$\gamma^*\gamma^*$ Cross Section at NLO and Properties of the BFKL Evolution at Higher Orders

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Photon-Induced Collisions at the LHC - CERN - Geneva
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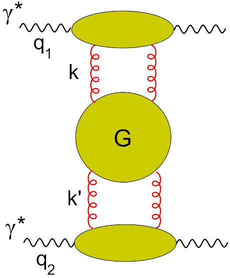
Production of two pairs of charmed quarks in $\gamma\gamma$ collisions

- The cross section for formation of two pairs of charmed quarks in $\gamma\gamma$ collisions was found in explicit form at LO within the Leading Logarithm Approximation (BFKL equation) in 1978 (I. Balitsky and L. N. Lipatov)

- Can we extend at NLO this result?

- Let us start with $\gamma^*\gamma^*$ cross section at NLO.



Outline

- Leading-Log-Approximation at high-energy: general case.
- BFKL in $\mathcal{N}$=4 SYM theory.
- Solution of the NLO BFKL equation in QCD.
- General Form of the Solution of Higher-Order BFKL equation.
- NLO photon impact factor and high-energy OPE
  $\Rightarrow$ $\gamma^*\gamma^*$ cross section.
- Numerical results.
- Conclusions.

Linear Evolution equations in QCD

DGLAP evolution equation

- Resum $\alpha_s \ln \frac{Q^2}{\Lambda_{\text{QCD}}}$
- Eigenfunctions at any order: Powers of x_{B}

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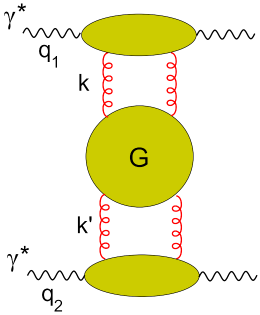
BFKL evolution equation

- LO: resum $(\alpha_s \ln s)^n$. NLO: resum $\alpha_s(\alpha_s \ln s)^n$
- Eigenfunctions:
  - LO: $(k^2)^{\gamma-1}$
  - NLO and higher order: Perturbative eigenfunctions (G.A.C. and Kovchegov).

Balitsky-Fadin-Kuraev-Lipatov equation



$$\frac{\partial}{\partial Y}G(k,k',Y) = \int d^2q\,K(k,q)\,G(q,k',Y)$$

$$G(k,k',Y=0) = \frac{1}{2\pi k}\delta(k-k')$$

$$k \equiv |\vec{k}_\perp| \;\text{ and }\; k' \equiv |\vec{k}'_\perp|$$

$$Y = \ln\frac{s}{k\,k'} \;\text{ and }\; s = (q_1+q_2)^2$$

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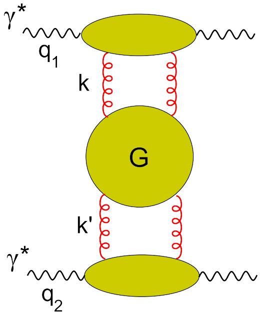

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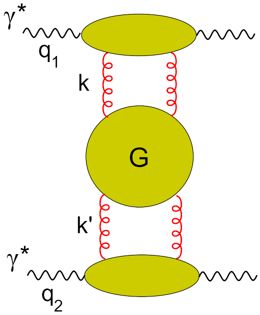

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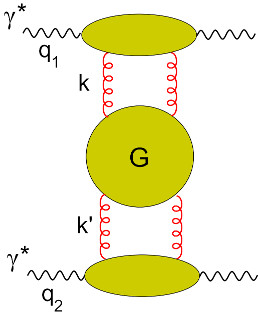

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- Resum $(\alpha_s\, Y)^n \longrightarrow$ LO BFKL eq.
- Resum $\alpha_s\,(\alpha_s\, Y)^n \longrightarrow$ NLO BFKL eq.
- The kernel is real and symmetric: $K(k, k') = K(k', k) \Rightarrow K(k, k')$ is Hermitian and the eigenvalues are real.

LO BFKL equation

$$\frac{\partial}{\partial Y} G(k, k', Y) = \int d^2q \, K^{\rm LO}(k, q) \, G(q, k', Y)$$

$$\int d^2q \, K^{\rm LO}(k, q) \; (q^2)^{-1+\gamma} = \bar{\alpha}_\mu \, \chi_0(\gamma)(k^2)^{-1+\gamma} \qquad \bar{\alpha}_\mu \equiv \frac{\alpha_\mu N_c}{\pi}$$

- $(k^2)^{-1+\gamma}$ are eigenfunctions.

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- $(k^2)^{-1+\gamma}$ are eigenfunctions.
- For $\gamma = \frac{1}{2} + i\nu$ and ν real parameter $\Rightarrow$ $(k^2)^{-1+\gamma}$ form a complete set.
- $\Rightarrow$ LO eigenvalues $\chi_0(\nu) = 2\,\psi(1) - \psi(\frac{1}{2}+i\nu) - \psi(\frac{1}{2}-i\nu)$ are real and sym. $\nu \leftrightarrow -\nu$
- LO BFKL is Conformal invariant.

$$G(k,k',Y) = \int \frac{d\nu}{2\pi^2\, k\, k'} \left(\frac{k^2}{k'^2}\right)^{i\nu} e^{\bar{\alpha}_\mu \chi_0(\nu)\, Y}$$

BFKL equation in the $\mathcal{N}$=4 SYM case

- In $\mathcal{N} = 4$ SYM theory the coupling constant does not run.
- $\Rightarrow (k^2)^{-\frac{1}{2}+i\nu}$ are eigenfunctions at any order.

$$K(q,k) = \alpha_{\rm SYM} K^{\rm LO}(q,k) + \alpha_{\rm SYM}^2 K^{\rm NLO}(q,k) + \dots$$

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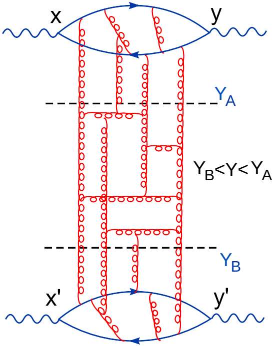
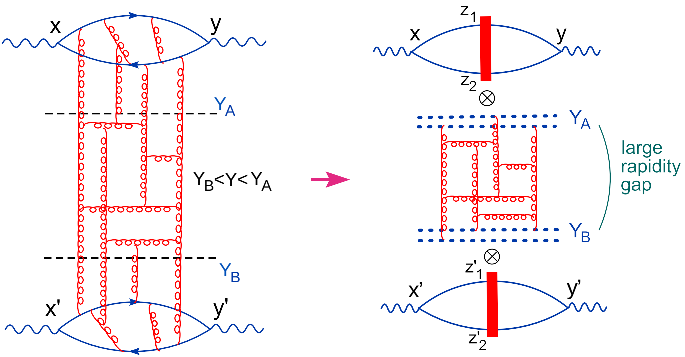
$$\int d^2q\, K(q,k)\, (q^2)^{-\frac{1}{2}+i\nu} = [\alpha_{\rm SYM}\chi_0(\nu) + \alpha^2_{\rm SYM}\chi_1(\nu)\ldots]\,(k^2)^{-\frac{1}{2}+i\nu}$$

$$G(k,k',Y) = \int \frac{d\nu}{2\pi^2 k\, k'}\, e^{[\alpha_{\rm SYM}\chi_0(\nu)+\alpha^2_{\rm SYM}\chi_1(\nu)\ldots]} \left(\frac{k^2}{k'^2}\right)^{i\nu}$$

