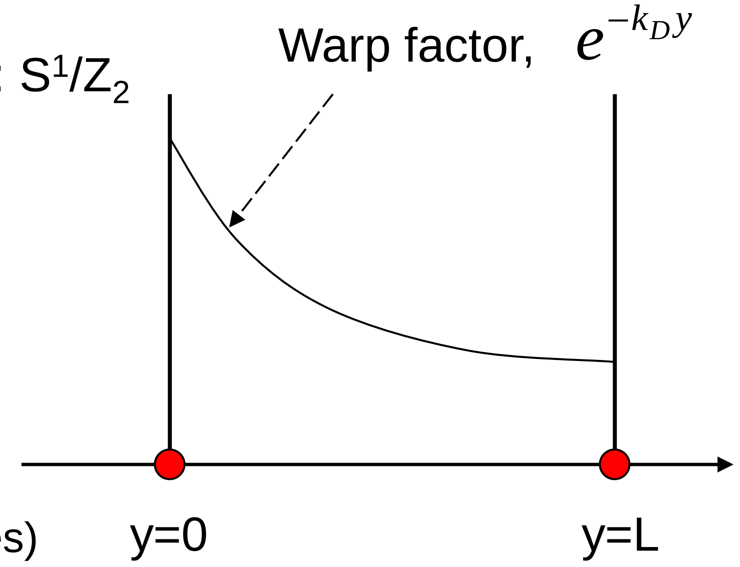
Randall-Sundrum braneworld models

Bulk geometry: 5-dimensional AdS spacetime

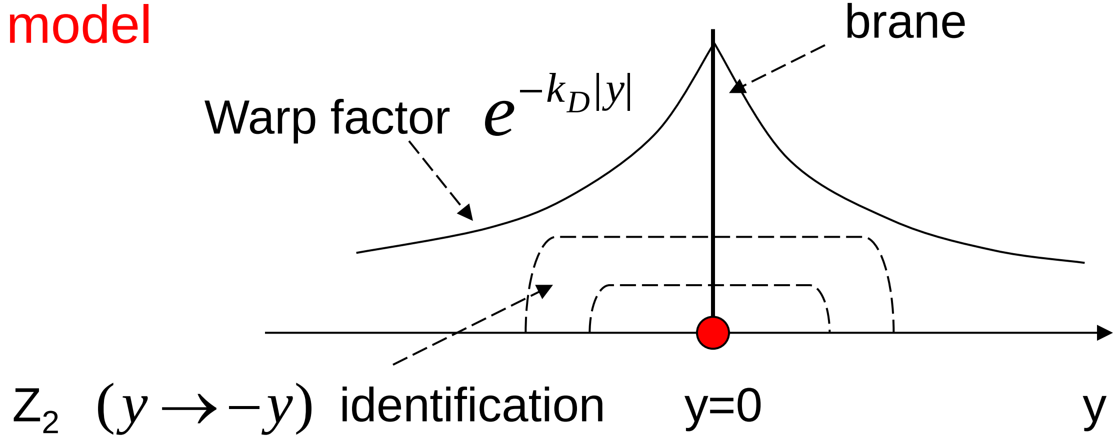
$$ds^2 = e^{-2ky}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2$$

RS two-brane model





RS single brane model



Energy-momentum tensor

- Decomposition of the EMT

$$\langle 0|T_{ik}|0\rangle = \langle T_{ik}\rangle_0 + \boxed{\text{Part induced by a single brane at } y=a} + \boxed{\text{Part induced by a single brane at } y=b} + \boxed{\text{Interference part}}$$

- Vacuum EMT is diagonal

$$\langle 0|T_i^k|0\rangle = \mathrm{diag}(\varepsilon,\ldots,-p_{||},\ldots,-p_{\perp}) \qquad p_{||} = -\varepsilon$$

