Unsafe but Calculable

Ratio Observables in Perturbative QCD

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Based on 1307.1699 with Andrew Larkoski

Boost 2013, Flagstaff — August 14, 2013

Outline

Ratio Observables: Ubiquitous & Unsafe

Ratios of Angularities: "Sudakov Safe"

Calculational Results

Ratio Observables: Ubiquitous & Unsafe

For jet substructure, often dimensionless ratios

Planar Flow	N-subjettiness	Energy Correlation Functions
$\mathrm{Pf} = \frac{4 \det S_\perp}{(\mathrm{tr} S_\perp)^2}$	$\tau_{N,N-1} = \frac{\tau_N}{\tau_{N-1}}$	$C_N = \frac{\mathrm{ECF}(N+1)\,\mathrm{ECF}(N-1)}{\mathrm{ECF}(N)^2}$

Motivated by quasi-scale-invariant nature of QCD

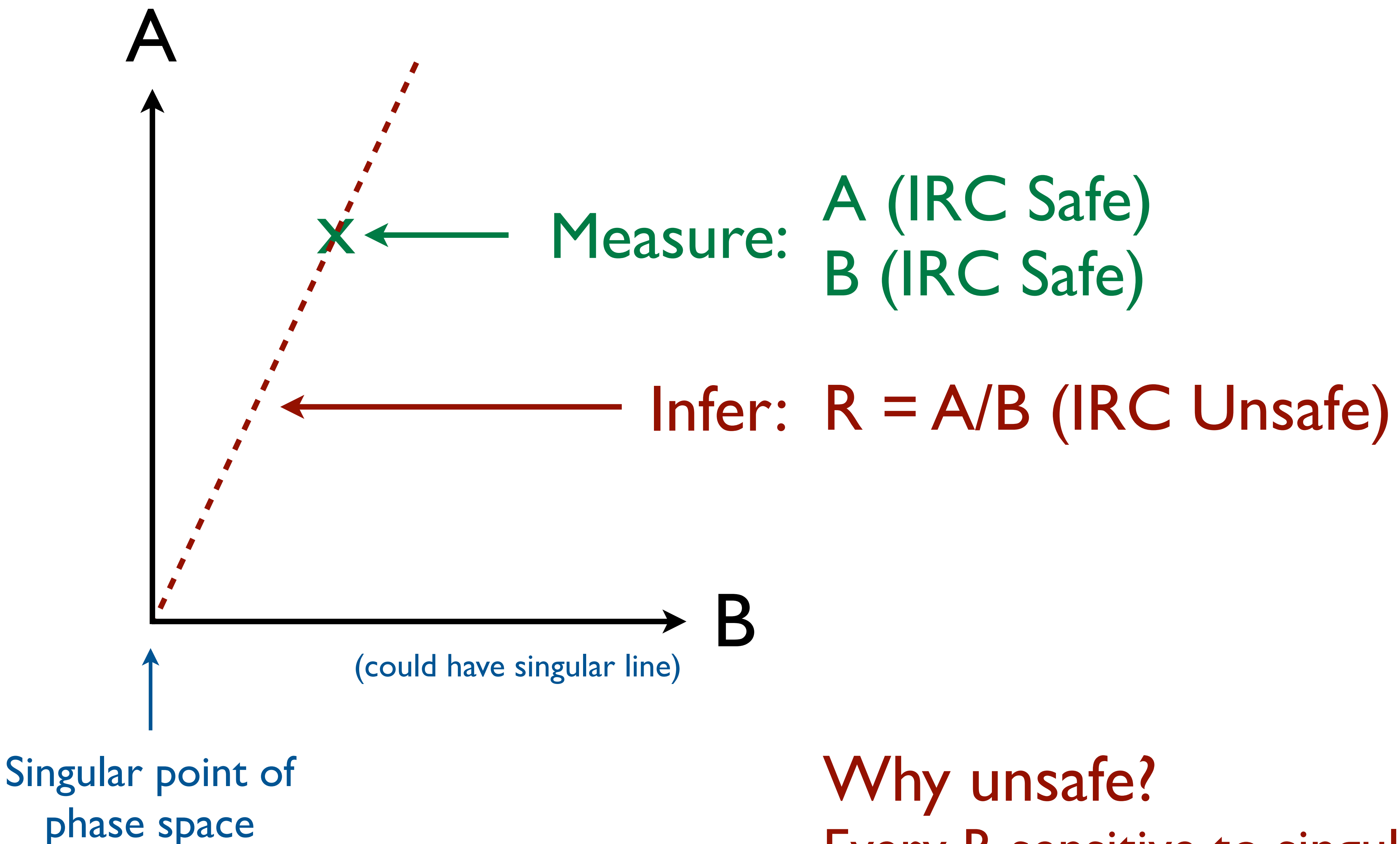
Surprising/Frustrating Fact:

$$\text{IRC Unsafe Ratio} = \frac{\text{IRC Safe Numerator}}{\text{IRC Safe Denominator}}$$

[observed by Soyez, Salam, Kim, Dutta, Cacciari]

WHAT?!

Safe/Safe = Unsafe?!

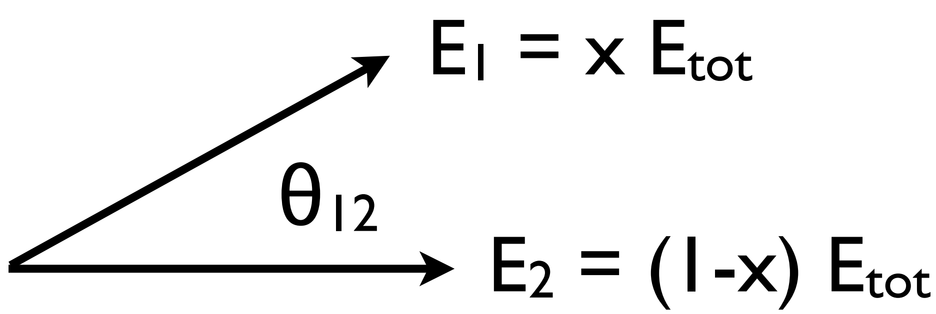


Why unsafe?

Every R sensitive to singularity

Simple Example

Mass/Energy = Unsafe



$$\frac{m^2}{E^2} = x(1-x)\,(\theta_{12})^2$$

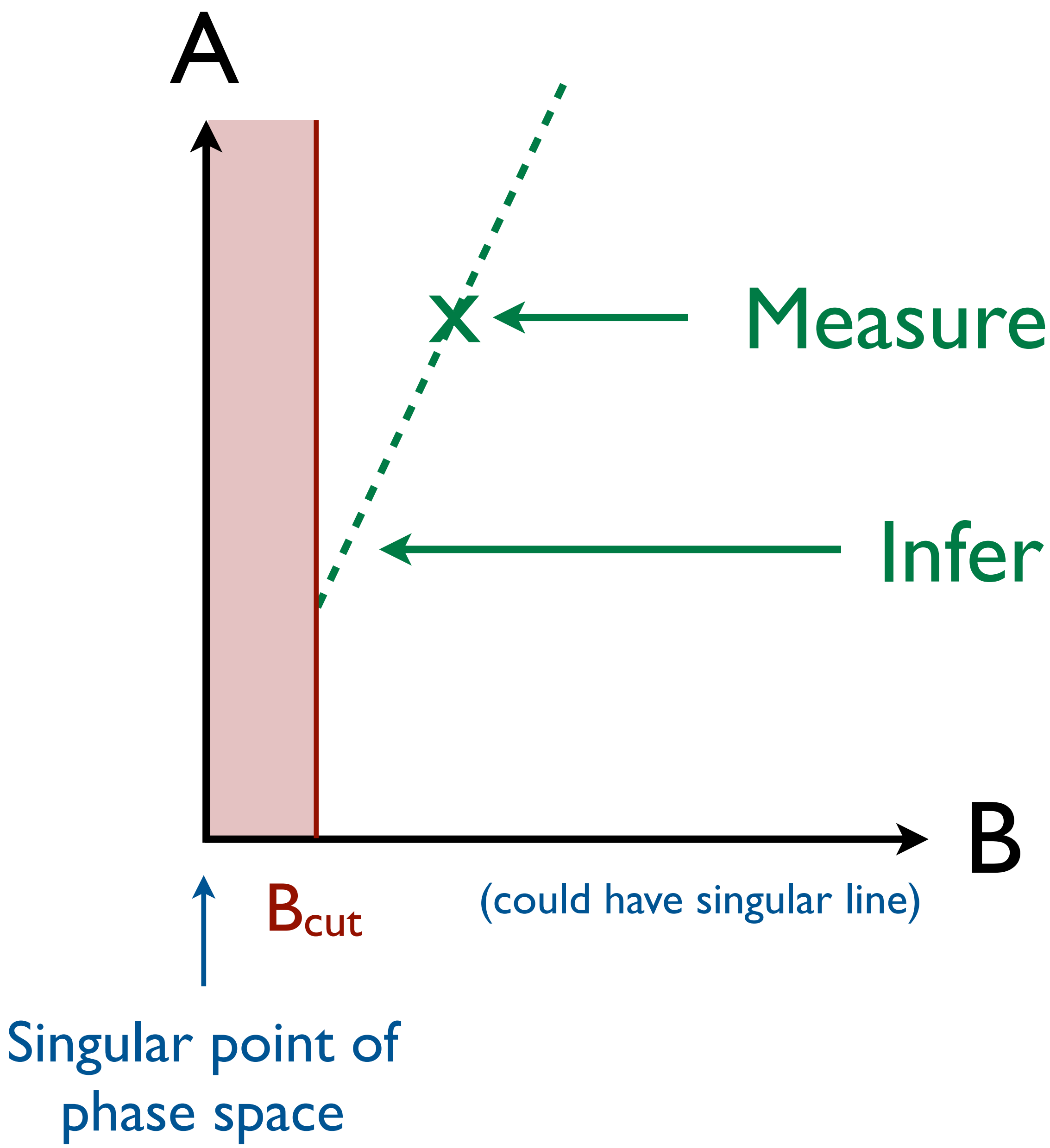
IRC Unsafe: Take soft limit $E_1, E_2 \to 0$ with x, θ_{12} fixed
Get arbitrary answer

Complaint: But if $E_1, E_2 \to 0$, no jet to measure

Solution: Require $E > E_{cut}$

Ah, ok.