- The eigenvalues $\bar{\alpha}_\mu\, \chi_0(\nu) + \bar{\alpha}^2_\mu\, \chi_1^{\rm SYM}(\nu) + \ldots$ are real and symmetric for $\nu \leftrightarrow -\nu$.

$\gamma^*\gamma^*$ scattering cross-section at LO





$$\langle j^\alpha(x) j^\beta(y) j^\rho(x') j^\lambda(y')\rangle \propto I_A^{\alpha\beta}(x,y;z_1,z_2)\, I_B^{\rho\lambda}(x',y';z'_1,z'_2) \otimes \langle \mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}^{Y_A} \mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}^{Y_B}\rangle$$

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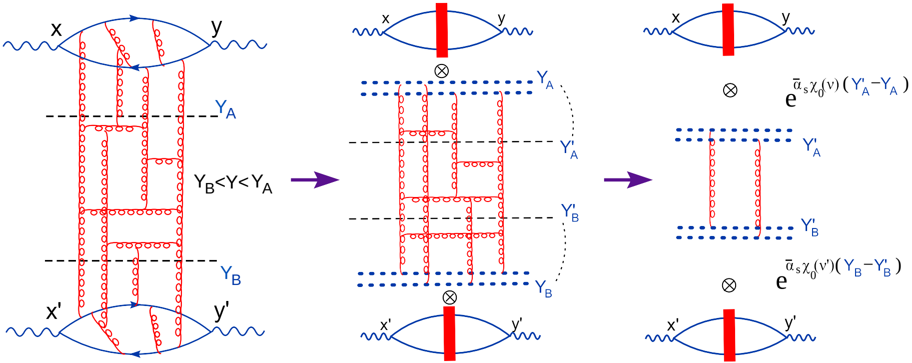

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$$\otimes\langle\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}^{Y_A}\rangle_A\langle\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}^{Y_B}\rangle_A$$

$\gamma^*\gamma^*$ scattering cross-section at LO

$$U_x = \mathrm{Pexp}\left\{ ig \int_{-\infty}^{+\infty} dx^+ A^-(x^+ + x_\perp) \right\}$$

$$\mathcal{A}^{\alpha\beta\,\rho\lambda}(q_1,q_2) \propto i\frac{\alpha_s^2}{Q_1Q_2}\int d\nu\, I^{\alpha\beta}_{\rm LO}(\nu)\, I^{\rho\lambda}_{\rm LO}(\nu)\, \left(\frac{Q_1^2}{Q_2^2}\right)^{i\nu} e^{\bar\alpha_\mu\,\chi_0(\nu)\,(Y_A - Y_B)}$$

$$Y_A = \frac{1}{2}\ln\frac{s}{Q_1^2}, \qquad Y_B = -\frac{1}{2}\ln\frac{s}{Q_2^2}, \qquad s = (q_1+q_2)^2$$

$$\mathcal{A}^{\alpha\beta\,\rho\lambda}(q_1,q_2) \propto i\frac{\alpha_s^2}{Q_1Q_2}\int d\nu\, I^{\alpha\beta}_{\rm LO}(\nu)\, I^{\rho\lambda}_{\rm LO}(\nu)\, \left(\frac{Q_1^2}{Q_2^2}\right)^{i\nu} e^{\bar\alpha_\mu\,\chi_0(\nu)\,\ln\frac{s}{Q_1Q_2}}$$

What about NLO?

- Can we repit the steps performed at LO also at NLO?
- Problems to be solved:
  - Solve NLO BFKL equation   G.A.C and Yu. Kovchegov (JHEP 2013)
  - Calculate NLO Impact Factor   I. Balitsky and G.A.C. (PRD 2012)
  - NLO Impact Factor has to be conformal invariant;
  - $\Rightarrow$ Energy dependence of NLO Impact Factor needs to be eliminated;
  - $\Rightarrow$ Composite Wilson line operators   I. Balisky and G.A.C. (NPB 2009)

BFKL equation at NLO in QCD

$$K^{\mathrm{LO+NLO}}(k,q) \equiv \bar{\alpha}_\mu\, K^{\mathrm{LO}}(k,q) + \bar{\alpha}_\mu^2\, K^{\mathrm{NLO}}(k,q)$$

$$\int d^2q\, K^{\mathrm{LO+NLO}}(k,q)\; q^{2\gamma-2} = \left[\bar{\alpha}_\mu\, \chi_0(\gamma) - \bar{\alpha}_\mu^2\, \beta_2\, \chi_0(\gamma)\, \ln\frac{k^2}{\mu^2} + \bar{\alpha}_\mu^2\, \frac{\delta(\gamma)}{4}\right] k^{2\gamma-2}$$

$$\bar{\alpha}_\mu = \frac{\alpha_\mu N_c}{\pi}, \qquad \beta_2 = \frac{11\,N_c - 2\,N_f}{12\,N_c}$$

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- $-\bar{\alpha}_\mu^2\, \beta_2\, \chi_0(\gamma)\, \ln\frac{k^2}{\mu^2}$ 1-loop running coupling.
- $\delta(\gamma) = -2\,\beta_2\,\chi_0'(\gamma) + 4\,\chi_1(\gamma)$ NLO Conformal terms Fadin-Lipatov (1998)
- $\chi_1(\gamma)$ Real and symmetric in $\gamma \leftrightarrow 1-\gamma$ $\gamma = \frac{1}{2} + i\nu$.
- $\frac{d}{d\gamma}\chi_0(\gamma) \equiv \chi_0'(\gamma)$ imaginary and not symmetric.

