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Canonical Solutions of Variational Problems and Canonical Equations of Mechanics

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Abstract

The canonical (non-parametric) solutions of the variational problems for integral functionals are considered and the canonical solutions of variational problems of mechanics in Minkowski spaces are derived. By combining the variational principles of least action, flow, and hyperflow canonically invariant equations for the energy–momentum variable are obtained. From these equations the equations for the action and wave functions as a general solution of the combined variational problems of mechanics are derived. These equations are applicable for describing different types of particles and interactions and are summarized within the approach of general relativity.

1. INTRODUCTION

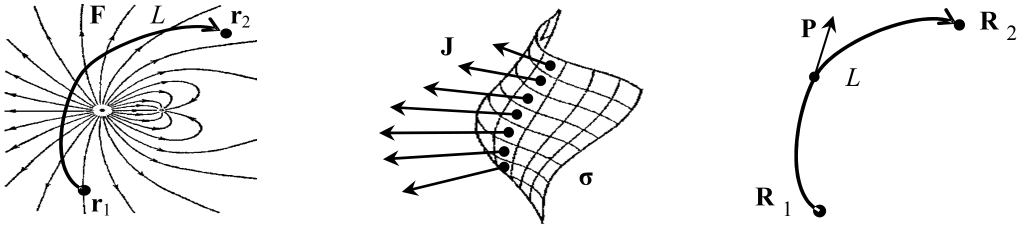
Behavior of a mechanical system is known to be determined by minimization of action S:

$$S = \int_{t_1}^{t_2} L(t, \mathbf{r}, \mathbf{v}) dt \to \min, \qquad (1)$$

functional of Lagrangian L, an expression composed of kinematical variables and dynamical constantsdescribing the system [1]. This approach reduces description and formulation of physical regularities todescription by only kinematical variables, generalized coordinate and its derivatives.

With introduction of notion of generalized coordinate and velocity a possibility appeared to describe in universal way different-kind physical phenomena and to reveal the system behavior by means of similarequations for generalized coordinate and velocity. Variational principle allowed formulating in mostgeneral form spatial-temporal properties of the system and introduced into physics one of basic concepts – the action. Different physical problems are formulated as variational problems for an integral functional of vector functions (fields). An example is the problem of finding the trajectory L of motion of a body in a given force field $\mathbf{F}(\mathbf{r})$ from the point $\mathbf{r}_1$ to the point $\mathbf{r}_2$ (Fig. 1a) where minimal work A is performed:

$$A = \int_{L(\mathbf{r}_1 \to \mathbf{r}_2)} \mathbf{F} \cdot d\mathbf{r} \to \min . \qquad (2)$$



(a) (b) (c)

Fig. 1. Variational problems of finding the (a) trajectory, (b) surface, and (c) four-dimensional curve.

Another example is the problem of finding a surface $\boldsymbol{\sigma}$ bounded by a closed curve L, with a given current (liquid) density $\mathbf{J}(\mathbf{r})$ (Fig. 1b) which ensures the minimal flux K:

$$K = \iint_{\boldsymbol{\sigma}} \mathbf{J} \cdot d\boldsymbol{\sigma} \to \min . \tag{3}$$

Generalization of the first example is the principle of least action in mechanics. If a system isdescribed by four-dimensional generalized momentum $\mathbf{P} = (\varepsilon, \mathbf{p})$ and coordinate $\mathbf{R} = (\tau, \mathbf{r})$ the trajectory L of motion from the point $\mathbf{R}_1$ to the point $\mathbf{R}_2$ (Fig. 1c) minimizes the action S:

$$S = -\int_{L(\mathbf{R}_1 \to \mathbf{R}_2)} \mathbf{P} \cdot d\mathbf{r} = -\int_{L(\tau_1, \mathbf{r}_1) \to (\tau_2, \mathbf{r}_2)} \left(\varepsilon d\tau - \mathbf{p} \cdot d\mathbf{r} \right) \to \min . \tag{4}$$

The known method for finding the general analytic solution of variational problem was proposed by Euler and then (in complemented and more general form) by Lagrange. The Euler solution was based on the possibility to represent the integral functional in parametrical form. So, in the last example above the action may be represented as

$$S = -\int_{\mathbf{R}_1}^{\mathbf{R}_2} \mathbf{P}\cdot d\mathbf{R} = -\int_{\mathbf{R}_1}^{\mathbf{R}_2} \left(\varepsilon d\tau - \mathbf{p}\cdot d\mathbf{r}\right) = -\int_{\tau_1}^{\tau_2} \left(\varepsilon - \mathbf{p}\cdot\frac{d\mathbf{r}}{d\tau}\right) d\tau = -\int_{\tau_1}^{\tau_2} \left(\varepsilon - \boldsymbol{\beta}\cdot\mathbf{p}\right) d\tau = \int_{\tau_1}^{\tau_2} L\, d\tau \,. \qquad (5)$$

Then the Euler–Lagrange equations may be used for obtaining Newton equations of motion in the parametrical form:

$$\frac{d}{d\tau}\frac{\partial L}{\partial \boldsymbol{\beta}} = \frac{\partial L}{\partial \mathbf{r}}\,, \qquad (6)$$

here three variables of parametrical representation, $\mathbf{r} = (x(\tau),\ y(\tau),\ z(\tau))$, are found from three second-order differential equations.

Introducing by Hamilton of the concept of canonical variables and creation of new (Hamiltonian)mechanics required both a new representation of the action and new solution of variational problem. Canonical variables, equally with kinematical variables, generalized coordinate and velocity, include as well the dynamical variables, generalized momentum and energy. The expression of energy introduced by Hamilton and called after him,

$$H = \mathbf{v}\cdot\mathbf{p} - L\left(t, \mathbf{r}, \mathbf{v}\right) \qquad (7)$$

and canonical equations

$$\mathbf{v} = \frac{\partial H}{\partial \mathbf{p}},\ \ \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{r}}\,. \qquad (8)$$

became the basis for construction of Hamiltonian and then quantum mechanics.

Hamilton equations (8) may be derived with use of variational principle by minimization of the action, the functional defined in terms of canonical variables,

$$S = -\int \left(H\,dt - \mathbf{p} \cdot d\mathbf{r} \right), \tag{9}$$

by independently varying coordinates and momentum [1]. The canonical variables turn to be canonically conjugate in transformations, hence they are mutually representable [2]. These properties are especially important in both relativistic approach, where time and spatial coordinates enter equally, and quantum mechanics, where the equivalence of coordinate and momentum representations of the Hamiltonian or the wave equation is formulated.