Have to put cut on the denominator

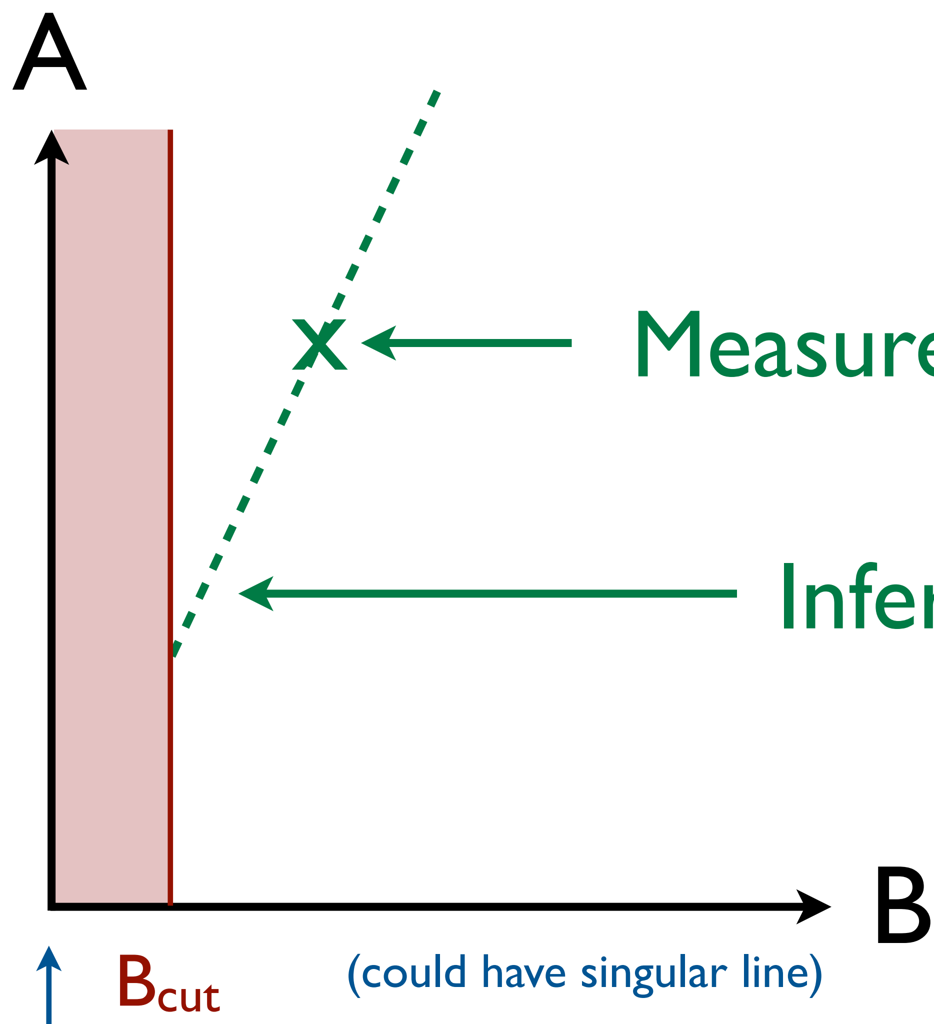


Measure: A (IRC Safe)
$B > B_{cut}$ (IRC Safe)

Infer: $R = A/B$ (now IRC Safe)

Ah, ok.

Have to put cut on the denominator

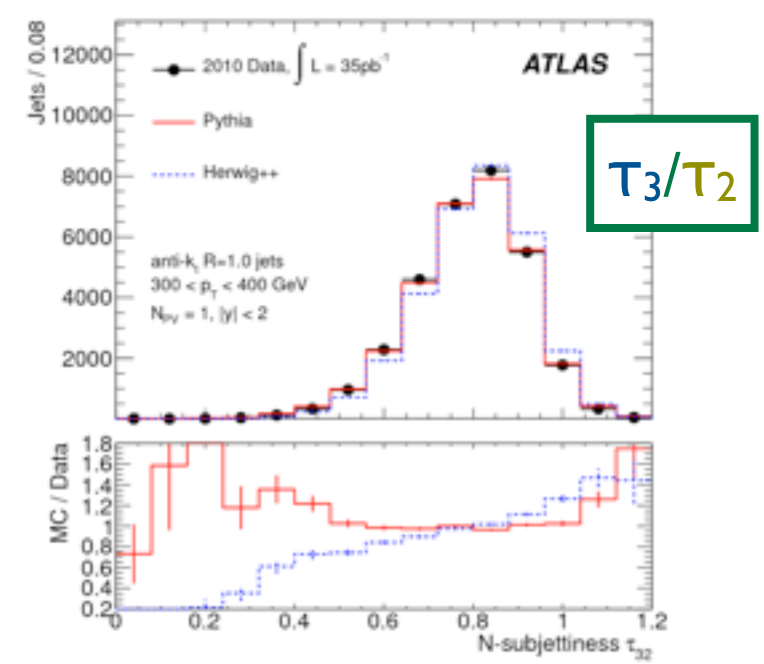


Measure: A (IRC Safe)
$B > B_{cut}$ (IRC Safe)

Infer: R = A/B (now IRC Safe)

Singular point of phase space

No cut on τ_2:
Strictly speaking
IRC unsafe



Surprising/Frustrating Fact:

$$\text{IRC Unsafe Ratio} = \frac{\text{IRC Safe Numerator}}{\text{IRC Safe Denominator}}$$

Ratios observables not calculable at any fixed order in α_s
(unless you cut on the denominator)

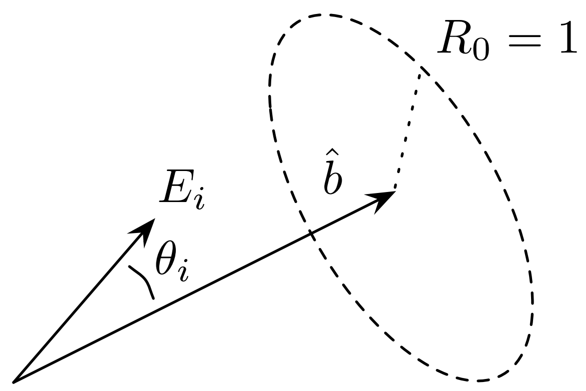
Surprising Resolution:

Ratios observables *are* calculable in QCD
if one accounts for *all orders* in α_s (even without a cut)

Corollary: Parton showers capture the dominant behavior of ratio observables

Ratios of Angularities

Simplest Test Case

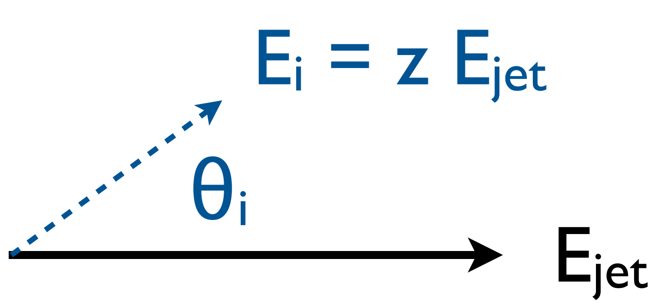


$$e_\beta = \frac{1}{E_{\text{jet}}} \sum_i E_i \left(\theta_i\right)^\beta$$

β = 1: Broadening
β = 2: Thrust ≈ m^2/E^2

Recoil subtlety: Measure angles with respect to "broadening axis"
Alternative: Energy-energy correlation function with same β, i.e. $C_1^{(\beta)}$

In soft & collinear limit (single emission):



