Quantum Field Theory and the Electroweak Standard Model

E.E. Boos
SINP MSU

Outline

1. Introduction
2. Elements of Quantum Field Theory
3. Construction of the Electroweak Standard Model Lagrangian
4. Phenomenology of the Electroweak Standard Model
5. Concluding remarks

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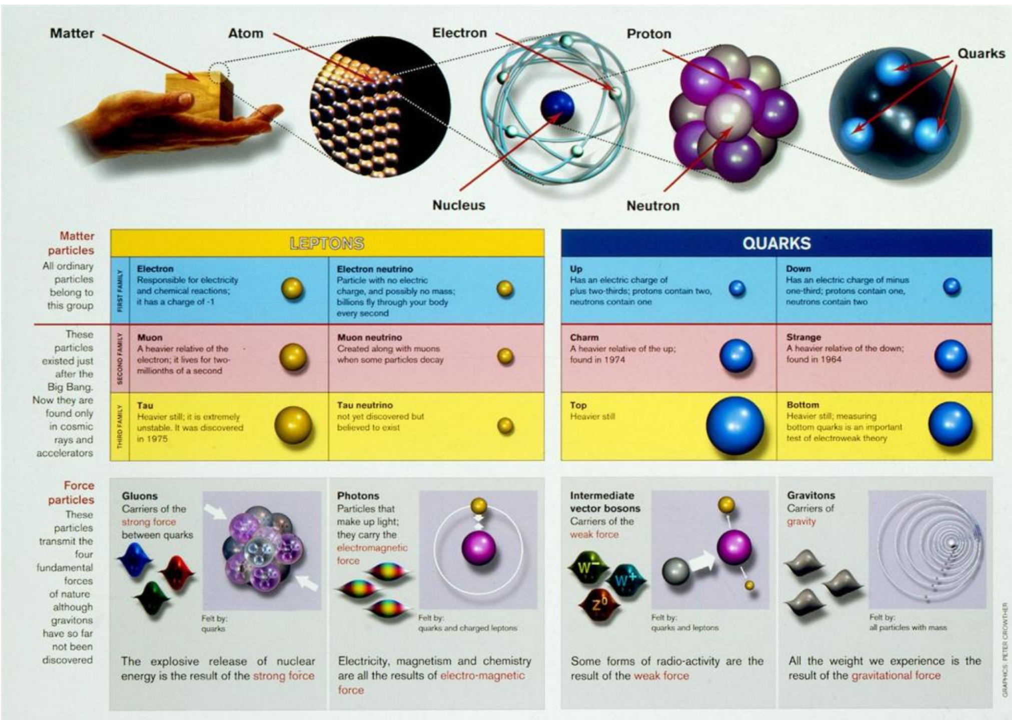
Standard Model of strong and electroweak interactions is the basis for understanding of nature at extremely small distances



In the system ħ=c=1

$1/\mathrm{GeV} \approx 2*10^{-14}$ cm

$\Delta X * \Delta P \geq 1/2$

More energy one transfers smaller distances one can probe

100 GeV -> 10^{-16} cm

1 TeV -> 10^{-17} cm

10 TeV -> 10^{-18} cm

LHC -> 10^{-17} - 10^{-18} cm

SM – the quantum field theory based on few principles and requirements :

- gauge invariance with lowest dimension (dimension 4) operators;
  SM gauge group: $SU(3)_c \times SU(2)_L \times U(1)_Y$

- correct electromagnetic neutral currents and (V-A) charged currents (Fermi);