From the mathematical point of view, introducing canonical (dynamical) variables is a way to represent the system of s Euler–Lagrange second-order differential equations as the system of 2s first-order linear differential equations [1].

The whole nonrelativistic quantum mechanics is constructed on the basis of only one canonical variable H with first-order time derivative and second-order derivatives with respect to spatial coordinates. Such situation appeared in nonrelativistic quantum mechanics as a consequence of violation of canonicity in representation of the system Hamiltonian. Specifically, instead of the canonical representation of the Hamiltonian in the form (7),

$$\widehat{\mathrm{H}} = \mathbf{v} \cdot \hat{\mathbf{p}} - L(t, \mathbf{r}, \mathbf{v}) = -\mathbf{v} \cdot \left(i\hbar \frac{\partial}{\partial \mathbf{r}} \right) - L(t, \mathbf{r}, \mathbf{v}), \tag{10}$$

it is transformed in nonrelativistic quantum mechanics into the expression

$$\begin{aligned} \widehat{\mathrm{H}} &= \mathbf{v} \cdot \hat{\mathbf{p}} + L(t, \mathbf{r}, \mathbf{v}) = \mathbf{v} \cdot \hat{\mathbf{p}} - \frac{m\mathbf{v}^2}{2} + U(t, \mathbf{r}) = \frac{\mathbf{p}}{m} \cdot \hat{\mathbf{p}} - \frac{\mathbf{p}^2}{2m} + U(t, \mathbf{r}) = \\ &\frac{\hat{\mathbf{p}}^2}{2m} + U(t, \mathbf{r}) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \mathbf{r}^2} + U(t, \mathbf{r}) \end{aligned} \tag{11}$$

and is given by a second-order operator. In this case one of canonical variables (velocity) is eliminated from the expression by incorrect replacement by another canonical variable of momentum. In the Hamilton approach the variational problem for the action is formulated on the basis of non-parameterized integral functional of four-dimensional vector function (4) and for solution of this variational problem by known methods it is needed to use some parametrical representation of variables and the functional (5). Poincare and later Minkowski, for preserving the equivalence of canonical variables and for invariant representation of the action, used as a parameter the four-dimensional interval [3]

$$S = -\int_{\mathbf{R}_1}^{\mathbf{R}_2} \mathbf{P} \cdot d\mathbf{R} = -\int_{\mathbf{R}_1}^{\mathbf{R}_2} \left(\varepsilon d\tau - \mathbf{p} \cdot d\mathbf{r} \right) = -\int_{\mathbf{R}_1}^{\mathbf{R}_2} \left(\varepsilon \frac{d\tau}{ds} - \mathbf{p} \cdot \frac{d\mathbf{r}}{ds} \right) = -\int_{s_1}^{s_2} \left(\mathbf{V} \cdot \mathbf{P} \right) ds = -\int_{s_1}^{s_2} I ds . \qquad (12)$$

In this case the Euler–Lagrange solution for the four-dimensional Poincare equations of motion are in parametrical form written as

$$\frac{d}{ds} \frac{\partial I}{\partial \mathbf{V}} = \frac{\partial I}{\partial \mathbf{R}} , \qquad (13)$$

where the four variables $\mathbf{R} = (\tau(s), x(s), y(s), z(s))$ of parametrical representation are found from the system of four second-order differential equations. Although the approach developed by Poincare preserves equivalence of the four coordinates and the action has a covariant representation, but the explicit equations for canonical variables are absent and it is impossible to reproduce them unambiguously from parametrical representations.

Procedures of elimination of the parameter from the solutions of the variational problem were the fundamental part of subsequent development of variational calculus and led to creation of Hamilton–Jacobi theory of integral invariants (Hilbert, Poincare, Cartan, and others) [4]. The essence of the developed approaches consisted in finding such algebraic, differential, or integral relations which would not, in spite of parametrical representation of variables, depend on the parameter (would be invariant with respect to parametrical representation).

Construction (reproducing) of the field for a functional from the family of the extremals in the parametrical representation on the basis of the integral invariant has been proposed by Hilbert in1900–1906 [4]. Properties of the Hilbert integral cause, in particular, vanishing of all Lagrange brackets composed with use of canonical variables corresponding to this family of extremals. Methods for constructing the field for a functional on the basis of integral invariants and their application to description of physical systems had further development in works of Poincare and Cartan [5]. Although these approaches display only implicitly the properties of canonical variables of a system, the commutation relations and integral invariants are now among the basic techniques for constructing modern physical theory.

However, if we consider, e.g., the Maxwell equations as a solution of the variational problem, this solution has a canonical form, i.e., it contains neither additional parameters, nor velocity and other total derivatives. The continuity equation and the wave equation have the same character. If these canonically given equations should be obtained from the solution of variational problem, an obvious necessity appears to either find the field from parametrical representation of the family of extremals by eliminating the parameter or find a way for the canonical (nonparametric) solution of at least given specific variational problem. Indeed, just for representation of Maxwell equations as solution of variational problem the direct way is used to solve this specific variational problem [6] which differs from the Euler–Lagrange parametrical approach. Here the variation of the integral functional and the necessary condition of its minimum are represented in the direct (canonical) form by applying the integral Gauss theorem without any parametrization procedure and correspondingly without using the Euler–Lagrange solution.

If there is a general canonical solution which contains no additional parameter and does not use one of variables as a parameter, the parametrization would become merely a required mathematical formulation of the final result, a suitable representation of solution for describing a specific system, rather than a principal necessity for solution of variational problem itself. Then the need in any parametrical invariants or commutation relations would fall away completely, since the canonical solution does already contain nothing else but independently and equivalently represented canonical variables.

Parametrical representation of the functionals and the Euler–Lagrange solution are, surely, very universal and powerful means to solve a very wide circle of physical and mathematical problems, but it is also necessary to have canonical solutions of canonically formulated problems. It should be a solution where entering explicitly and equivalently is only the canonical variables (without extra parameters or separation of a variable as a parameter), which would allow formulating in a unified way the variational principles and solution of physical problems in canonically-invariant form.