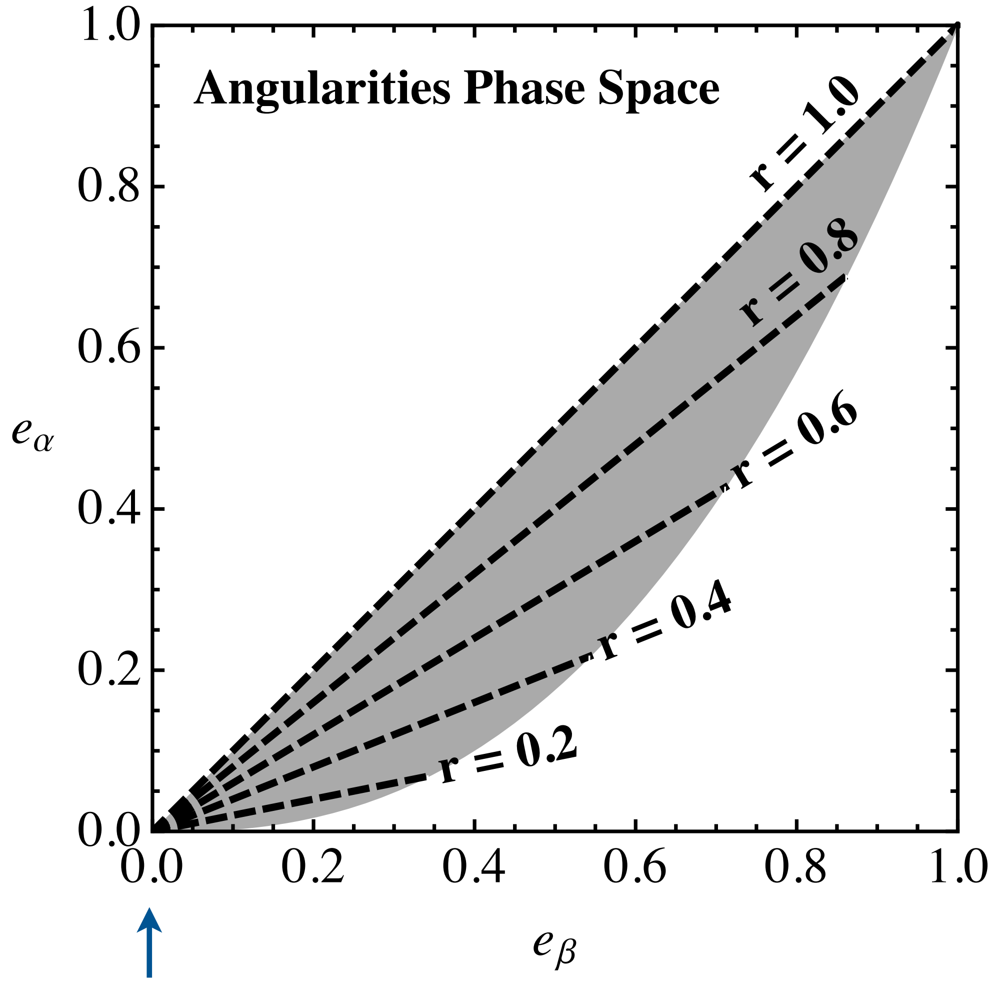
$$e_\beta = z\, \theta^\beta$$

IRC Safe

$$r = \frac{e_\alpha}{e_\beta} = \theta^{\alpha-\beta}$$

IRC Unsafe:
Arbitrary answer in z → 0 limit

Angularities Phase Space



Singular point of phase space

$$e_\beta = \sum_i z_i (\theta_i)^\beta \qquad r = \frac{e_\alpha}{e_\beta}$$

For α > β (set R = 1):

$$0 < e_\alpha < e_\beta \quad (0 < r < 1)$$

Single emission:

$$(e_\alpha)^\beta > (e_\beta)^\alpha$$

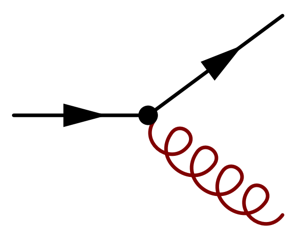
IRC Unsafe:

Each r has real emission singularity,
but virtual diagrams only have "r = 0" singularity

Fixed-Order vs. Strongly-Ordered

$$\frac{d\sigma}{dr} = \int de_\alpha \, de_\beta \, \frac{d^2\sigma}{de_\alpha de_\beta} \, \delta\left(r - \frac{e_\alpha}{e_\beta}\right)$$

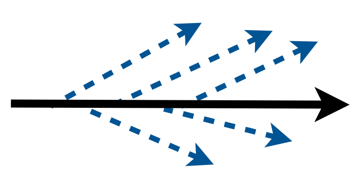
IRC Unsafe
Infinity at $O(\alpha_s)$



IRC Safe
"I can simultaneously measure e_α and e_β"

vs.

"Sudakov Safe"
Finite answer with
all orders in α_s



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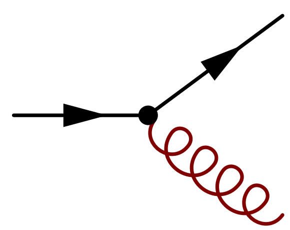

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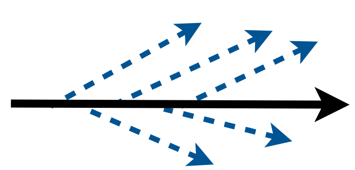
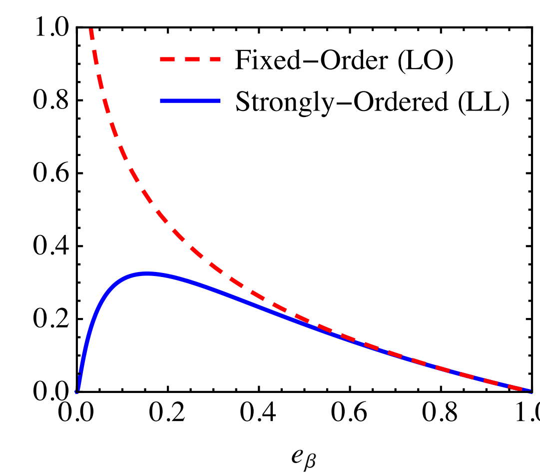


Two Options for a Finite Result:

Put explicit cut on e_β

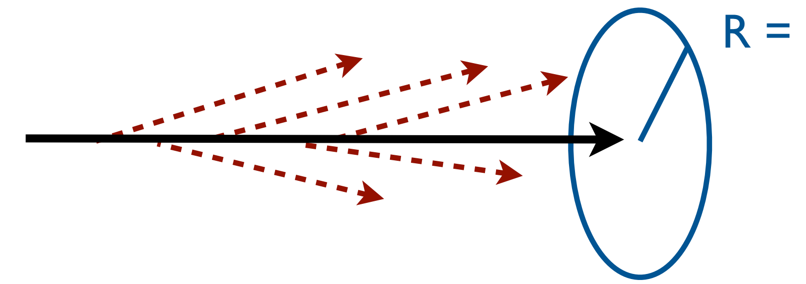
vs.

Sudakov factor effectively cuts on e_β



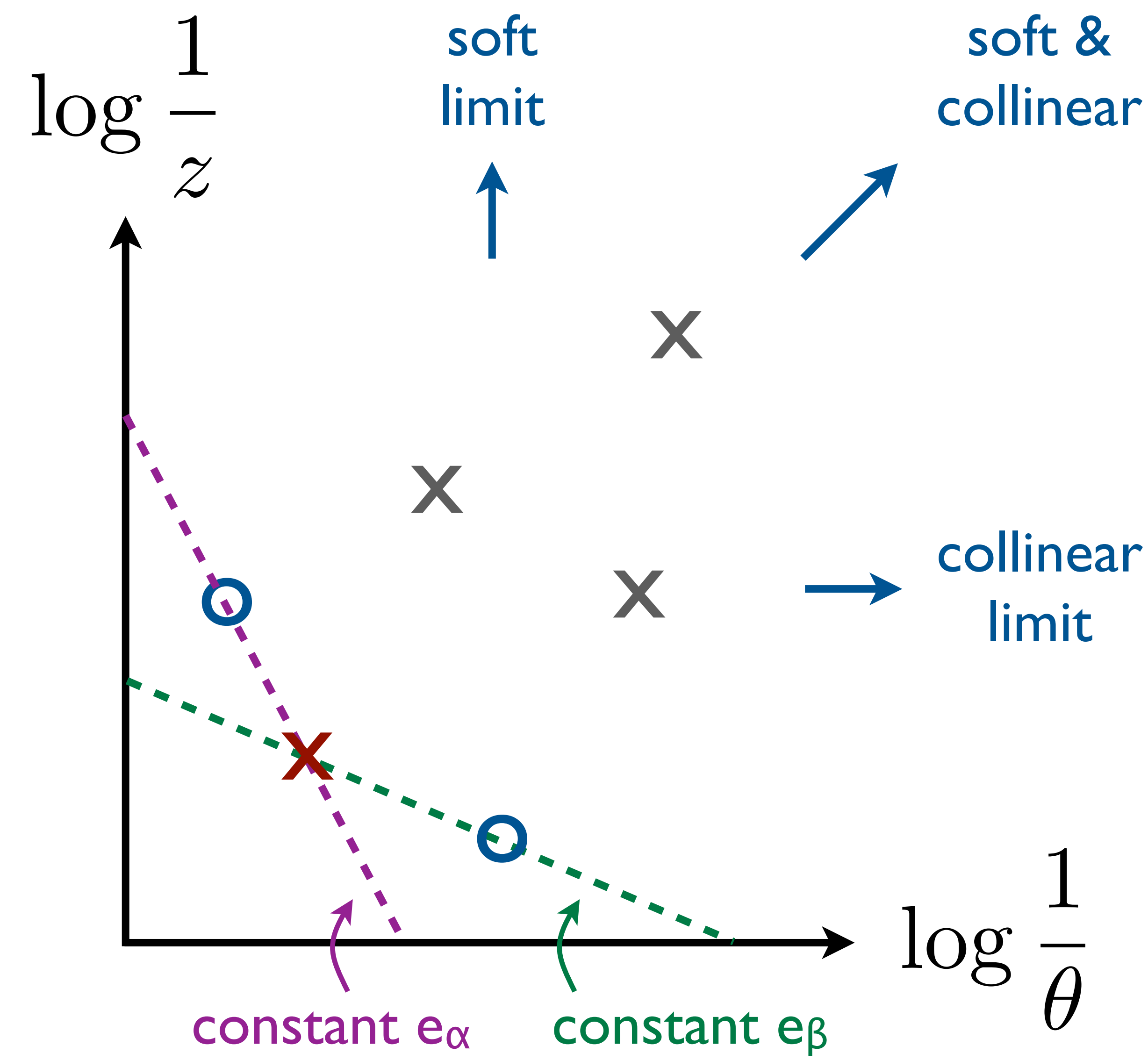
In Strongly-Ordered Limit

Recall introductory talk (set R = 1)