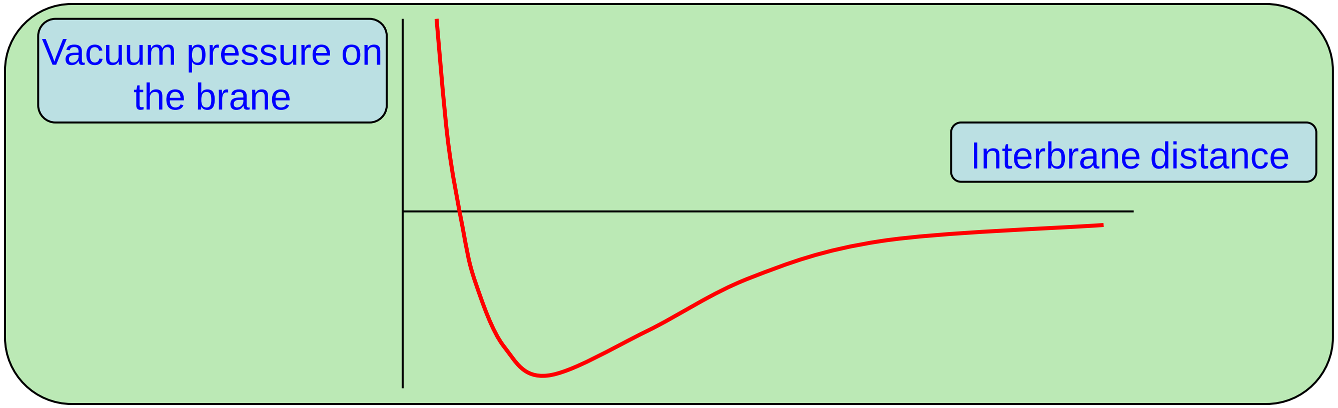
Vacuum energy density

Vacuum pressures in directions parallel to the branes

Vacuum pressure perpendicular to the branes

Vacuum forces

- At separations between the branes larger than the AdS curvature radius the vacuum forces are exponentially suppressed
- In dependence of the coefficients in the boundary conditions the vacuum interaction forces between the branes can be either attractive or repulsive



- Vacuum forces are repulsive for small distances and are attractive for large distances

Stabilization of the interbrane distance (radion field) by vacuum forces

Surface energy-momentum tensor

- Total vacuum energy per unit coordinate surface on the brane

$$E = \frac{1}{2}\sum_{\alpha} \omega_{\alpha}$$

- Volume energy in the bulk

$$E^{(v)} = \int d^{D+1}x\sqrt{|g|}\langle 0|T_0^{(v)0}|0\rangle$$

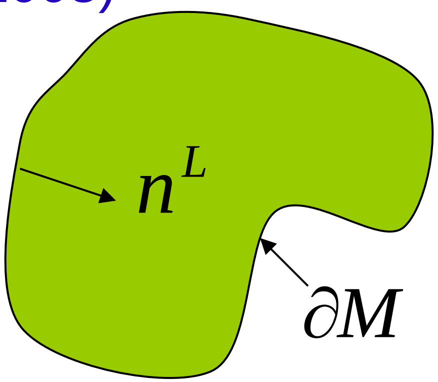
- In general $E^{(v)} \neq E$
- Difference is due to the presence of the surface energy located on the boundary (A.Romeo, A.A.Saharian, J.Phys.A,2002)
- For a scalar field on manifolds with boundaries the energy-momentum tensor in addition to the bulk part contains a contribution located on the boundary (A.A.Saharian, Phys.Rev.D,2003)

$$T_{ik}^{(s)} = \delta(x;\partial M)[\zeta\varphi^2 K_{ik} - (2\zeta - 1/2)h_{ik}\varphi n^l \nabla_l \varphi]$$

Extrinsic curvature tensor

Induced metric

Unit normal



Surface EMT and induced cosmological constant

- Vacuum expectation value of the surface EMT on the brane at *y=j*

$$\langle 0|T_M^{(s)N}|0\rangle = \mathrm{diag}(\varepsilon_j^{(s)},\ldots,-p_j^{(s)},\ldots), \quad \varepsilon_j^{(s)} = -p_j^{(s)}$$