2. CANONICAL SOLUTION OF VARIATIONAL PROBLEMS

Let an integral functional $S = \int\limits_{L(\mathbf{R}_1 \to \mathbf{R}_2)} \mathbf{F} \cdot d\mathbf{R}$ of a vector function $\mathbf{F}(\mathbf{R})$ – $\mathbf{R} \equiv (x_1, x_2, ..., x_n)$ $\mathbf{F}(\mathbf{R}) \equiv (f_1(\mathbf{R}), f_2(\mathbf{R}), ..., f_n(\mathbf{R}))$, in the region $\mathbf{R} \subset D \subset \mathbb{R}^n$, be given which is defined on a n-dimensional curve L. The variation of the functional, δS, on the curves L' and L'' connecting the points $\mathbf{R}_1$ and $\mathbf{R}_2$ (Fig.2a,b,c), is

$$\delta S = \int\limits_{L'(\mathbf{R}_1 \to \mathbf{R}_2)} \mathbf{F} \cdot d\mathbf{R} - \int\limits_{L''(\mathbf{R}_1 \to \mathbf{R}_2)} \mathbf{F} \cdot d\mathbf{R} = \int\limits_{L'(\mathbf{R}_1 \to \mathbf{R}_2)} \mathbf{F} \cdot d\mathbf{R} + \int\limits_{L''(\mathbf{R}_2 \to \mathbf{R}_1)} \mathbf{F} \cdot d\mathbf{R} = \oint\limits_{L = L' \cup L''} \mathbf{F} \cdot d\mathbf{R} . \tag{14}$$

where L is a closed curve consisting of curves L' and L''.

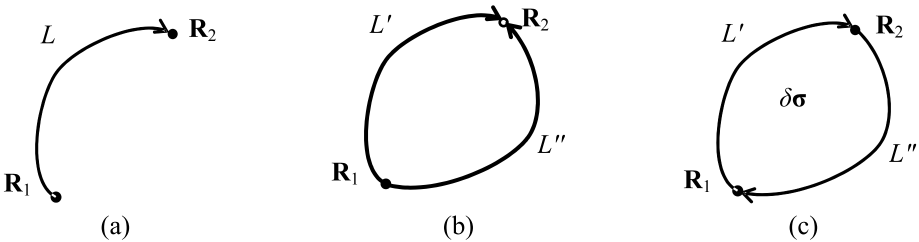


Fig. 2. (a) Curve L where the functional is defined, (b) curves L' and L'' where the functional variation is defined, and (c) closed curve $L = L' + L''$ where the functional variation is represented as an integral over the closed curve L.

The generalized Stokes theorem for multidimensional curvilinear integral over the closed curve Land an arbitrary *n*-dimensional surface $\delta\boldsymbol{\sigma}$ bounded by this curve yields

$$\oint_L \mathbf{F}\cdot d\mathbf{R} = \oint_L f_i dx_i = \frac{1}{2}\iint_{\delta\boldsymbol{\sigma}}\left(\frac{\partial f_k}{\partial x_i} - \frac{\partial f_i}{\partial x_k}\right)\cdot d\sigma_{ik}\,. \tag{15}$$

So, the variation of the curvilinear integral functional equals the integral over a closed-curve-supported surface. This allows determining the variation of the curvilinear integral in the vicinity of an arbitrarily chosen point **R** of the given (sought for) curve (Fig. 3a).

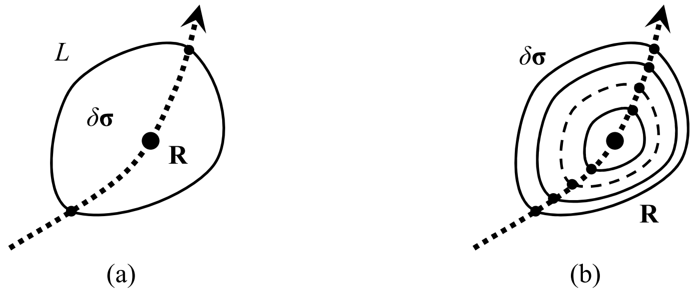


Fig. 3. (a) Curve *L* and (b) a series of closed curves around the point **R**.

By considering a succession of closed curves *L* around the point **R** (Fig. 3b) and contracting to this point, it is obtained in the limit:

$$\delta S = \oint_L \mathbf{F}\cdot d\mathbf{R} = \frac{1}{2}\iint_{\delta\boldsymbol{\sigma}}\left(\frac{\partial f_k}{\partial x_i} - \frac{\partial f_i}{\partial x_k}\right)d\sigma_{ik} \simeq \frac{1}{2}\left(\frac{\partial f_k}{\partial x_i} - \frac{\partial f_i}{\partial x_k}\right)\delta\sigma_{ik}\,, \tag{16}$$

which is the definition of rotor. It is seen that for this functional the variable of varying is the area $\delta\boldsymbol{\sigma}$ and the part of variation linear in $\delta\boldsymbol{\sigma}$ is determined by the last expression in (16). Vanishing of the rotor components gives just equation of the curve sought for where the functional of the curvilinear integral has an extremum. These, generally tensor, relationships may be written as

$$\left[\frac{\partial}{\partial\mathbf{R}}\times\mathbf{F}\right]=0\,. \tag{17}$$

For integral functionals in n-dimensional spaces with the element of integration of lower order k corresponding integral theorems are available. For example, the variation of the n-dimensional integral functional over $n-1$ manifolds $d\boldsymbol{\Sigma}=(dx_2dx_3dx_4...dx_n,\ dx_1dx_3dx_4...dx_n,\ ...,\ dx_1dx_2dx_3...dx_{n-1})$

$$\overbrace{\int\limits_{\boldsymbol{\Sigma}}...\int}^{n-1}\mathbf{F}\cdot d\boldsymbol{\Sigma}=\min;$$

$$\delta D=\overbrace{\int\limits_{\boldsymbol{\Sigma}'}...\int}^{n-1}\mathbf{F}\cdot d\boldsymbol{\Sigma}-\overbrace{\int\limits_{\boldsymbol{\Sigma}''}...\int}^{n-1}\mathbf{F}\cdot d\boldsymbol{\Sigma}=\overbrace{\int\limits_{\boldsymbol{\Sigma}'}...\int}^{n-1}\mathbf{F}\cdot d\boldsymbol{\Sigma}+\overbrace{\int\limits_{-\boldsymbol{\Sigma}''}...\int}^{n-1}\mathbf{F}\cdot d\boldsymbol{\Sigma}=\overbrace{\int\limits_{\boldsymbol{\Sigma}}...\int}^{n-1}\mathbf{F}\cdot d\boldsymbol{\Sigma}=\overbrace{\int\limits_{\delta\Omega}...\int}^{n}\frac{\partial F_i}{\partial x_i}\cdot d\Omega \tag{18}$$

equals, according to the Gauss theorem, the integral of the divergence over n-dimensional volume $d\Omega=dx_1dx_2dx_3...dx_n$. Then the solution to the variational problem is expressed as the equation of the curve sought for:

$$\frac{\partial\mathbf{F}}{\partial\mathbf{R}}=\frac{\partial f_i}{\partial x_i}=0\,. \tag{19}$$