$$e_\beta = z\,\theta^\beta \qquad r = \frac{e_\alpha}{e_\beta}$$

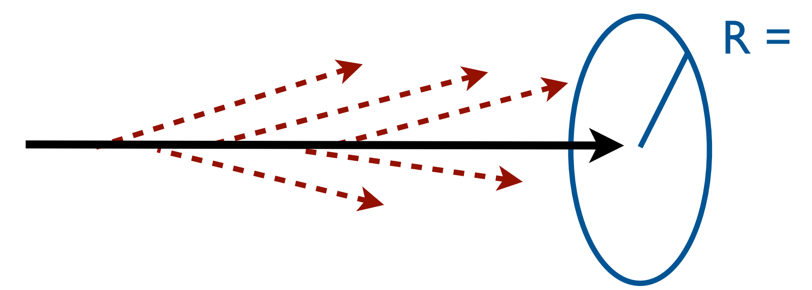
One emission (x) can set both e_α & e_β,
or can be set by two different (o) emissions.



$$\log \frac{1}{e_\beta} = \log \frac{1}{z} + \beta \log \frac{1}{\theta}$$

In Strongly-Ordered Limit

Recall introductory talk (set R = 1)



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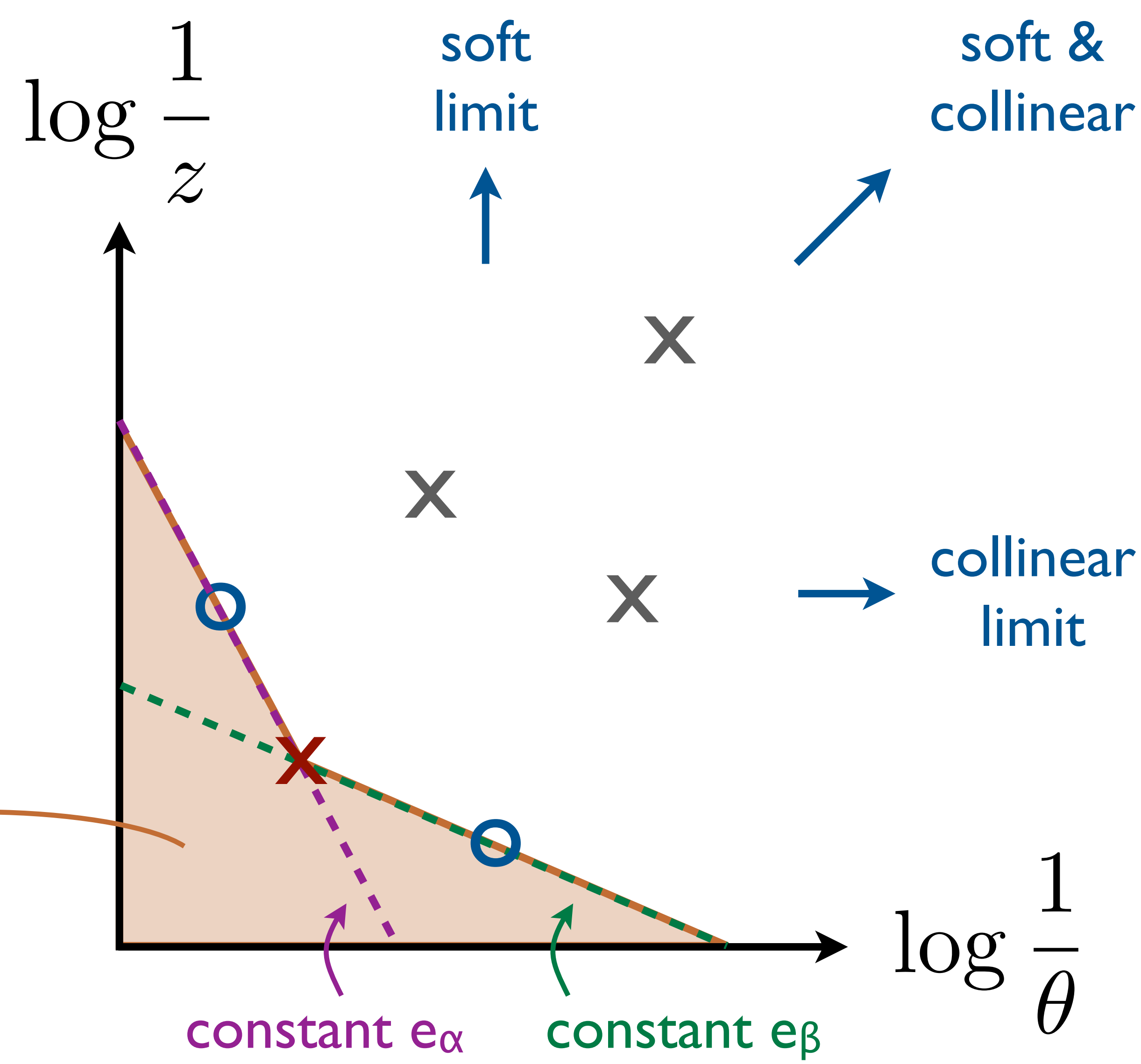
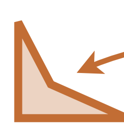


Sudakov Factor

(probability to get measurement below
a certain value of both e_α & e_β)

$$\Delta(e_\alpha, e_\beta) = e^{-\frac{2\alpha_s C_F}{\pi}}$$

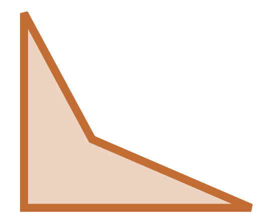


Captures leading logs of
e_α & e_β distribution

$$\log \frac{1}{e_\beta} = \log \frac{1}{z} + \beta \log \frac{1}{\theta}$$

Turning the Crank

Sudakov Factor:

$\Delta(e_\alpha, e_\beta) = \exp\Big[-\frac{2\alpha_s C_F}{\pi}$  $\Big]$

Double Differential Cross Section:

$$\frac{d^2\sigma^{\mathrm{LL}}}{de_\alpha\, de_\beta} = \frac{\partial}{\partial e_\alpha}\frac{\partial}{\partial e_\beta}\Delta(e_\alpha, e_\beta)$$

Ratio Cross Section:

$$\frac{d\sigma^{\mathrm{LL}}}{dr} = \int de_\alpha\, de_\beta\, \frac{d^2\sigma^{\mathrm{LL}}}{de_\alpha de_\beta}\,\delta\left(r - \frac{e_\alpha}{e_\beta}\right)$$

Turning the Crank

Sudakov Factor:

$$\Delta(e_\alpha, e_\beta) = \exp\left[-\frac{\alpha_s}{\pi} C_F \left(\frac{1}{\beta}\log^2 e_\beta + \frac{1}{\alpha - \beta}\log^2 \frac{e_\alpha}{e_\beta}\right)\right]$$

Double Differential Cross Section:

$$\frac{d^2\sigma^{\mathrm{LL}}}{de_\alpha\, de_\beta} =$$

Ratio Cross Section:

$$\frac{d\sigma^{\mathrm{LL}}}{dr} =$$

Turning the Crank

Sudakov Factor:

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Double Differential Cross Section:

$$\frac{d^2\sigma^{\mathrm{LL}}}{de_\alpha\, de_\beta} = \left(\frac{2\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\frac{1}{e_\alpha e_\beta} + \frac{4\alpha_s^2}{\pi^2}\frac{C_F^2}{\beta(\alpha-\beta)^2}\frac{1}{e_\alpha e_\beta}\log\frac{e_\beta}{e_\alpha}\log\frac{e_\alpha^\beta}{e_\beta^\alpha}\right)\Delta(e_\alpha, e_\beta)$$

(Cross check: Reduces to known single differential)

Ratio Cross Section:

$$\frac{d\sigma^{\mathrm{LL}}}{dr} =$$

Turning the Crank

Sudakov Factor:

$$\Delta(e_\alpha, e_\beta) = \exp\left[-\frac{\alpha_s}{\pi} C_F \left(\frac{1}{\beta}\log^2 e_\beta + \frac{1}{\alpha-\beta}\log^2\frac{e_\alpha}{e_\beta}\right)\right]$$

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(Cross check: Reduces to known single differential)

Ratio Cross Section:

$$\frac{d\sigma^{\mathrm{LL}}}{dr} = \frac{\sqrt{\alpha_s C_F \beta}}{\alpha-\beta}\frac{1}{r}\left(1 - 2\frac{\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\log^2 r\right)\left(\mathrm{erf}\left[\frac{\sqrt{\alpha_s C_F \beta}}{\sqrt{\pi}(\alpha-\beta)}\log r\right] + 1\right)e^{-\frac{\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\log^2 r} - 2\frac{\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\frac{\log r}{r}e^{-\frac{\alpha_s}{\pi}C_F\frac{\alpha}{(\alpha-\beta)^2}\log^2 r}$$