Perturbative eigenfunctions $H_\gamma(k) = k^{2\gamma-2} + \bar{\alpha}_\mu F_\gamma(k)$

- we have to determine $F_\gamma(k)$ so that

$$\int d^2q\, K^{\rm LO+NLO}(k,q)\, H_\gamma(q) = \Delta(\gamma)\, H_\gamma(k)$$

- $\Delta(\gamma)$ eigenvalues to be determined.
- $F_\gamma(k)$ has to satisfy:

$$-\chi_0(\gamma)\, F_\gamma(k) + \int d^2q\, K^{\rm LO}(k,q) F_\gamma(q) - \beta_2\, \chi_0(\gamma)\, H_\gamma(k)\, \ln\frac{k^2}{\mu^2} = c(\gamma)\, H_\gamma(k)$$

For some function $c(\gamma)$.

NLO eigenfunctions ansatz

At this order we can write the condition for $F_\gamma(k)$ as:

$$-\chi_0(\gamma)F_\gamma(k) + \int d^2q\, K^{\mathrm{LO}}(k,q)F_\gamma(q) - \beta_2\,\chi_0(\gamma)\,k^{2\gamma-2}\,\ln\frac{k^2}{\mu^2} = c(\gamma)\,k^{2\gamma-2}$$

- Ansatz

$$F_\gamma(k) = k^{2\gamma-2}\left[c_0(\gamma) + c_1(\gamma)\,\ln\frac{k^2}{\mu^2} + c_2(\gamma)\ln^2\frac{k^2}{\mu^2} + c_3(\gamma)\ln^3\frac{k^2}{\mu^2} + \ldots\right]$$

NLO eigenfunctions

- Truncate the series at $n = 2$ $\Rightarrow$

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- $\Rightarrow$ For $c_2(\gamma) = \frac{\beta_2\,\chi_0(\gamma)}{2\,\chi_0'(\gamma)}$ and for any $c_0(\gamma)$ and $c_1(\gamma)$ the eigenfunction is

$$H_\gamma(k) = k^{2\gamma-2}\ \left[1 + \bar{\alpha}_\mu\left(\frac{\beta_2\,\chi_0(\gamma)}{2\,\chi_0'(\gamma)}\ \ln^2\frac{k^2}{\mu^2} + c_1(\gamma)\ln\frac{k^2}{\mu^2} + c_0(\gamma)\right)\right]$$

and eigenvalues

$$\Delta(\gamma) = \bar{\alpha}_\mu\,\chi_0(\gamma) + \bar{\alpha}_\mu^2\left(-\frac{1}{2}\,\beta_2\,\chi_0'(\gamma) + \chi_1(\gamma) + c_1(\gamma)\chi_0'(\gamma) + \frac{\beta_2\,\chi_0(\gamma)}{2\,\chi_0'(\gamma)}\right)$$

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- Next step is to impose Completeness relation.

Completeness of the NLO eigenfunctions

$$\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{d\gamma}{2\pi i} H_\gamma(k)\, H^*_\gamma(k') = \delta(k^2 - k'^2)$$

- Completeness relation has to be satisfied order-by-order in α_μ.

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- Completeness relation at LO is satisfied with $\sigma = \frac{1}{2}$ and $\gamma = \frac{1}{2} + i\nu$ with ν real parameter.
  $\Rightarrow$ we have to impose α_μ-order to be 0 $\Rightarrow$

$$\mathrm{Re}[c_1(\nu)] = \frac{\beta_2}{2} \frac{\partial}{\partial\nu} \frac{\chi_0(\nu)}{\chi_0'(\nu)} \qquad \frac{\partial}{\partial\nu}\mathrm{Im}[c_1(\nu)] + 2\,\mathrm{Re}[c_0(\nu)] = 0$$

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- If completeness relation is satisfied then orthogonality relation is also satisfied.

Completeness of the NLO H-eigenfunctions

- provided that $\frac{\partial}{\partial\nu}\mathrm{Im}[c_1(\nu)] + 2\,\mathrm{Re}[c_0(\nu)] = 0$ the eigenfunctions are

$$H_{\frac{1}{2}+i\nu}(k) = k^{-1+2i\nu}\left[1 + \bar{\alpha}_\mu\left(i\frac{\beta_2\chi_0(\nu)}{2\chi_0'(\nu)}\ln^2\frac{k^2}{\mu^2} + \frac{\beta_2}{2}\left(\frac{\partial}{\partial\nu}\frac{\chi_0(\nu)}{\chi_0'(\nu)}\right)\ln\frac{k^2}{\mu^2} + i\mathrm{Im}[c_1(\nu)]\ln\frac{k^2}{\mu^2} + \mathrm{Re}[c_0(\nu)]\right)\right]$$

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- and the eigenvalues are real ($\mathrm{Im}[c_1(\nu)]\,\chi_0'(\nu)$ is real function of ν)

$$\Delta(\nu) = \bar{\alpha}_\mu\,\chi_0(\nu) + \bar{\alpha}_\mu^2\left[\chi_1(\nu) + \mathrm{Im}[c_1(\nu)]\,\chi_0'(\nu)\right]$$

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$$\Delta(\nu) = \bar{\alpha}_\mu\,\chi_0(\nu) + \bar{\alpha}_\mu^2\left[\chi_1(\nu) + \mathrm{Im}[c_1(\nu)]\,\chi_0'(\nu)\right]$$

- The imaginary term $-2\,\beta_2\,\bar{\alpha}_\mu^2\chi_0'(\gamma)$ has canceled.
- We have freedom to chose $\mathrm{Re}[c_0(\nu)]$ and $\mathrm{Im}[c_1(\nu)]$
- This freedom will not affect the solution. It is just an artifact of the phase and ν-reparametrization freedom of the $H_{\frac{1}{2}+i\nu}$ function.

Phase freedom and ν-reparametrization

- Phase freedom of the H-functions:

$$H_{\frac{1}{2}+i\nu}(k) \to e^{-i\bar{\alpha}_\mu \left(\mathrm{Im}[c_0(\nu)] - \mathrm{Im}[c_1(\nu)] \ln \mu^2\right)} H_{\frac{1}{2}+i\nu}(k)$$

- ν-reparametrization: $\nu' = \nu + \bar{\alpha}_\mu \mathrm{Im}[c_1(\nu)]$

$\Rightarrow$
The Phase freedom and ν-reparametrization allow us to put $c_0(\nu) = 0$ and $\mathrm{Im}[c_1(\nu)] = 0$.

Solution of NLO BFKL equation

- NLO eigenfunctions: perturbation around the conformal LO eigenfunctions

$$H_{\frac{1}{2}+i\nu}(k) = k^{-1+2i\nu}\left[1+\bar{\alpha}_\mu\,\beta_2\left(i\,\frac{\chi_0(\nu)}{2\,\chi_0'(\nu)}\,\ln^2\frac{k^2}{\mu^2}+\frac{1}{2}\left(\frac{\partial}{\partial\nu}\frac{\chi_0(\nu)}{\chi_0'(\nu)}\right)\ln\frac{k^2}{\mu^2}\right)\right]$$

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Solution of NLO BFKL equation **G.A.C. and Yu. Kovchegov (JHEP 2013)**

$$G(k, k', Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2}\, e^{\left[\bar{\alpha}_\mu\,\chi_0(\nu) + \bar{\alpha}_\mu^2\,\chi_1(\nu)\right] Y}\, H_{\frac{1}{2}+i\nu}(k)\,\left[H_{\frac{1}{2}+i\nu}(k')\right]^*$$

- The perturbative expansion is in both the exponent and in the eigenfunctions (contrary to DGLAP case and $\mathcal{N}$=4 BFKL).