In other cases of multidimensional integral functionals it is not possible to suggest some general form of integral theorems. They have different formulations and representations depending on measure and parity of the space.

For illustration let us apply the obtained results for solution of the two example problems above, (2) and (3). For the first one it is obtained rot $\mathbf{F} = 0$. Motion along these trajectories forms a family of extremals (a field)

$$\mathbf{F} = \frac{\partial \varphi}{\partial \mathbf{r}}, \tag{20}$$

were $\varphi = \varphi(\mathbf{r})$ – is an arbitrary scalar function (scalar potential). For example, for a force field $\mathbf{F}$ given by the expression

$$\mathbf{F} = \left(ay^3 + by, \quad cx^3 + dx, \quad 0\right); \Rightarrow \text{rot}\,\mathbf{F} = \left(0, \quad 0, \quad 3cx^2 - 3ay^2 + d - b\right),$$

trajectories sought for are determined by the equation

$$x^2 - \frac{a}{c}y^2 = \frac{b-d}{3c},$$

Note that this determines the points (curves) of equilibrium. The family of the same points (or curves)may be given also in the form of the system of equations

$$ay^3 + by = \frac{\partial \varphi}{\partial x}; \qquad cx^3 + dx = \frac{\partial \varphi}{\partial y}; \qquad 0 = \frac{\partial \varphi}{\partial z};\cdot$$

For the second example, (3), it is obtained div $\mathbf{J} = 0$, nd the surfaces sought for make a family of extremals

$$\mathbf{J} = \text{rot}\,\mathbf{A}, \tag{21}$$

were $\mathbf{A} = \mathbf{A}(\mathbf{r})$ – is an arbitrary vector function (vector potential). For example, in the case where $\mathbf{J} = J_0(x^3/a^2 - x, y^3/b^2 - y, z^3/c^2 - z)$ it is obtained $\operatorname{div}\mathbf{J} = 3J_0(x^2/a^2 + y^2/b^2 + z^2/c^2 - 1) = 0$.

The surfaces sought for are

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$$

This equation defines the (surface) regions of laminar (vortex-free) flow. The family of curves (or points) is given in the form of the system of equations

$$(x^3/a^2 - x, y^3/b^2 - y, z^3/c^2 - z) = \operatorname{rot}\mathbf{A}.$$

3. INTEGRAL FUNCTIONALS IN MINKOWSKI SPACES

Because of importance of physical applications, consider in more detail the integral functionals in four-dimensional Minkowski spaces. In these spaces four physically meaningful types of integration are possible [6].

1) Integral over a curve in the four-space (curvilinear integral). The element of integration is a length element $d\mathbf{R} = \{dx^i\} = (dx^0, dx^1, dx^2, dx^3)$. Differential form of integration $d\mathbf{S}$ is a scalar (covariant) product of the four-vector function $\mathbf{A}(\mathbf{R})$ and the four-vector $d\mathbf{R}$:

$$dS = \mathbf{A}\cdot d\mathbf{R} = A^0dx^0 - A^1dx^1 - A^2dx^2 - A^3dx^3 = A_i dx^i = A^i dx_i ,$$

$$S = \int \mathbf{A}\cdot d\mathbf{R} = \int\left(A^0dx^0 - A^1dx^1 - A^2dx^2 - A^3dx^3\right) = \int A_i dx^i = \int A^i dx_i . \qquad (22)$$

2) Integral over a surface (two-dimensional) in four-space (surface integral). Surface element in four-space is defined as antisymmetric second-rank tensor $df^{ik} = dx^i dx'^k - dx^k dx'^i$. Another tensor may be constructed, $d\sigma^{ik} = \frac{1}{2}e^{iklm}df_{lm}$, which is dual to tensor df^{ik}, and has six significant elements:

$$d\sigma^{ik} = \begin{pmatrix} 0 & dx^1dx^0 & dx^2dx^0 & dx^3dx^0 \\ -dx^1dx^0 & 0 & -dx^2dx^1 & dx^3dx^1 \\ -dx^2dx^0 & dx^2dx^1 & 0 & -dx^3dx^2 \\ -dx^3dx^0 & -dx^3dx^1 & dx^3dx^2 & 0 \end{pmatrix} = (d\mathbf{p}, d\mathbf{a}), \qquad (23)$$

it may be represented in the form of a polar and axial dual vectors, $d\mathbf{p} = (dx^1dx^0, dx^2dx^0, dx^3dx^0)$ and $d\mathbf{a} = (dx^2dx^1, dx^3dx^1, dx^3dx^2)$. Differential form of integration dD is represented as covariant product for an antisymmetric tensor F^{ik}

$$dD = F^{ik} d\sigma_{ik}; \qquad D = \iint F^{ik} d\sigma_{ik} . \qquad (24)$$

3) Integration over a hypersurface, i.e., over a three-dimensional manifold. Element of integration is anelement of hypersurface (three-volume), $d\mathbf{\Sigma} = dS^i = \{dx^i dx'^j dx''^k\} = (dx^1dx^2dx^3, dx^0dx^2dx^3, dx^0dx^1dx^3, dx^0dx^1dx^2)$. The differential form of integration dS is represented as scalar (covariant) product of the four-vector function $\mathbf{A}(\mathbf{R})$ and the four-vector $d\mathbf{\Sigma}$

$$d\Phi = \mathbf{A}\cdot d\mathbf{\Sigma} = A^0 dS_0 - A^1 dS_1 - A^2 dS_2 - A^3 dS_3 = A^i dS_i = A_i dS^i ,$$

$$\Phi = \iiint \mathbf{A}\cdot d\mathbf{\Sigma} = \iiint \left(A^0 dS_0 - A^1 dS_1 - A^2 dS_2 - A^3 dS_3 \right) = \iiint A^i dS_i = \iiint A_i dS^i . \qquad (25)$$

4) For an integral over four-volume the element of integration is the scalar $d\Omega = dx^0dx^1dx^2dx^3$.