Turning the Crank

Ratio Cross Section:

$$\frac{d\sigma^{\rm LL}}{dr} = \frac{\sqrt{\alpha_s C_F \beta}}{\alpha-\beta}\frac{1}{r}\left(1-2\frac{\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\log^2 r\right)\left(\text{erf}\left[\frac{\sqrt{\alpha_s C_F\beta}}{\sqrt{\pi}(\alpha-\beta)}\log r\right]+1\right)e^{-\frac{\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\log^2 r} - 2\frac{\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\frac{\log r}{r}e^{-\frac{\alpha_s}{\pi}C_F\frac{\alpha}{(\alpha-\beta)^2}\log^2 r}$$

Expanded in small α_s:

$$\frac{d\sigma^{\rm LL}}{dr} = \sqrt{\alpha_s}\frac{\sqrt{C_F\beta}}{\alpha-\beta}\frac{1}{r} + \mathcal{O}(\alpha_s)$$

Turning the Crank

Ratio Cross Section:

$$\frac{d\sigma^{\mathrm{LL}}}{dr} = \frac{\sqrt{\alpha_s C_F \beta}}{\alpha - \beta}\frac{1}{r}\left(1 - 2\frac{\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\log^2 r\right)\left(\mathrm{erf}\left[\frac{\sqrt{\alpha_s C_F \beta}}{\sqrt{\pi}(\alpha-\beta)}\log r\right]+1\right)e^{-\frac{\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\log^2 r} - 2\frac{\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\frac{\log r}{r}e^{-\frac{\alpha_s}{\pi}C_F\frac{\alpha}{(\alpha-\beta)^2}\log^2 r}$$

Expanded in small α_s:

$$\frac{d\sigma^{\mathrm{LL}}}{dr} = \sqrt{\alpha_s}\frac{\sqrt{C_F\beta}}{\alpha-\beta}\frac{1}{r} + \mathcal{O}(\alpha_s)$$

Not a valid Taylor expansion

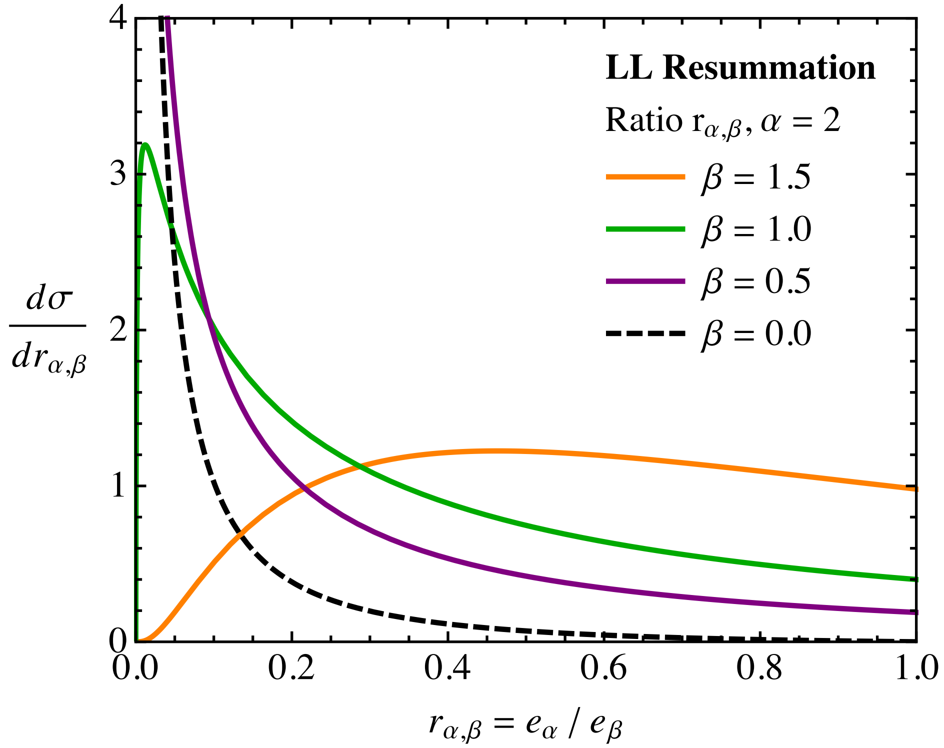
Finite cross section for all r (even r = 0)