μ-independence of the NLO solution

- The $H_\gamma(k)$ eigenfunctions diagonalize the LO+NLO BFKL kernel

$$\bar{\alpha}_\mu K^{\mathrm{LO}}(k,q) + \bar{\alpha}_\mu^2 K^{\mathrm{NLO}}(k,q) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\gamma}{2\,\pi^2\,i}\,\Delta(\gamma)\,H_\gamma(k) H^*_\gamma(q)$$

- LO+NLO BFKL kernel is μ-independent up to $\mathcal{O}(\alpha_\mu^3)$ $\Rightarrow$
- So is its diagonalization through $H_\gamma(k)$ eigenfunctions.

$$
\begin{aligned}
G(k,k',Y) &= \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\, k\, k'}\, e^{\left[\bar{\alpha}_\mu\, \chi_0(\nu) + \bar{\alpha}_\mu^2\, \chi_1(\nu)\right] Y} H_\gamma(k) H_\gamma^*(q) \\
&= \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\, k\, k'}\, e^{\left[\bar{\alpha}_\mu\, \chi_0(\nu) + \bar{\alpha}_\mu^2\, \chi_1(\nu)\right] Y} \left(\frac{k^2}{k'^2}\right)^{i\nu} \left(1 - \bar{\alpha}_\mu^2\, \beta_2\, \chi_0(\nu)\, Y\, \ln\frac{kk'}{\mu^2}\right)
\end{aligned}
$$

- $\Rightarrow G(k,k',Y)$ is μ-independent up to order $\mathcal{O}(\alpha_\mu^3)$.

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\, k\, k'}\, e^{\left[\bar{\alpha}_\mu\, \chi_0(\nu) + \bar{\alpha}_\mu^2\, \chi_1(\nu)\right]\, Y} H_\gamma(k) H_\gamma^*(q)$$

$$= \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\, k\, k'}\, e^{\left[\bar{\alpha}_\mu\, \chi_0(\nu) + \bar{\alpha}_\mu^2\, \chi_1(\nu)\right]\, Y} \left(\frac{k^2}{k'^2}\right)^{i\nu} \left(1 - \bar{\alpha}_\mu^2\, \beta_2\, \chi_0(\nu)\, Y\, \ln\frac{kk'}{\mu^2}\right)$$

- $\Rightarrow G(k,k',Y)$ is μ-independent up to order $\mathcal{O}(\alpha_\mu^3)$.
- At NLO we may write the solution as

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\, k\, k'}\, e^{\left[\bar{\alpha}_s(k\, k')\, \chi_0(\nu) + \bar{\alpha}_s^2(k\, k')\, \chi_1(\nu)\right]\, Y} \left(\frac{k^2}{k'^2}\right)^{i\nu}$$

- At this order the scale $\bar{\alpha}_s^{\lambda}(k^2)\, \bar{\alpha}_s^{\lambda}(k'^2) \bar{\alpha}_s^{1-2\lambda}(k\, k')$ (for real λ) works as well.

General Form of the Solution of All-Order BFKL equation

Ansatz

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2}\, e^{\left[\bar\alpha_s(k\,k')\,\chi_0(\nu)+\bar\alpha_s^2(k\,k')\,\chi_1(\nu)+\bar\alpha_s^3(k\,k')\,\chi_2(\nu)+\ldots\right]Y} \left(\frac{k^2}{k'^2}\right)^{i\nu}$$

- $\chi_2(\nu)$ and higher-order coefficients indicated by the ellipsis in the exponent are the scale-invariant (conformal) $(\nu \leftrightarrow -\nu)$-even (real-valued) parts of the prefactor function generated by the action of the next-to-next-to-leading-order (NNLO) (and higher-order) kernels on the LO eigenfunctions.

General Form of the Solution of All-Order BFKL equation

Ansatz

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2}\, e^{\left[\bar\alpha_s(kk')\,\chi_0(\nu)+\bar\alpha_s^2(kk')\,\chi_1(\nu)+\bar\alpha_s^3(kk')\,\chi_2(\nu)+\ldots\right]Y}\left(\frac{k^2}{k'^2}\right)^{i\nu}$$

- To check the ansatz we have to plug it in the evolution eq. and we need the two-loop beta-function β_3:

$$\mu^2\frac{d\bar\alpha_\mu}{d\mu^2} = -\beta_2\bar\alpha_\mu^2 + \beta_3\bar\alpha_\mu^3$$

and

$$\int d^2q\, K^{\rm LO+NLO+NNLO}(k,q)\; q^{-1+2i\nu} = \Big\{\bar\alpha_\mu\,\chi_0(\nu)\left[1-\bar\alpha_\mu\,\beta_2\,\ln\frac{k^2}{\mu^2}+\bar\alpha_\mu^2\beta_2^2\,\ln^2\frac{k^2}{\mu^2}+\bar\alpha_\mu^2\,\beta_3\,\ln\frac{k^2}{\mu^2}\right]$$
$$+\bar\alpha_\mu^2\left[\frac{i}{2}\,\beta_2\,\chi_0'(\nu)+\chi_1(\nu)\right]\left[1-2\,\bar\alpha_\mu\,\beta_2\,\ln\frac{k^2}{\mu^2}\right]+\bar\alpha_\mu^3\,\left[\chi_2(\nu)+i\,\delta_2(\nu)\right]\Big\}\;k^{-1+2i\nu}$$

- The ansatz does not work, but it allows us to recover the structure of the NNLO solution

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2 k\, k'}\, e^{\left[\bar{\alpha}_s(k\,k')\,\chi_0(\nu)+\bar{\alpha}_s^2(k\,k')\chi_1(\nu)+\bar{\alpha}_s^3(k\,k')\,\chi_2(\nu)\right]Y} \left(\frac{k^2}{k'^2}\right)^{i\nu}$$

$$\times \left\{1 + (\bar{\alpha}_\mu \beta_2)^2 \left[-\frac{1}{24}(\bar{\alpha}_\mu Y)^3 \chi_0(\nu)^2 \chi_0''(\nu) + \frac{1}{4}(\bar{\alpha}_\mu Y)^2 \chi_0(\nu)\left(\frac{\chi_0'(\nu)^2}{2\chi_0(\nu)} - \chi_0''(\nu)\right) + \bar{\alpha}_\mu Y \frac{\chi_0''(\nu)}{4}\right]\right\}$$

provided that the imaginary part $i\delta_2(\gamma)$ is

$$i\delta_2(\nu) = -\frac{i}{2}\chi_0'(\nu)\beta_3 + i\chi_1'(\nu)\beta_2$$

This solution satisfies also the initial condition: the solution is unique so it is the right one.