Correspondingly, there exist integral theorems of transformation between these integrals.

a) Integral over a four-dimensional closed curve is transformed into an integral over the enclosed surface by a replacement $dx^i \to df^{ki}\dfrac{\partial}{\partial x^k}$:

$$\oint \mathbf{A}\cdot d\mathbf{R} = \oint A_i dx^i = \iint \frac{\partial A_i}{\partial x^k} df^{ki} = \frac{1}{2}\iint \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} \right) df^{ik} , \qquad (26)$$

which is generalization of the Stokes theorem.

b) Integral over a closed four-dimensional surface is transformed into an integral over the enclosed hypersurface (three-volume) by replacement of the integration element, $d\sigma^{ik} \to dS_i \dfrac{\partial}{\partial x^k} - dS_k \dfrac{\partial}{\partial x^i}$:

$$\frac{1}{2}\oiint F^{ik} d\sigma_{ik} = \frac{1}{2}\iiint\left(dS_i \frac{\partial F^{ik}}{\partial x^k} - dS_k \frac{\partial F^{ik}}{\partial x^i}\right) = \iiint \frac{\partial F^{ik}}{\partial x^k} dS_i \,. \tag{27}$$

c) Integral over a closed hypersurface (three-volume) may be transformed into an integral over the enclosed four-volume by replacement of the integration element $dS^i \to d\Omega \frac{\partial}{\partial x^i}$:

$$\oiiint \mathbf{A}\cdot d\mathbf{S} = \oiiint A^i dS_i = \iiiint \frac{\partial A^i}{\partial x^i} d\Omega = \iiiint \frac{\partial \mathbf{A}}{\partial \mathbf{R}} d\Omega \,, \tag{28}$$

which is a generalization of the Gauss theorem.

For physical applications variational problems on the basis of listed integral functionals can be formulated for the four-vector (energy-momentum four-vector **P**) and for the antisymmetric four-tensor (four-tensors of electromagnetic field $F^{ik} = \left[\frac{\partial}{\partial \mathbf{R}} \times \mathbf{A}\right]$ = (**E**, –**B**) and angular momentum M^{ik} = [**R**×**P**] = $\{x_i p_k - x_k p_i\}$ = (**T**, –**M**)).

4. CANONICAL SOLUTION OF VARIATIONAL PROBLEMS OF MECHANICS

A. Principle of least action. Hamilton's expression for the action is represented as

$$S = -\int_{\mathbf{R}_1}^{\mathbf{R}_2} \mathbf{P}\cdot d\mathbf{R} = -\int_{(\tau_1,\mathbf{r}_1)}^{(\tau_2,\mathbf{r}_2)} \left(\varepsilon d\tau - \mathbf{p}\cdot d\mathbf{r}\right) \to \min \,, \tag{29}$$

where the variable $\mathbf{P} = (\varepsilon, \mathbf{p})$ – is the four-dimensional vector of energy-momentum. The variation, according to formula (14), is

$$\delta S = -\int_{L'(\mathbf{R}_1\to\mathbf{R}_2)} \mathbf{P}\cdot d\mathbf{R} + \int_{L''(\mathbf{R}_1\to\mathbf{R}_2)} \mathbf{P}\cdot d\mathbf{R} = -\left(\int_{L'(\mathbf{R}_1\to\mathbf{R}_2)} \mathbf{P}\cdot d\mathbf{R} + \int_{L''(\mathbf{R}_2\to\mathbf{R}_1)} \mathbf{P}\cdot d\mathbf{R} \right) = -\oint_L \mathbf{P}\cdot d\mathbf{R}\,. \tag{30}$$

From the generalized Stokes theorem for a four-integral over a closed curve $L = L' + L''$ with a length l and the surface area $\delta\boldsymbol{\sigma}$ it is obtained

$$\delta S = \oint_L P_i dx^i = \iint_{\delta\boldsymbol{\sigma}} \frac{\partial P_k}{\partial x^i} d\sigma^{ik} = \frac{1}{2}\iint_{\delta\boldsymbol{\sigma}} \left(\frac{\partial P_k}{\partial x^i} - \frac{\partial P_i}{\partial x^k} \right) d\sigma^{ik} \,. \tag{31}$$

Variation of action δS for true trajectories is zero, hence for points of a true trajectory we have

$$\delta S = \oint_L P_i dx^i = \frac{1}{2}\iint_{\delta\boldsymbol{\sigma}} \left(\frac{\partial P_k}{\partial x^i} - \frac{\partial P_i}{\partial x^k} \right) d\sigma^{ik} \simeq \left(\frac{\partial P_k}{\partial x^i} - \frac{\partial P_i}{\partial x^k} \right) d\sigma^{ik} = 0\,. \tag{32}$$

Independence of the variation components $\delta\sigma^{ik}$ yields vanishing of every term of the expression composed of components of four-rotor and corresponding to $\delta\sigma^{ik}$. Vanishing of components of four-rotor of generalized momentum can be expressed as vector relationships

$$\left[\frac{\partial}{\partial \mathbf{R}} \times \mathbf{P}\right] = \begin{cases} \dfrac{\partial \mathbf{p}}{\partial \tau} + \dfrac{\partial \varepsilon}{\partial \tau} = 0, \\ -\mathrm{rot}\,\mathbf{p} = 0. \end{cases} \tag{33}$$

Note that in this case the variational problem is formulated without parametrization of integral functional and the solution is represented as a system of six first-order differential equation. Corresponding parametrization of these equations or parametrization of integral functional yield Euler–Lagrange equations – system of three second-order differential equations. Thus, instead of canonization of Euler–Lagrange equations by introducing new variables of energy and momentum, the solution of variational problem is found in canonical form. This means that integral functional is determined and the solution is obtained directly for canonical variables of momentum and energy. It is principally important for a physical problem, since representation of second-order parametrical Euler–Lagrange equations inthe form of the system of first-order equations is not explicit and unambiguous. Among all representations of the Euler–Lagrange equation in the form of the system of first-order equations, it is just the explicit representation (33) that is canonical.