Unsafe... ...but Calculable

Calculational Results

Leading-Log Result is Systematically Improvable

LL



strongly-ordered

LL $\Rightarrow$ $(\alpha_s \log^2 r)^{n/2}$

Three representative ratios

$\frac{e_2}{e_{1.5}}$ $\frac{e_2}{e_1}$ $\frac{e_2}{e_{0.5}}$

Usual Angularity

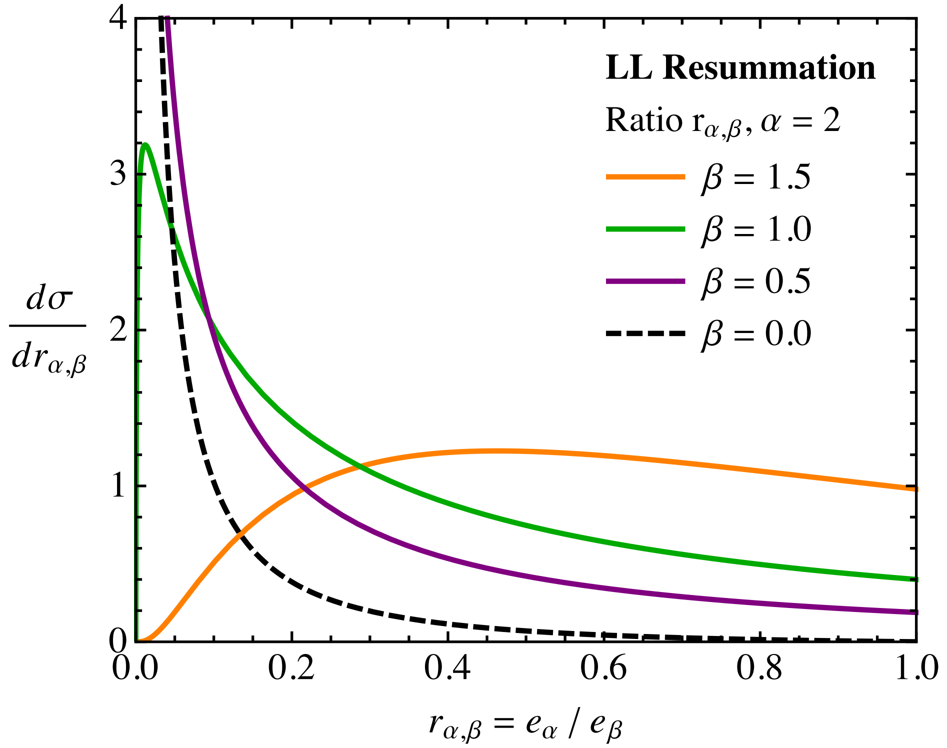
$$\frac{e_2}{e_0} = e_2$$

In following plots,
dashed always corresponds to
previous level of accuracy

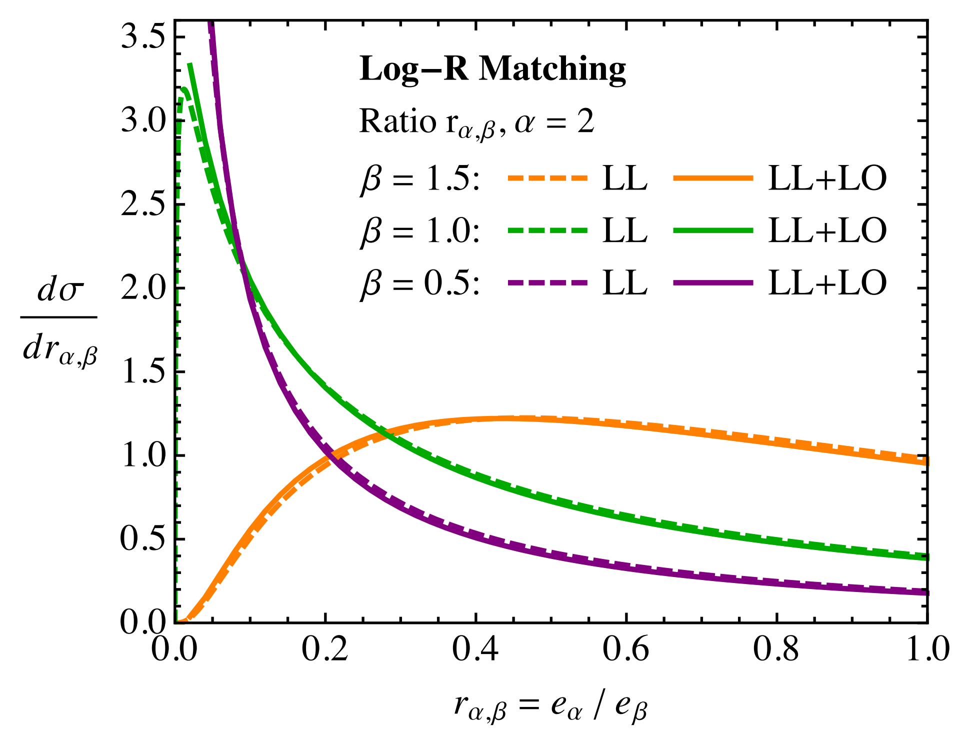
Fixed-Order Matching

Mild impact from exact matrix element

LL ⇒ LL+LO



strongly-ordered

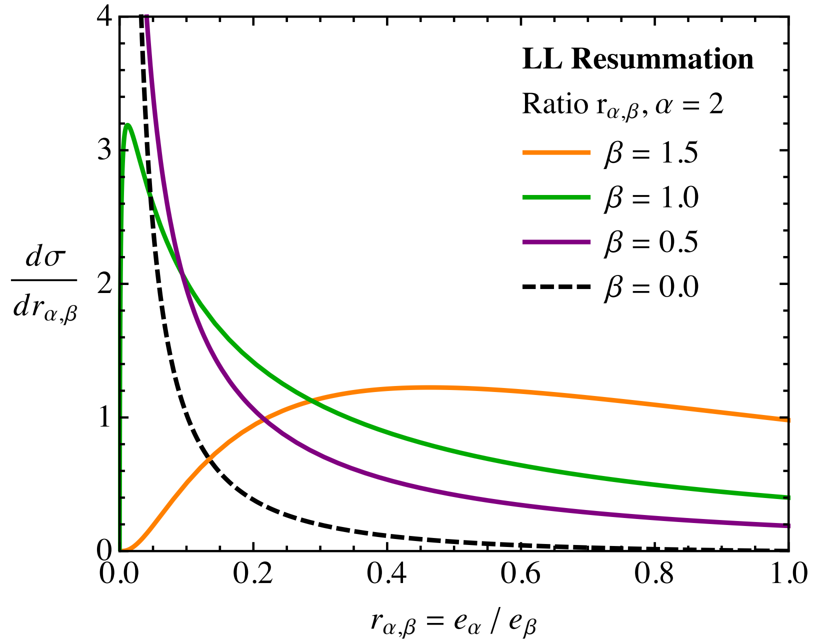


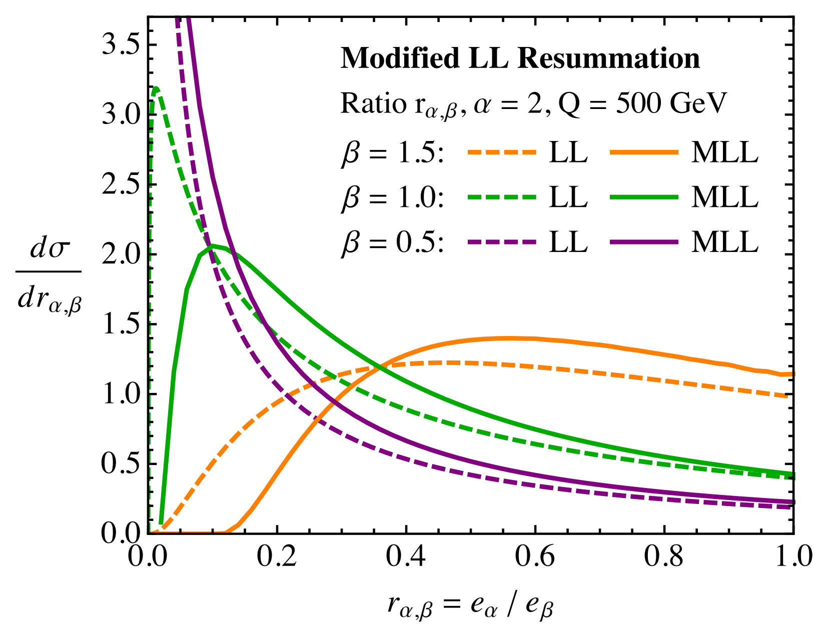
tree-level matching

Higher-Order Resummation

Large impact from running coupling: "Modified Leading Log"

LL ⇒ MLL





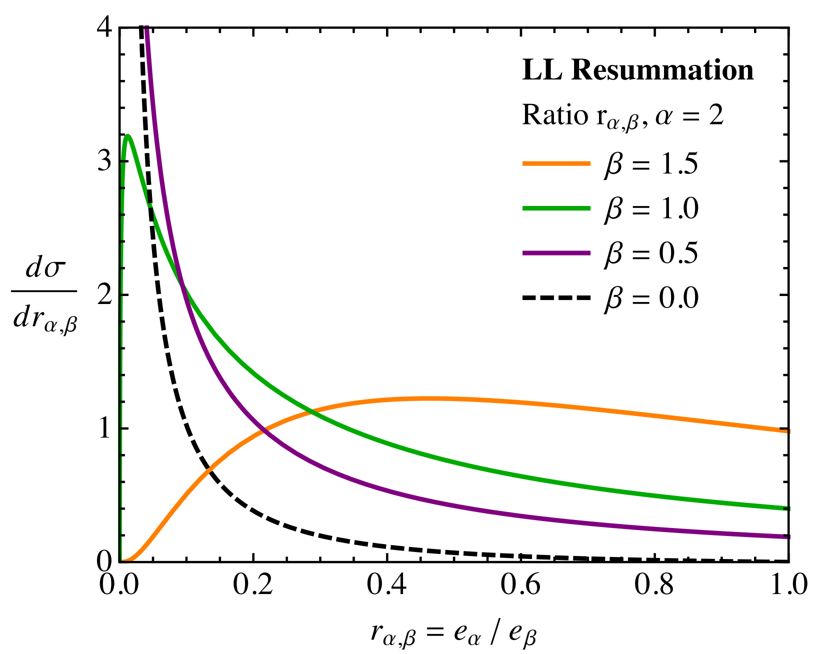
strongly-ordered

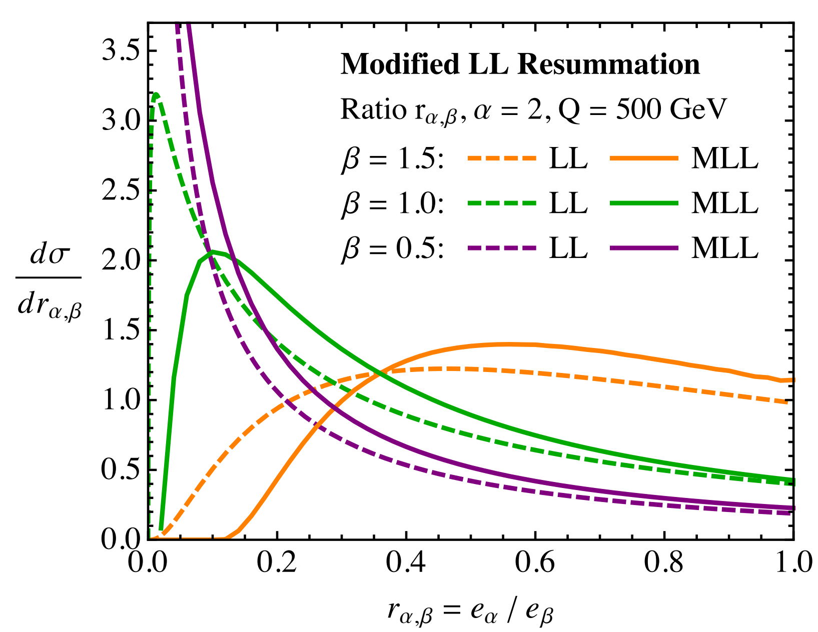
running αs,
subleading splitting functions

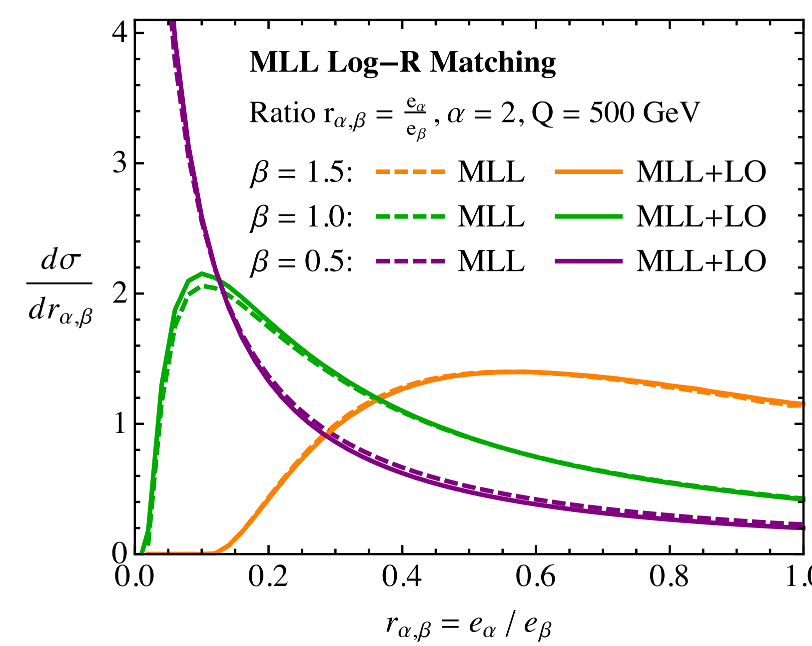
Higher-Order Resummation

Large impact from running coupling: "Modified Leading Log"