General Form of the Solution of All-Order BFKL equation

- The ansatz does not work, but it allows us to recover the structure of the NNLO solution

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2 k\,k'}\, e^{\left[\bar{\alpha}_s(k\,k')\,\chi_0(\nu) + \bar{\alpha}_s^2(k\,k')\chi_1(\nu) + \bar{\alpha}_s^3(k\,k')\,\chi_2(\nu)\right]Y} \left(\frac{k^2}{k'^2}\right)^{i\nu}$$

$$\times \left\{ 1 + (\bar{\alpha}_\mu\beta_2)^2 \left[-\frac{1}{24}(\bar{\alpha}_\mu Y)^3 \chi_0(\nu)^2 \chi_0''(\nu) + \frac{1}{4}(\bar{\alpha}_\mu Y)^2 \chi_0(\nu) \left(\frac{\chi_0'(\nu)^2}{2\chi_0(\nu)} - \chi_0''(\nu) \right) + \bar{\alpha}_\mu Y \frac{\chi_0''(\nu)}{4} \right] \right\}$$

provided that the imaginary part $i\delta_2(\gamma)$ is

$$i\delta_2(\nu) = -\frac{i}{2}\chi_0'(\nu)\beta_3 + i\chi_1'(\nu)\beta_2$$

This solution satisfies also the initial condition: the solution is unique so it is the right one.

- The imaginary part $i\delta_2(\nu)$ has to be confirmed from the explicit calculation of the NNLO eigenfunction.

- So, let us calculate explicitly the NNLO eigenfunction.

General Form of the Solution of All-Order BFKL equation

The eigenfunction of the NNLO BFKL equation is

$$H_{\frac{1}{2}+i\nu}(k) = k^{-1+2i\nu}\left[1 \quad + \quad \bar{\alpha}_\mu\,\beta_2\left(i\,\frac{\chi_0(\nu)}{2\,\chi_0'(\nu)}\,\ln^2\frac{k^2}{\mu^2} + \frac{1}{2}\left(\frac{\partial}{\partial\nu}\frac{\chi_0(\nu)}{\chi_0'(\nu)}\right)\ln\frac{k^2}{\mu^2}\right)\right.$$
$$\left. + \quad \bar{\alpha}_\mu^2\,f_2\left(\frac{k}{\mu},\nu\right) + \dots\right]$$

- The function $f_2(k/\mu,\nu)$ denotes the NNLO corrections to the eigenfunctions.
- Ansatz for $f_2(k/\mu,\nu)$:

$$f_2(k/\mu,\nu) = c_0^{(2)}(\nu) + c_1^{(2)}(\nu)\,\ln\frac{k^2}{\mu^2} + c_2^{(2)}(\nu)\ln^2\frac{k^2}{\mu^2} + c_3^{(2)}(\nu)\ln^3\frac{k^2}{\mu^2} + \dots$$

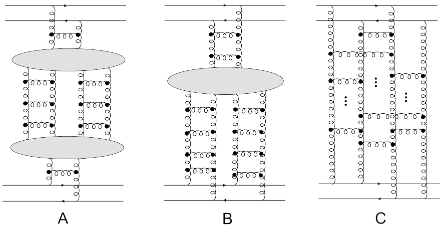
- We have completely determined the $c_0^{(2)}(\nu), c_1^{(2)}(\nu), c_2^{(2)}(\nu), c_3^{(2)}(\nu)$ and confirmed the ansatz for the structure of the NNLO solution of the NNLO BFKL equation.

Definition of BFKL to all-orders

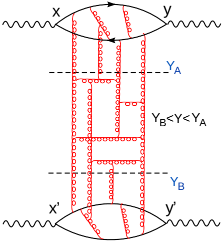
- BFKL to all-orders: linearization of the evolution equation for the color Wilson line composite operator to all-orders

$$\frac{d}{d\eta}\mathrm{tr}\{U^{\eta}(x_\perp)U^{\eta\dagger}(y_\perp)\} = \Big[\alpha_s K_{\rm LO} + \alpha_s^2 K_{\rm NLO} + \dots\Big] \otimes \mathrm{tr}\{U^{\eta}(x_\perp)U^{\eta\dagger}(y_\perp)\}$$

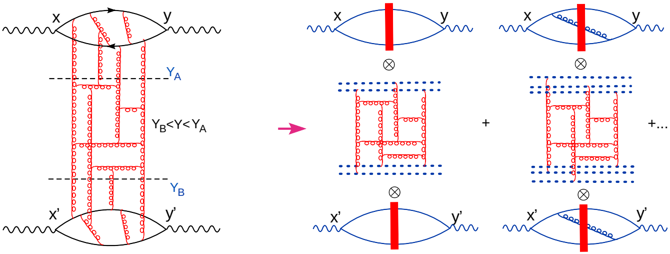
- No Pomeron loops (Diagram A).
- No Pomeron vertices (Diagram B).
- No Reggions interactions (Diagram C).



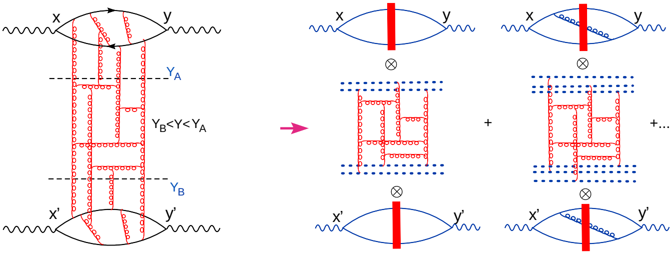
$\gamma^*\gamma^*$ scattering cross-section at NLO



$\gamma^*\gamma^*$ scattering cross-section at NLO



Using high-energy Operator Product Expansion in composite Wilson line operator we get NLO Impact Factor that does not scale with energy.