B. Principle of lowest flow (tensor fields F^{ik} = (**E, –B**), and the angular momentum M^{ik} = [**R×P**]). If in the four-space a system is described by a field given by antisymmetric tensor F^{ik} = (**E, –B**), the flow of the field through the surface bounded by a given closed curve L is, according to formula (24), determined by integral

$$D = \iint F^{ik} d\sigma_{ik} \rightarrow \min. \tag{34}$$

Variational problem of finding the surface which minimizes the functional (34) is solved by representation of variation of integral functional on a four-dimensional closed surface S as an integral over enclosed hypersurface Σ (three-volume), according to formula (27):

$$\delta D = \frac{1}{2}\iint\limits_{S'} F^{ik} d\sigma_{ik} - \frac{1}{2}\iint\limits_{S''} F^{ik} d\sigma_{ik} = \frac{1}{2}\iint\limits_{S'} F^{ik} d\sigma_{ik} + \frac{1}{2}\iint\limits_{-S''} F^{ik} d\sigma_{ik} = \frac{1}{2}\oint\limits_{S} F^{ik} d\sigma_{ik} = \iiint\limits_{\Sigma} \frac{\partial F^{ik}}{\partial x^k} d\mathbf{\Sigma}_i \,, \quad (35)$$

which is expressed, with use for the fields of dual vectors F^{ik} = **(E, –B)**, as the field equations

$$\frac{\partial F^{ik}}{\partial x^k} = \begin{cases} \dfrac{\partial \mathbf{E}}{\partial \mathbf{r}} = 0, \\ \dfrac{\partial \mathbf{E}}{\partial \tau} - \operatorname{rot}\mathbf{B} = 0. \end{cases} \quad (36)$$

If the antisymmetric tensor F^{ik} is represented by the components of four-rotor of the vector **P** as $F^{ik} = \partial P^k / \partial x_i - \partial P^i / \partial x_k$ and the fields are represented as

$$\mathbf{E} = -\frac{\partial \mathbf{p}}{\partial \tau} - \frac{\partial \varepsilon}{\partial \mathbf{r}}, \qquad \mathbf{B} = \operatorname{rot}\mathbf{p}\,, \quad (37)$$

then it is obtained

$$\frac{\partial \mathbf{E}}{\partial \tau} = \operatorname{rot}\mathbf{B}, \quad \frac{\partial \mathbf{E}}{\partial \mathbf{r}} = 0, \quad \frac{\partial \mathbf{B}}{\partial \tau} = -\operatorname{rot}\mathbf{E}, \quad \frac{\partial \mathbf{B}}{\partial \mathbf{r}} = 0, \quad (38)$$

and the generalized momentum appears as

$$\frac{\partial^2 \varepsilon}{\partial \tau^2} - \frac{\partial^2 \varepsilon}{\partial \mathbf{r}^2} = \frac{\partial}{\partial \tau}\left(\frac{\partial \varepsilon}{\partial \tau} + \frac{\partial \mathbf{p}}{\partial \mathbf{r}}\right),$$
$$\frac{\partial^2 \mathbf{p}}{\partial \tau^2} - \frac{\partial^2 \mathbf{p}}{\partial \mathbf{r}^2} = -\frac{\partial}{\partial \mathbf{r}}\left(\frac{\partial \varepsilon}{\partial \tau} + \frac{\partial \mathbf{p}}{\partial \mathbf{r}}\right). \tag{39}$$

C. Principle of lowest hyperflow (transfer and redistribution over three-volumes). Continuity equation is a canonical solution to the variational problem of finding the lowest flow of the energy–momentum four-vector P through a hypersurface, i.e., the lowest hyperflow:

$$\Phi = \iiint_{\Sigma'} \mathbf{P}\cdot d\boldsymbol{\Sigma} \to \min,$$
$$\delta\Phi = \iiint_{\Sigma'} \mathbf{P}\cdot d\boldsymbol{\Sigma} - \iiint_{\Sigma''} \mathbf{P}\cdot d\boldsymbol{\Sigma} = \iiint_{\Sigma'} \mathbf{P}\cdot d\boldsymbol{\Sigma} + \iiint_{-\Sigma''} \mathbf{P}\cdot d\boldsymbol{\Sigma} = \oiiint_{\Sigma} \mathbf{P}\cdot d\boldsymbol{\Sigma} = \int\int\int\int \frac{\partial P_i}{\partial x_i} d\Omega = 0, \tag{40}$$

which in the form of continuity equation for generalized momentum,

$$\frac{\partial \mathbf{P}}{\partial \mathbf{R}} = \frac{\partial \varepsilon}{\partial \tau} + \operatorname{div}\mathbf{p} = 0 \tag{41}$$

expresses the energy and charge conservation law and the Lorentz gauge relation for the field potentials.

5. REPRESENTATION OF SYSTEMS IN KINEMATICAL AND DYNAMICAL SPACES

Summarizing what has been stated above and the results of work [7], the representation of mechanical systems may in general case be described as follows. Physical systems are displayed in coordinate and momentum ($\mathbf{R} = (\tau, \mathbf{r})$ and $\mathbf{Q} = (\varsigma, \mathbf{q})$, kinematical and dynamical) spaces and are described kinematical and dynamical (canonical) variables ($\mathbf{R}, \mathbf{V}, \ldots$) and ($\mathbf{Q}, \mathbf{U}, \ldots$) with dimensions, respectively, [cm] and [erg s/cm]. These spaces should be mutually representable and equivalent in description of properties of a physical system.

More general description with use of the metric tensor has the form

$$ds^2 = d\varsigma^2 - d\mathbf{q}^2 = \frac{\varepsilon^2}{\lambda^2} g_{ik} dx^i dx_k = I^2(x) \frac{\varepsilon^2}{\lambda^2}\left(d\tau^2 - d\mathbf{r}^2\right), \quad g_{00} = I^2(x); \quad \left(g_{ik} \to I^2(x) g_{ik}\right). \quad (42)$$

The action is a linear element in the momentum space and the principle of least action is reduced to the minimization of interval in the momentum space. Essentially, the characteristic physical quantity is the system invariant $I(x)$, which displays spatial-temporal properties (curvature of the space in a given point, $R = I(x)$) for the considered particle. Here, as in general relativity, the problem is reduced to that properties of the space–time geometry and properties of mapping (coordinate representation) of momentum space (metric correspondence). This correspondence is metric, kinematical correspondence. Conjugation of spaces should be expressed by also the correspondence of differential (variational) properties of lines (trajectories), surfaces, and hypersurfaces describing the dynamical characteristics of the physical system in the state of motion. Such a correspondence is a differential, dynamical correspondence. Respectively, representation of a physical system in kinematical and dynamical spaces should reflect fully the metric and differential correspondence of kinematical and dynamical spaces. As shown above, the physical and geometrical properties of these conjugate spaces can completely be described within variational approaches.