LL ⇒ MLL ⇒ MLL+LO







strongly-ordered

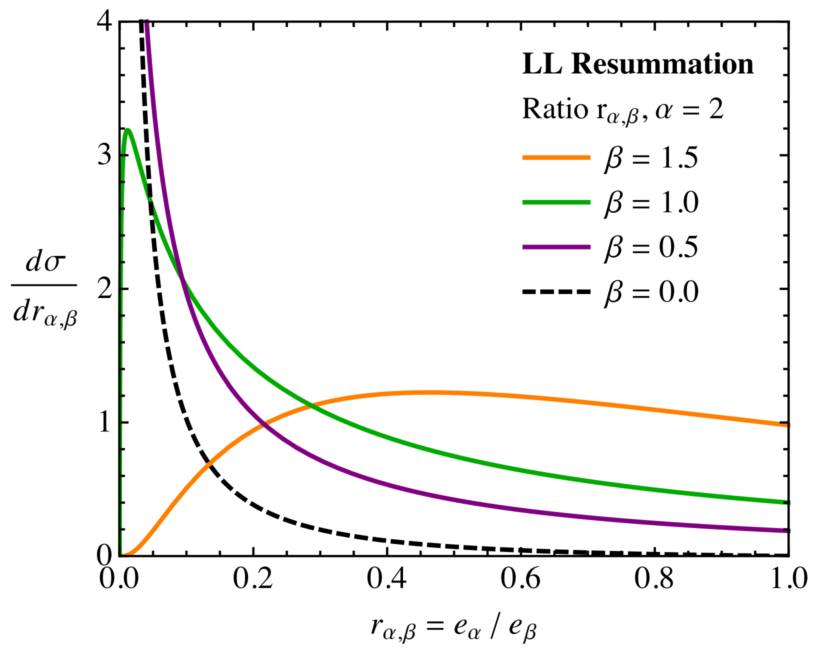
running αs,
subleading splitting functions