Composite Wilson line operators

I. Balitsky and G.A.C. (PRB 2009)

$$[\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} = \mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$+ \frac{\alpha_s}{4\pi}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\frac{1}{N_c}\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - \mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2} + O(\alpha_s^2)$$

k_T-factorization form

I. Balitsky and G.A.C. (PRD 2012)

$$\int d^4x\, e^{iq\cdot x}\langle p|\mathrm{T}\{\hat{j}_\mu(x)\hat{j}_\nu(0)\}|p\rangle = \frac{s}{2}\int\frac{d^2k_\perp}{k_\perp^2}I_{\mu\nu}(q,k_\perp)\mathcal{V}_{a=x_B}(k_\perp)$$

where the evolution of the dipole gluon distribution at NLO is

$$\begin{aligned}
2a\frac{d}{da}\mathcal{V}_a(k) \;=\; & \frac{\alpha_s N_c}{\pi^2}\int d^2k'\Big\{\Big[\frac{\mathcal{V}_a(k')}{(k-k')^2} - \frac{(k,k')\mathcal{V}_a(k)}{k'^2(k-k')^2}\Big] \\
\times & \Big(1+\frac{\alpha_s b}{4\pi}\Big[\ln\frac{\mu^2}{k^2}+\frac{N_c}{b}\Big(\frac{67}{9}-\frac{\pi^2}{3}-\frac{10n_f}{9N_c}\Big)\Big]\Big) - \frac{b\alpha_s}{4\pi} \\
\times & \Big[\frac{\mathcal{V}_a(k')}{(k-k')^2}\ln\frac{(k-k')^2}{k'^2} - \frac{k^2\mathcal{V}_a(k)}{k'^2(k-k')^2}\ln\frac{(k-k')^2}{k^2}\Big] \\
+ & \frac{\alpha_s N_c}{4\pi}\Big[-\frac{\ln^2(k^2/k'^2)}{(k-k')^2} + F(k,k') + \Phi(k,k')\Big]\;\mathcal{V}_a(k')\Big\} + 3\frac{\alpha_s^2 N_c^2}{2\pi^2}\zeta(3)\mathcal{V}_a(k)
\end{aligned}$$

$$\mathcal{V}(k_\perp) \equiv \int d^2z_\perp e^{-i(k,z)_\perp}\mathcal{V}(z_\perp) \qquad \mathcal{V}(z_\perp) = -\partial_\perp^2\big[1-\tfrac{1}{N_c}\mathrm{Tr}\{U(z_\perp)U^\dagger(0)\}\big]$$

k_T-factorization form

I. Balitsky and G.A.C. (PRD 2012)

$$I^{\mu\nu}(q,k_\perp) = \frac{N_c}{32}\int\frac{d\nu}{\pi\nu}\frac{\sinh\pi\nu}{(1+\nu^2)\cosh^2\pi\nu}\left(\frac{k_\perp^2}{Q^2}\right)^{\frac{1}{2}-i\nu}\Big\{\Big[\Big(\frac{9}{4}+\nu^2\Big)\Big(1+\frac{\alpha_s}{\pi}+\frac{\alpha_s N_c}{2\pi}\mathcal{F}_1(\nu)\Big)P_1^{\mu\nu} + \Big(\frac{11}{4}+3\nu^2\Big)\Big(1+\frac{\alpha_s}{\pi}+\frac{\alpha_s N_c}{2\pi}\mathcal{F}_2(\nu)\Big)P_2^{\mu\nu}\Big]\Big\}$$

$$P_1^{\mu\nu} = g^{\mu\nu} - \frac{q_\mu q_\nu}{q^2}$$

$$P_2^{\mu\nu} = \frac{1}{q^2}\Big(q^\mu - \frac{p_2^\mu q^2}{q\cdot p_2}\Big)\Big(q^\nu - \frac{p_2^\nu q^2}{q\cdot p_2}\Big)$$

k_T-factorization form

I. Balitsky and G.A.C. (PRD 2012)

$$\mathcal{F}_{1(2)}(\nu) \;=\; \Phi_{1(2)}(\nu) + \chi_\gamma \Psi(\nu), \quad \mathcal{F}_3(\nu) \;=\; F_6(\nu) + \Big(\chi_\gamma - \frac{1}{\bar\gamma\gamma}\Big)\Psi(\nu),$$

$$\Psi(\nu) \;\equiv\; \psi(\bar\gamma) + 2\psi(2-\gamma) - 2\psi(4-2\gamma) - \psi(2+\gamma),$$

$$F_6(\gamma) \;=\; F(\gamma) - \frac{2C}{\bar\gamma\gamma} - 1 - \frac{2}{\gamma^2} - \frac{2}{\bar\gamma^2} - 3\frac{1+\chi_\gamma - \frac{1}{\gamma\bar\gamma}}{2+\bar\gamma\gamma},$$

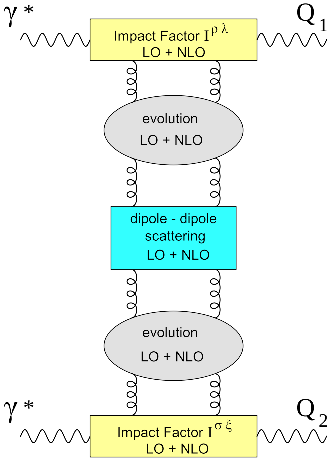
$$\Phi_1(\nu) \;=\; F(\gamma) + \frac{3\chi_\gamma}{2+\bar\gamma\gamma} + 1 + \frac{25}{18(2-\gamma)} + \frac{1}{2\bar\gamma} - \frac{1}{2\gamma} - \frac{7}{18(1+\gamma)} + \frac{10}{3(1+\gamma)^2}$$

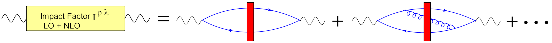
$$\Phi_2(\nu) \;=\; F(\gamma) + \frac{3\chi_\gamma}{2+\bar\gamma\gamma} + 1 + \frac{1}{2\bar\gamma\gamma} - \frac{7}{2(2+3\bar\gamma\gamma)} + \frac{\chi_\gamma}{1+\gamma} + \frac{\chi_\gamma(1+3\gamma)}{2+3\bar\gamma\gamma},$$

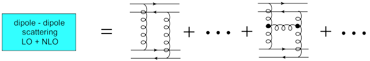
$$F(\gamma) \;=\; \frac{2\pi^2}{3} - \frac{2\pi^2}{\sin^2\pi\gamma} - 2C\chi_\gamma + \frac{\chi_\gamma - 2}{\bar\gamma\gamma}$$

$\gamma = \frac{1}{2} + i\nu$

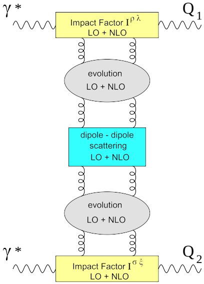
$\gamma^*\gamma^*$ scattering cross-section at NLO