6. COMBINATION OF VARIATIONAL PROBLEMS OF MECHANICS. EQUATION FOR ACTION AND WAVE FUNCTION

Physical characteristics of a system and the correspondence of kinematical and dynamical variables to the same system (particle) are expressed in the form of allowable mutual representations and their properties caused by the properties of the group of allowable mapping of spaces. Once such a representation of variables is given, determination of spatial-temporal properties means assignment of properties of representation (of variables) on a line, surface, hypersurface, and in a volume, as properties of one and the same physical system (particle).

These fundamental spatial-temporal mapping properties for a physical system should be united and considered as a whole. Such variational problems or corresponding differential equations displaying the properties of representation on a line, surface, hypersurface, and in a volume, should be formulated for one and the same variable. Usually this variable is the generalized momentum P of the system in the coordinate representation and its above-listed properties are formulated in the coordinate space. Because of what has been said above, equations (33), (39), and (41) should be solved together under condition of invariance of representation of generalized momentum **P** [7]. Since the condition of vanishing of the four-rotor of **P**, (33), means that the integrand in (29) is total differential, the generalized momentum may be represented as a gradient of action in the form $\mathbf{P} = (\varepsilon, \mathbf{p}) = (-\partial S/\partial\tau, \partial S/\partial\mathbf{r}) = -\partial S/\partial x_i$. Then instead of system of equations (33), (39), and (41) it is obtained

$$\frac{\partial^2 S}{\partial \tau^2} - \frac{\partial^2 S}{\partial \mathbf{r}^2} = 0, \tag{43}$$

$$\left(\frac{\partial S}{\partial \tau}\right)^2 - \left(\frac{\partial S}{\partial \mathbf{r}}\right)^2 = \varepsilon^2 - \mathbf{p}^2 = I^2 = \frac{\left(mc^2 + q\varphi\right)^2 - \left(q\mathbf{A}\right)^2}{c^2} = \mathrm{inv}\,, \tag{44}$$

– which is the wave equation for the action and the condition of invariance of representation of the energy–momentum vector. The latter is just the Hamilton–Jacobi equation for action which constitutes, combined with the wave equation, the eikonal equation. Equations (43) and (44) display the wave property of the energy–momentum variable, the fundamental property of the system independent of the specific form of representation of generalized momentum and the character of interaction. Equation (43) determines the fundamental class of solutions describing wave behavior of any physical system at arbitrarily given interaction (44). Note that physically sound are only the derivatives of the function of action, rather than the function itself, therefore finding the function and specification of properties higher than the first derivatives is senseless.

Difficulty of combined solution of equations (43) and (44) consists in that from a very wide class of solutions of wave equation (43), those should be selected which satisfy the condition of invariance (44). It would be much more convenient, if the condition of invariance of representation would be contained in the wave equation narrowing thus the class of solutions of wave equation down to limits where they would unambiguously satisfy the condition of invariance of representation (44). For this purpose let the equation (44) be divided by a dimensional constant □2 and the sum with the equation (43) be composed. It is then obtained

$$\frac{\partial^2 S}{\partial \tau^2} - \frac{\partial^2 S}{\partial \mathbf{r}^2} + \frac{1}{\hbar^2}\left(\left(\frac{\partial S}{\partial \tau}\right)^2 - \left(\frac{\partial S}{\partial \mathbf{r}}\right)^2\right) = \frac{I^2}{\hbar^2} \tag{45}$$

Multiplying the equation by a function $A\exp(iS/\hbar)$ or $B\exp(-iS/\hbar)$, it may be represented as

$$\frac{\partial^2 e^{iS/\hbar}}{\partial \tau^2} - \frac{\partial^2 e^{iS/\hbar}}{\partial \mathbf{r}^2} = -\frac{I^2}{\hbar^2} e^{iS/\hbar} \quad \text{или} \quad \frac{\partial^2 e^{-iS/\hbar}}{\partial \tau^2} - \frac{\partial^2 e^{-iS/\hbar}}{\partial \mathbf{r}^2} = -\frac{I^2}{\hbar^2} e^{-iS/\hbar}. \tag{46}$$

This means that for the function

$$\Psi = Ae^{iS/\hbar} + Be^{-iS/\hbar}, \tag{47}$$

usually termed wave function, the following equation is valid:

$$\frac{\partial^2 \Psi}{\partial \tau^2} - \frac{\partial^2 \Psi}{\partial \mathbf{r}^2} = -\frac{I^2}{\hbar^2}\Psi . \tag{48}$$

It is apparent that the wave equation for the wave function Ψ contains an expression which allows explicitly taking into account the additional condition (44). Substitution of solutions Ψ of the wave equation into the condition of invariance of gradient (44) gives equations of curves sought for (trajectories) where the solutions to the wave equations provide the condition of invariance. In this representation the equations indicate explicitly that if equation (48) is solved by the method of separation of variables (Ψ is written as a product of functions of independent variables), this corresponds to solving equation (43) by writing S as a sum of functions of independent variables. If the wave function is chosen as $\Psi = A\exp(i\,S/\hbar)$, i.e., $S = -ih\ln\Psi + S_0$, the equations (43) and (44) will be written as

$$\begin{gathered} \frac{\partial^2 \Psi}{\partial \tau^2} - \frac{\partial^2 \Psi}{\partial \mathbf{r}^2} = -\frac{I^2}{\hbar^2}\Psi, \\ \left(\frac{\partial \Psi}{\partial \tau}\right)^2 - \left(\frac{\partial \Psi}{\partial \mathbf{r}}\right)^2 = -\frac{I^2}{\hbar^2}\Psi^2. \end{gathered} \tag{49}$$