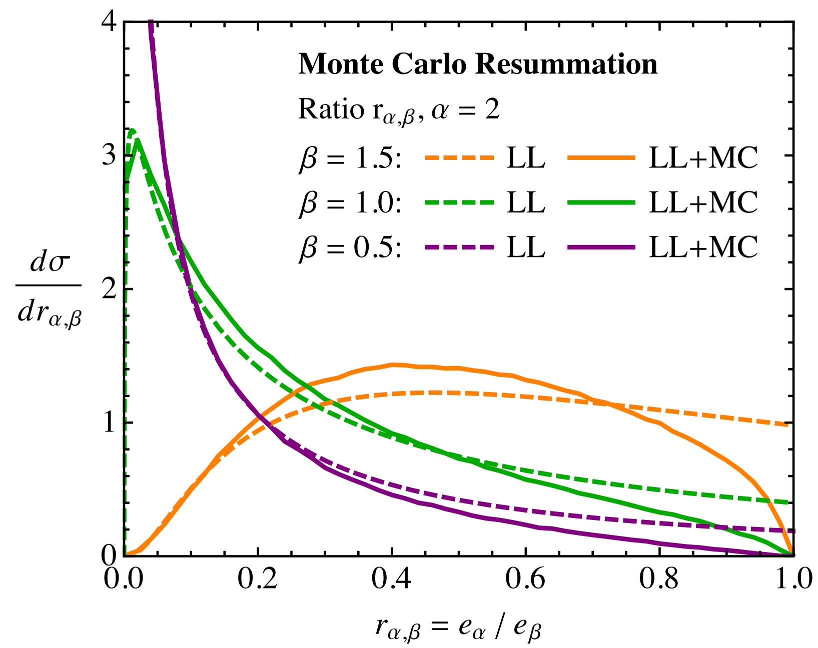
tree-level matching

Multiple Emissions

Change in tail region from Monte-Carlo-style resummation

LL ⇒ LL+MC





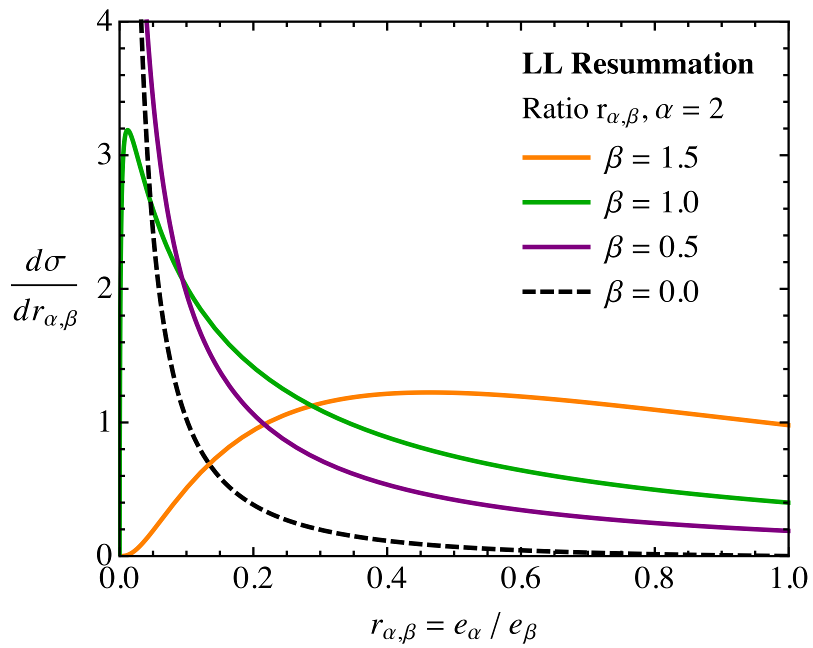
strongly-ordered

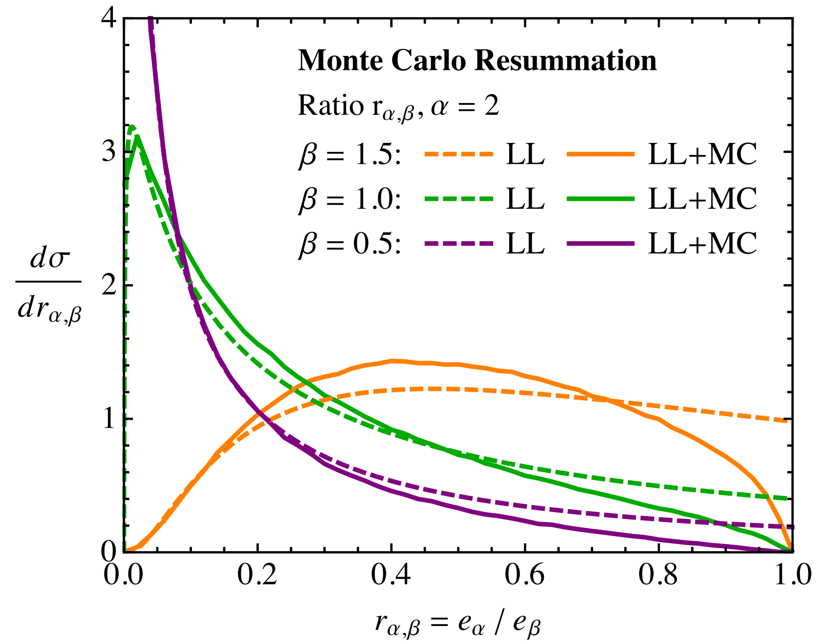
multiple emissions

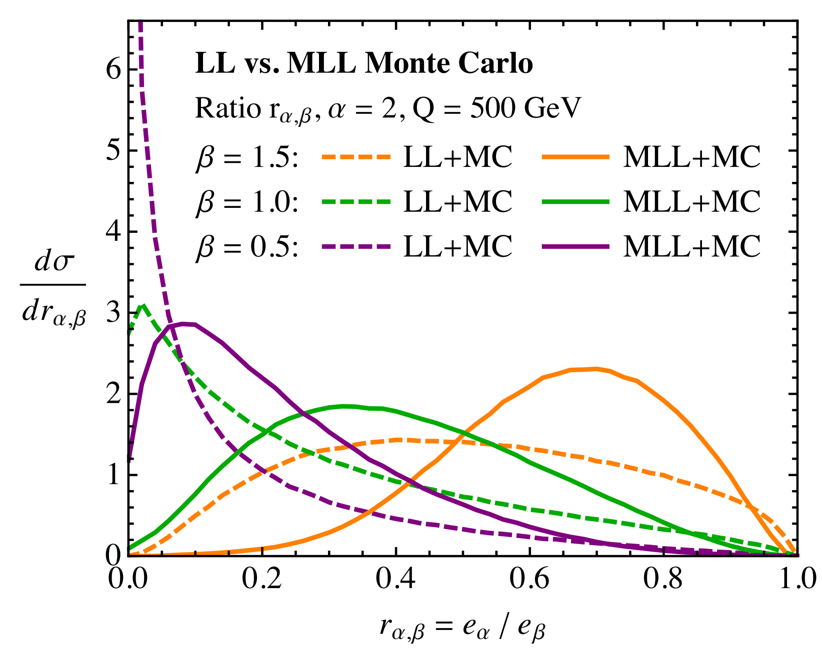
Multiple Emissions

Change in tail region from Monte-Carlo-style resummation

LL ⇒ LL+MC ⇒ $\text{MLL}_{+\text{LO}}$+MC







strongly-ordered

multiple emissions

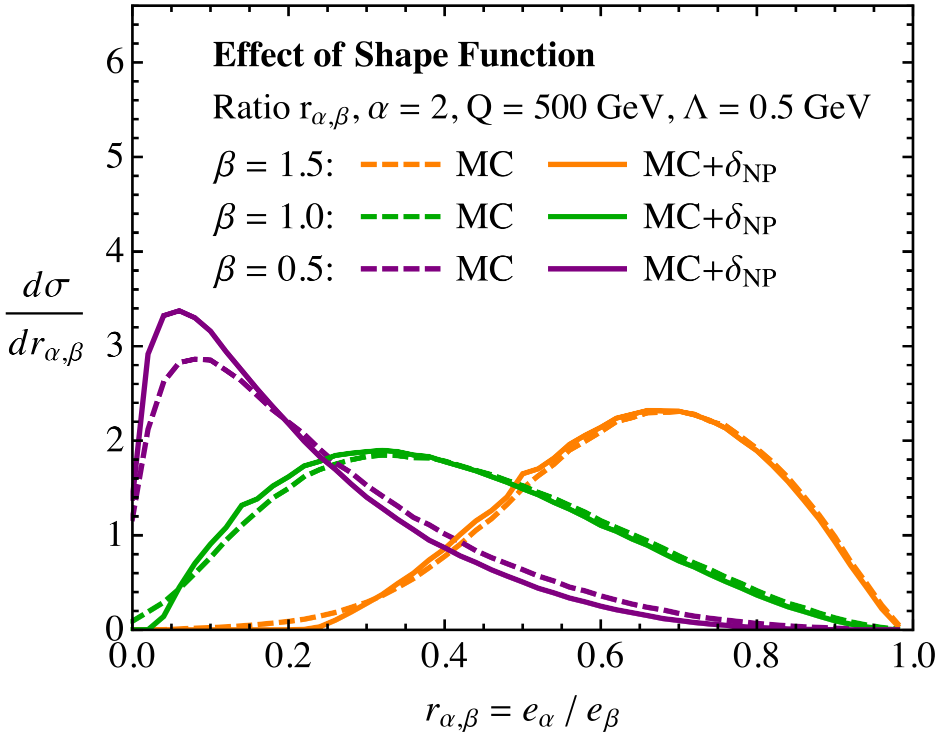
running α_s,
AP splitting functions

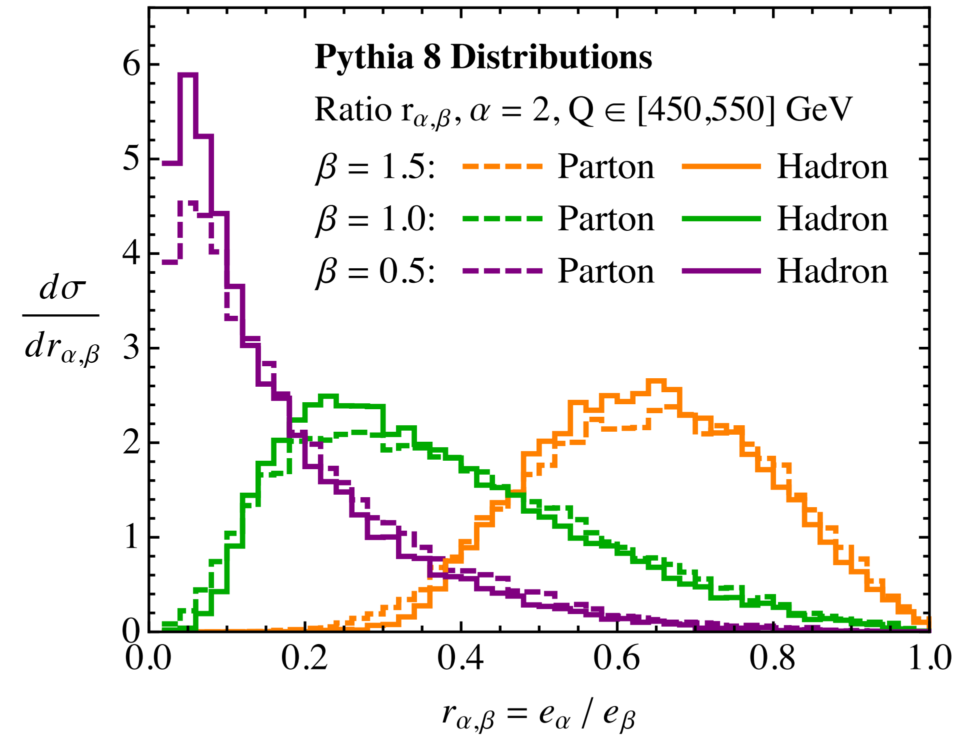
includes tree-level matching

Hadronization Corrections

Power Corrections vs. Hadronization Model

MLL+LO+MC+δ_{NP} vs. Pythia 8





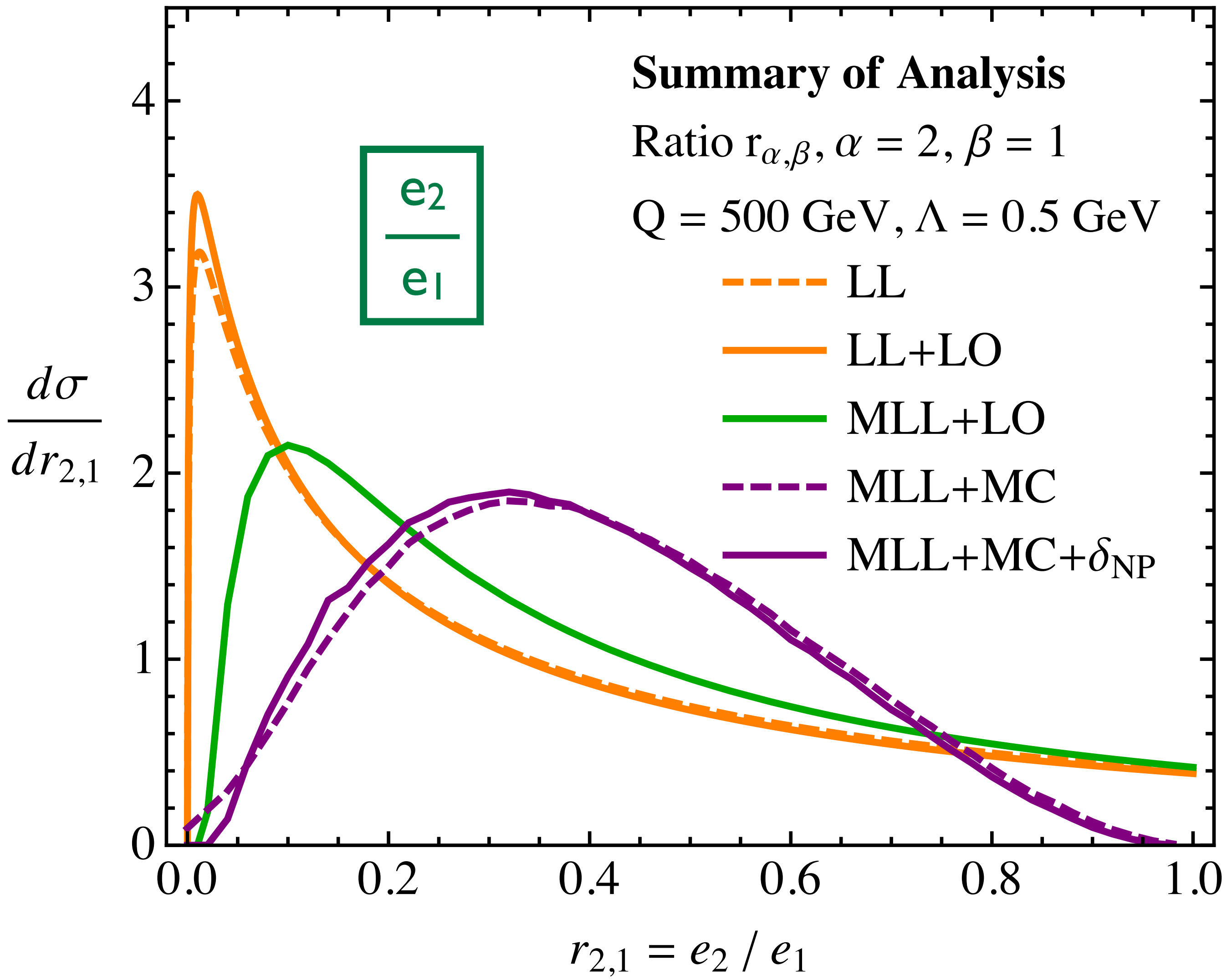
power corrections

hadronization model

momentum conservation,
color coherence, ...

Building up a Realistic Distribution

All of these (and more) in existing parton shower programs



= strongly ordered
+ tree-level matching
+ running α_s
+ multiple emissions
+ hadronization correction

Expect ratio observables to be reasonably well-described in Monte Carlo

Summary

Ratio Observables: Ubiquitous & Unsafe

Even if A,B are IRC safe, R = A/B has singularities in the denominator B
Cannot calculate to any fixed order in α_s

Ratios of Angularities: "Sudakov Safe"

Sudakov factor regulates singularities
Ratio distribution well-defined with all orders in α_s
Amusing expansion in $\sqrt{\alpha_s}$

Calculational Results

Can systematically improve strongly-ordered limit
Many improvements already in parton shower Monte Carlos
Ongoing work to calculate 2-prong ratio observables like τ_2/τ_1

Broader Lesson

If an IRC unsafe observable is
at all related to an IRC safe one,
then probably a way to calculate

Hadronic Cross Sections = PDFs ⊗ IRC Safe
Track-based Observables = Track Functions ⊗ IRC Safe
Ratio Observables = IRC Safe / IRC Safe