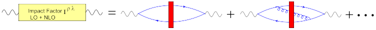


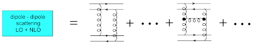




$\gamma^*\gamma^*$ scattering cross-section at NLO



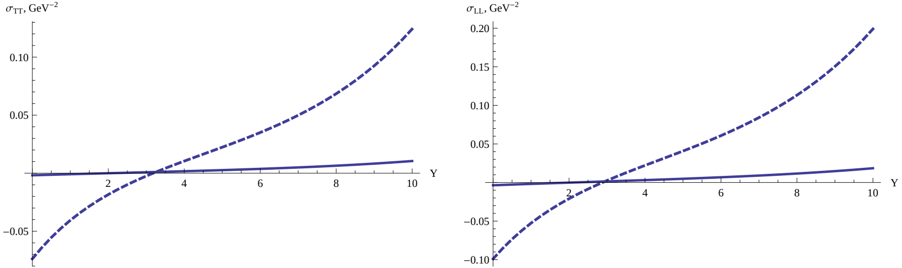




$$
\begin{aligned}
\sigma^{\gamma^*\gamma^*}_{\mathrm{LO+NLO}}(TT) \quad &= \frac{1}{4}\sum_{\lambda_1,\lambda_2=\pm1} 32\pi^6 \varepsilon^{\lambda_1 *}_{\rho_1}(q_1)\varepsilon^{\lambda_1}_{\sigma_1}(q_1)\varepsilon^{\lambda_2 *}_{\rho_2}(q_2)\varepsilon^{\lambda_2}_{\sigma_2}(q_2)\frac{N_c^2-1}{N_c^2}\frac{\alpha_s(Q_1^2)\alpha_s(Q_2^2)}{Q_1Q_2} \\
&\times \int_{-\infty}^{\infty} d\nu \left(\frac{Q_1^2}{Q_2^2}\right)^{i\nu}\left(\frac{s}{Q_1Q_2}\right)^{\bar\alpha_s(Q_1Q_2)\,\chi_0(\nu)+\bar\alpha_s^2(Q_1Q_2)\chi_1(\nu)} \tilde I^{\rho_1\sigma_1}_{\mathrm{LO+NLO}}(q_1,\nu)\tilde I^{\rho_2\sigma_2}_{\mathrm{LO+NLO}}(q_2,-\nu) \\
&\times \left[1+\bar\alpha_s(Q_1Q_2)\mathrm{Re}[F(\nu)]\right]
\end{aligned}
$$

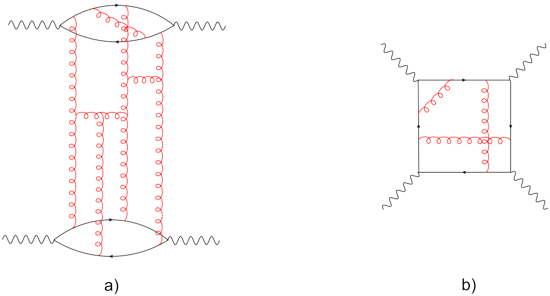
G.A.C. and Yu. Kovchegov (JHEP 2014)

$\mathrm{Re}[F(\nu)]$ is the NLO dipole-dipole scattering projected on the LO eigenfunctions.



- NLO $\gamma^*\gamma^*$ cross section with $Q_1 = Q_2 = 5$ GeV (dashed line) and $Q_1 = Q_2 = 10$ GeV (solid line) plotted as functions of rapidity Y.

Numerical results



- Diagram a): Feynman diagram that contributes to the high-energy asymptotics of $\gamma^*\gamma^*$ scattering.
- Diagram b): example of the "box" diagram that are suppressed at high energies by a power of energy and, therefore, can be neglected. However, such types of diagrams become relevant, and therefore not anymore negligible at low rapidity.

Conclusions

- The NLO BFKL eigenfunctions have been constructed: they satisfy completeness and orthogonality condition $\Rightarrow$ NLO BFKL solution.
- NNLO Eigenfunctions has also been presented $\Rightarrow$ The structure of the NNLO solution has been found up to the still unknown conformal contribution $\chi_2(\nu)$; but we predicted the imaginary part of the NNLO BFKL equation.
- Procedure to construct the solution of the BFKL equation to any order is now available.
- NLO $\gamma^*\gamma^*$ cross-section using the solution of the NLO BFKL equation and the NLO impact factor calculated through the composite Wilson line operators.
- Outlook: To extend the NLO $\gamma^*\gamma^*$ cross-section to the NLO $\gamma\gamma$ cross-section with charm production one has to calculate the NLO Impact Factor including the mass of the charm quark.

NNLO BFKL Solution: the NNLO eigenfunction

- This time we truncated the series up to $c_4^{(2)}$ and proceeding similarly to the NLO case and we get

$$\mathrm{Re}[c_1^{(2)}] = \beta_2 \left(\frac{\chi_1'(\nu))}{2\chi_0'(\nu)} - \frac{\chi_1(\nu)\chi_0''(\nu)}{\chi_0'^2(\nu)} - \frac{\chi_1''(\nu)\chi_0(\nu)}{2\chi_0'^2(\nu)} + \frac{\chi_0(\nu)\chi_1'(\nu)\chi_0''(\nu)}{\chi_0'^3(\nu)} \right) - \beta_3 \left(\frac{1}{2} - \frac{\chi_0(\nu)\chi_0''(\nu)}{2\chi_0'^2(\nu)} \right)$$

$$c_2^{(2)} = \beta_2^2 \left(\frac{5}{8}\frac{\chi_0''^2(\nu)\chi_0^2(\nu)}{\chi_0'^4(\nu)} - \frac{\chi_0(\nu)\chi_0''(\nu)}{4\chi_0'^2(\nu)} - \frac{\chi_0^2(\nu)\chi_0'''(\nu)}{4\chi_0'^3(\nu)} - \frac{1}{8} \right) + i\beta_2 \left(\frac{\chi_1(\nu)}{\chi_0'(\nu)} - \frac{\chi_0(\nu)\chi_1'(\nu)}{2\chi_0'(\nu)} \right) - i\beta_3 \frac{\chi_0(\nu)}{2\chi_0'(\nu)}$$

$$c_3^{(2)} = i\beta_2^2 \left(-\frac{5}{12}\frac{\chi_0''(\nu)\chi_0^2(\nu)}{\chi_0'^3(\nu)} + \frac{\chi_0(\nu)}{4\chi_0'(\nu)} \right)$$

$$c_4^{(2)} = -\beta_2^2 \frac{\chi_0^2(\nu)}{8\chi'^2(\nu)}$$

The $\mathrm{Im}[c_1^{(2)}(\nu)]$ is fixed to be 0 using again the ν-reparametrization.