This representation is identical from the point of view of definition of generalized momentum via the function of action S and wave function Ψ. Since

$$\mathbf{P} = -\frac{\partial S}{\partial \mathbf{R}}, \quad \Rightarrow \quad \varepsilon = -\frac{\partial S}{\partial \tau}, \quad \mathbf{p} = \frac{\partial S}{\partial \mathbf{r}}, \tag{50}$$

the mathematical identity

$$\frac{\partial S}{\partial \mathbf{R}}=\frac{1}{A\exp(iS/\hbar)}\left(A\exp(iS/\hbar)\frac{\partial S}{\partial \mathbf{R}}\right)=-\frac{i\hbar}{A\exp(iS/\hbar)}\frac{\partial}{\partial \mathbf{R}}\left(A\exp(iS/\hbar)\right)=-i\hbar\frac{1}{\Psi}\frac{\partial \Psi}{\partial \mathbf{R}} \qquad (51)$$

yields

$$\mathbf{P}=-i\hbar\frac{1}{\Psi}\frac{\partial \Psi}{\partial \mathbf{R}}=\frac{\partial}{\partial \mathbf{R}}\left(-i\hbar\ln\Psi\right)=\frac{\partial S}{\partial \mathbf{R}}, \quad \Rightarrow \quad \varepsilon=i\hbar\frac{1}{\Psi}\frac{\partial \Psi}{\partial \tau}=-\frac{\partial S}{\partial \tau}, \quad \mathbf{p}=-i\hbar\frac{1}{\Psi}\frac{\partial \Psi}{\partial \mathbf{r}}=\frac{\partial S}{\partial \mathbf{r}}. \qquad (52)$$

The wave function Ψ may always be represented also in the form

$$\begin{gathered}\Psi=2\sqrt{AB}\,\frac{\sqrt{A/B}\exp(iS/\hbar)+\sqrt{B/A}\exp(-iS/\hbar)}{2}= \\ C\,\frac{\exp\left(i(S-S_0)/\hbar\right)+\exp\left(-i(S-S_0)/\hbar\right)}{2}=C\cos\left(\frac{S-S_0}{\hbar}\right), \quad S_0=i\hbar\ln\sqrt{A/B},\end{gathered} \qquad (53)$$

where unessential constant C may be omitted. If the solutions for the wave function are chosen in the form $\Psi=\cos((S-S_0)/\hbar)$ ($S=\hbar\arccos\Psi+S_0$ or $S=\hbar\arcsin\Psi+S_0$), equations (48) are written as

$$\begin{gathered}\frac{\partial^2 \Psi}{\partial \tau^2}-\frac{\partial^2 \Psi}{\partial \mathbf{r}^2}=-\frac{I^2}{\hbar^2}\Psi, \\ \left(\frac{\partial \Psi}{\partial \tau}\right)^2-\left(\frac{\partial \Psi}{\partial \mathbf{r}}\right)^2=-\frac{I^2}{\hbar^2}\left(\Psi^2-1\right).\end{gathered} \qquad (54)$$

Convenience of solving the wave equation in complex representation is retained and the analytic continuation of functions in the complex region may be written as

$$S = \pm\hbar \arccos\Psi + S_0 \equiv -i\hbar\ln\left(\Psi \pm i\sqrt{1-\Psi^2}\right) + S_0,$$
$$\Psi = \cos\left(\frac{S-S_0}{\hbar}\right) \equiv \frac{\exp\left(i\left(S-S_0\right)/\hbar\right) + \exp\left(-i\left(S-S_0\right)/\hbar\right)}{2}. \qquad (55)$$

Let us write down also the covariant representation of equations (49):

$$\begin{cases} \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(g_{ik}\sqrt{-g}\,\frac{\partial S}{\partial x^k}\right) = 0; \\ g_{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} = I^2; \end{cases} \quad \begin{cases} \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(g_{ik}\sqrt{-g}\,\frac{\partial \Psi}{\partial x^k}\right) = -\frac{I^2}{\hbar^2}\Psi; \\ g_{ik}\frac{\partial \Psi}{\partial x^i}\frac{\partial \Psi}{\partial x^k} = -\frac{I^2}{\hbar^2}\Psi^2; \end{cases} \qquad (56)$$

Now, if these equations will be generalized within approach of general relativity, any system can be represented as the result of transformation of initial (unperturbed) momentum space:

$$\begin{cases} \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(g_{ik}\sqrt{-g}\,\frac{\partial S}{\partial x^k}\right) = 0; \\ g_{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} = 0, \pm 1; \end{cases} \quad \begin{cases} \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(g_{ik}\sqrt{-g}\,\frac{\partial \Psi}{\partial x^k}\right) = -\left(0, \pm 1\right)\Psi; \\ g_{ik}\frac{\partial \Psi}{\partial x^i}\frac{\partial \Psi}{\partial x^k} = -\left(0, \pm 1\right)\Psi^2; \end{cases} \qquad (57)$$

Correspondingly, by representation of metric tensor in the form $g_{i,k} = T_{i,j}\ T_{j,k}\ /R^2 = T_{i,j}\ T_{j,k}\ /I^2$, где $T_{i,j}$ is the matrix of invariant representation of generalized momentum, from the Poincare group, theequations (43) and (49) are obtained. On the other hand, starting from more general considerations of symmetry and specific properties of representation of momentum space via metric tensor or Poincare-group matrix, it is possible to find from equations (57) representations of invariants and interactions allowable for a physical system.

5. CONCLUSION

Canonical solutions of variational problem for integral functionals have been proposed and canonical solutions of variational problems of mechanics in Minkowski spaces were given. By combining variational principles of the least action, lowest flow, and lowest hyperflow canonically invariant equations were obtained for the energy–momentum variable. From these equations those for the action and the wave function were derived as general solution of united variational problem of mechanics. Equations are applicable to different types of particles and interactions and generalized within approach of general relativity.

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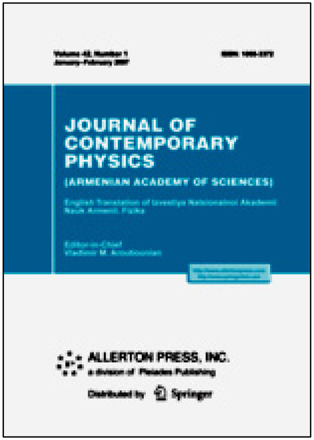

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