- This corresponds to the generation of the cosmological constant on the branes by quantum effects
- Induced cosmological constant is a function on the interbrane distance, AdS curvature radius, and on the coefficients in the boundary conditions
- In dependence of these parameters the induced cosmological constant can be either positive or negative

Physics for an observer on the brane

- *D*-dimensional Newton's constant G_{Dj} measured by an observer on the brane at *y=j* is related to the fundamental *(D+1)*-dimensional Newton's constant G_{D+1} by the formula

$$G_{Dj} = \frac{(D-2)kG_{D+1}}{e^{(D-2)k(b-a)}-1} e^{(D-2)k(b-j)}$$

- For large interbrane distances the gravitational interactions on the brane *y=b* are exponentially suppressed. This feature is used in the Randall-Sundrum model to address the hierarchy problem
- Same mechanism also allows to obtain a naturally small cosmological constant on the brane generated by vacuum fluctuations

$$\Lambda_{Dj} = 8\pi G_{Dj}\varepsilon_j^{(s)} \sim 8\pi G_{Dj} M_{Dj}^D e^{-(D+2\nu)k(b-a)}$$

Cosmological constant on the brane *y=j*

Effective Planck mass on the brane *y=j*

$$\nu = \sqrt{D^2/4 - D(D+1)\zeta + m^2/k^2}$$

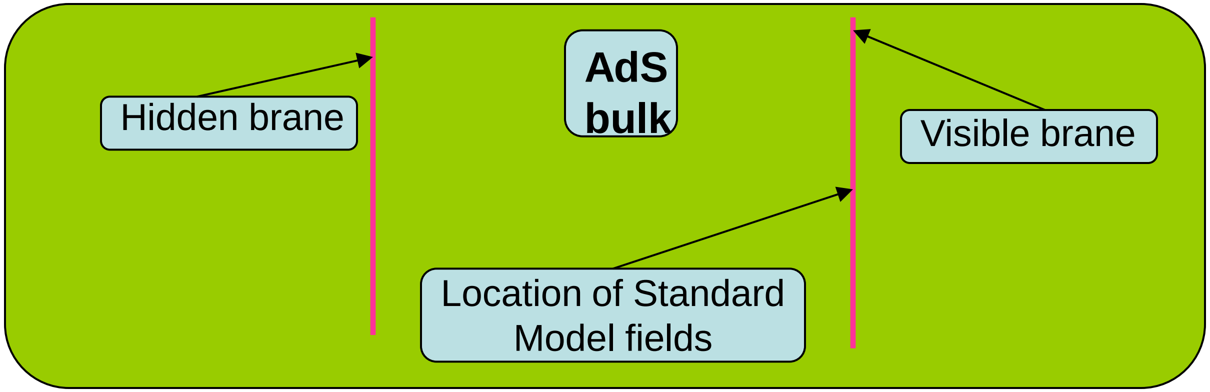
Induced cosmological constant in the Randall-Sundrum brane model

- For the Randall-Sundrum brane model D=4, the brane at *y=a* corresponds to the hidden brane and the brane at *y=b* corresponds to the visible brane

$$M_{D+1} \sim TeV, \quad M_{Db} = M_{Pl} \sim 10^{16} TeV$$

- Observed hierarchy between the gravitational and electrowek scales is obtained for $k(b-a) \approx 40$

- For this value of the interbrane distance the cosmological constant induced on the visible brane is of the right order of magnitude with the value implied by the cosmological observations



Conclusions

- For mixed boundary conditions the vacuum forces between the branes in AdS bulk can be either repulsive or attractive

- There is a region in the space of Robin parameters in which the forces are repulsive for small distances and are attractive for large distances, providing a possibility to stabilize interbrane distance by using vacuum forces

- Quantum fluctuations of a bulk scalar field induce surface densities of the cosmological constant type localized on the branes

- In the original Randall-Sundrum model for interbrane distances solving the hierarchy problem, the value of the cosmological constant on the visible brane by order of magnitude is in agreement with the value suggested by current cosmological observations without an additional fine tuning of the parameters

Thank You!