Structure of the NNLO BFKL Solution

- From explicit calculation of $f_2(\gamma)$ we not only confirm the imaginary part

$$i\delta_2(\nu) = -\frac{i}{2}\chi_0'(\nu)\beta_3 + i\chi_1'(\nu)\beta_2$$

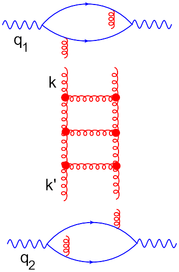
- but also confirm the structure of the NNLO BFKL solution obtained above in an indirect way

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\, k\, k'}\, e^{\left[\bar{\alpha}_s(k\,k')\,\chi_0(\nu)+\bar{\alpha}_s^2(k\,k')\,\chi_1(\nu)+\bar{\alpha}_s^3(k\,k')\,\chi_2(\nu)\right]\,Y}\left(\frac{k^2}{k'^2}\right)^{i\nu}$$

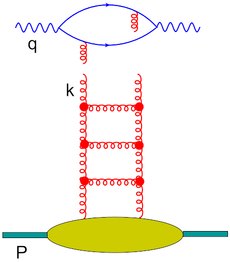
$$\times\left\{1+(\bar{\alpha}_\mu\beta_2)^2\left[-\frac{1}{24}(\bar{\alpha}_\mu\,Y)^3\chi_0(\nu)^2\chi_0''(\nu)+\frac{1}{4}(\bar{\alpha}_\mu\,Y)^2\chi_0(\nu)\left(\frac{\chi_0'(\nu)^2}{2\chi_0(\nu)}-\chi_0''(\nu)\right)+\bar{\alpha}_\mu Y\frac{\chi_0''(\nu)}{4}\right]\right\}$$

- It looks like QCD is not just conformal part and running coupling contributions.

BFKL at NNLO and higher orders

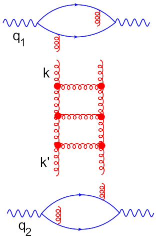


VS.



- At NNLO there are Pomeron loop contributions which are inevitable in the symmetric case of γ^*-γ^* case $\Rightarrow$ a simple generalization of BFKL at NNLO (and higher) does not exist unless we consider large N_c limit.
- In the asymmetric case of DIS, there are no pomeron loops. $\Rightarrow$ Linearization (with large N_c limit) of the Balitsky-Kovchegov equation at any order provides a systematic procedure to consistently define a BFKL type of evolution equation at any order.
- BFKL to all order: linearization of all order BK equation in large N_c limit.

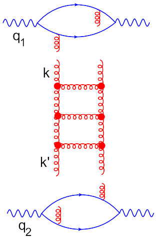
BFKL equation in DIS case



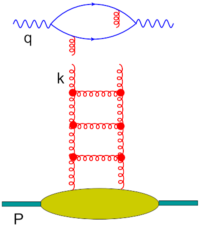
K_{BFKL} is $k \leftrightarrow k'$ symmetric

$Y^{\text{sym}} = \ln \frac{s}{k\,k'}$

BFKL equation in DIS case



vs.

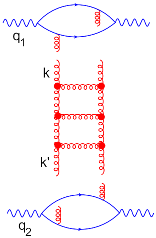


$K_{\rm BFKL}$ is $k \leftrightarrow k'$ symmetric

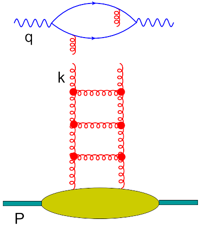
$$Y^{\rm sym} = \ln\frac{s}{k\,k'}$$

NLO $K^{\rm DIS}_{\rm BFKL}$ is not Symmetric

$$Y^{\rm DIS} = \ln\frac{s}{k^2} \simeq \ln\frac{1}{x_B}$$



vs.



$K_{\rm BFKL}$ is $k \leftrightarrow k'$ symmetric

$Y^{\rm sym} = \ln \frac{s}{k\,k'}$

NLO $K^{\rm DIS}_{\rm BFKL}$ is not Symmetric

$Y^{\rm DIS} = \ln \frac{s}{k^2} \simeq \ln \frac{1}{x_{\rm B}}$

$$K^{\rm DIS}_{\rm NLO} = K^{\rm sym}_{\rm NLO} - \frac{1}{2}\int d^{D-2}q'\,K_{\rm LO}(q_1,q')\ln\frac{q'^2}{q^2}\,K_{\rm LO}(q,q_2)$$ Fadin – Lipatov(1998)

- $K^{\rm DIS}_{BFKL}$ is not symmetric ⇒ eigenvalues not $\gamma \leftrightarrow 1-\gamma$ symmetric.
- ⇒ Eigenvalues get an extra term: $\Delta^{\rm DIS}(\gamma) = \Delta^{\rm sym}(\gamma) - \frac{1}{2}\,\bar\alpha^2_\mu\,\chi_0(\gamma)\chi'(\gamma)$
- Reproduced lower order and predicted (and later confirmed) the 3-loop DGLAP anomalous dimension (Fadin-Lipatov (1998))

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2}\, e^{\left[\bar{\alpha}_\mu\, \chi_0(\nu) + \bar{\alpha}_\mu^2\, \chi_1(\nu)\right]\,(Y^{\rm DIS} + \ln\frac{k}{k'})}\, H_{\frac{1}{2}+i\nu}(k)\, \left[H_{\frac{1}{2}+i\nu}(k')\right]^*$$

$$\simeq \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2}\, e^{\left[\bar{\alpha}_\mu\, \chi_0(\nu) + \bar{\alpha}_\mu^2\, \chi_1(\nu)\right]\, Y^{\rm DIS}}\, H_{\frac{1}{2}+i\nu}(k)\, \left[H_{\frac{1}{2}+i\nu}(k')\right]^* \left(1 + \bar{\alpha}_\mu \chi_0(\nu) \ln\frac{k}{k'}\right)$$

$\Rightarrow$ perform partial integration and exponentiate the $Y^{\rm DIS}$-dependent terms $\Rightarrow$

$$G(k,k',Y^{\rm DIS}) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2}\, e^{\left[\bar{\alpha}_\mu\, \chi_0(\nu) + \bar{\alpha}_\mu^2\, \left(\chi_1(\nu) + 2\,i\,\chi_0(\nu)\chi_0'(\nu)\right)\right]\, Y^{\rm DIS}} H_{\frac{1}{2}+i\nu}(k)\, \left[H_{\frac{1}{2}+i\nu}(k')\right]^*$$
$$\times \left(1 + \frac{i}{2}\,\bar{\alpha}_\mu\, \chi_0'(\nu)\right)$$

- $\Rightarrow$ $\quad \Delta^{\rm DIS}(\gamma) = \bar{\alpha}_\mu\, \chi_0(\gamma) + \bar{\alpha}_\mu^2\, \chi_1(\gamma) - \frac{1}{2}\,\bar{\alpha}_\mu^2\, \chi_0(\gamma)\chi_0'(\gamma)$
- Agrees with DGLAP 3-loop anomalous dimension.