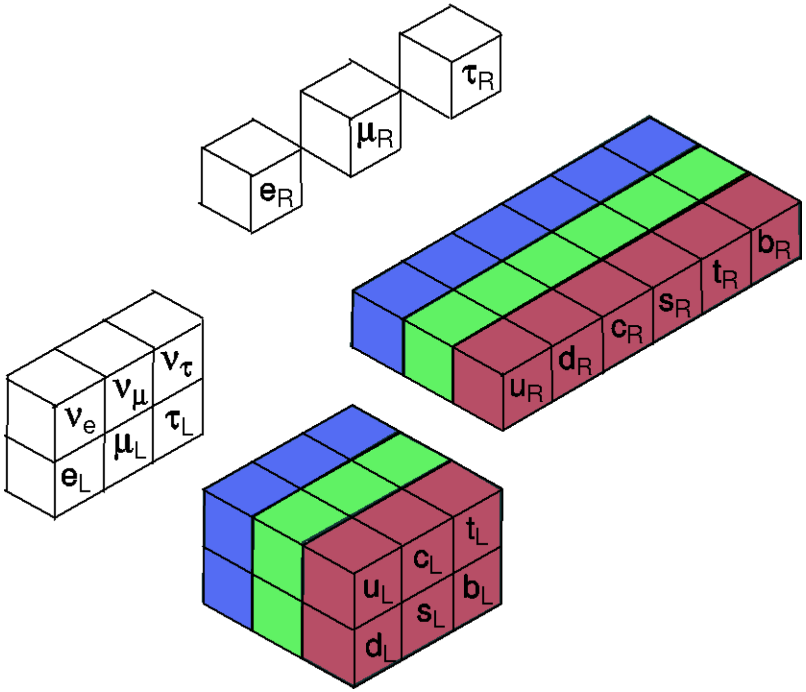
$$\frac{G_F}{\sqrt{2}} \cdot [\overline{\nu}_\mu \cdot \gamma_\alpha (1-\gamma_5) \cdot \mu] \cdot [\overline{e} \cdot \gamma_\alpha (1-\gamma_5) \cdot \overline{\nu}_e] + h.c.$$

- 3 generations without chiral anomalies

- Higgs mechanism of spontaneous symmetry breaking

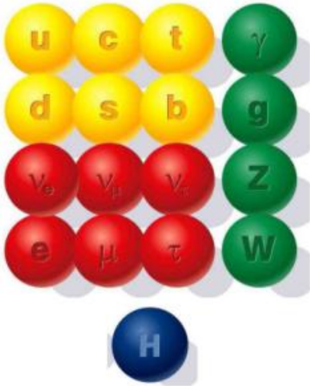
Standard Model – one of the main intellectual achievement for about last 50 years, a result of many theoretical and experimental studies

Fermions are combined into 3 generations forming left doublets and right singlets with respect to weak isospin

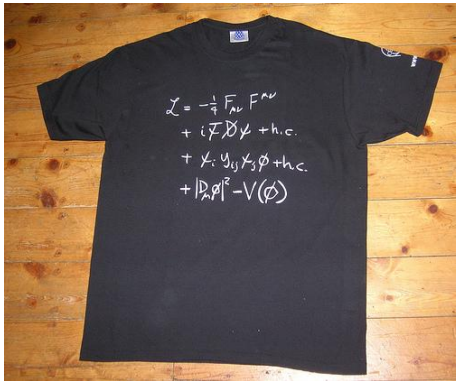


$$f_{L,R} = \frac{1}{2}(1 \mp \gamma_5) f$$

$$I_f^{3L,3R} = \pm\frac{1}{2}, 0 : \quad \begin{array}{l} L_1 = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L , \; e_{R_1} = e_R^- , \; Q_1 = \begin{pmatrix} u \\ d \end{pmatrix}_L , \; u_{R_1} = u_R , \; d_{R_1} = d_R \\ L_2 = \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L , \; e_{R_2} = \mu_R^- , \; Q_2 = \begin{pmatrix} c \\ s \end{pmatrix}_L , \; u_{R_2} = c_R , \; d_{R_2} = s_R \\ L_3 = \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L , \; e_{R_3} = \tau_R^- , \; Q_3 = \begin{pmatrix} t \\ b \end{pmatrix}_L , \; u_{R_3} = t_R , \; d_{R_3} = b_R \end{array}$$



Standard Model



$SU(3)_c \times SU(2)_L \times U(1)_Y$

$$
\begin{aligned}
\mathcal{L}_{\rm SM} &= -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a - \frac{1}{4} W^a_{\mu\nu} W^{\mu\nu}_a - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \\
&\quad + \bar{L}_i \, i D_\mu \gamma^\mu \, L_i + \bar{e}_{Ri} \, i D_\mu \gamma^\mu \, e_{R_i} \\
&\quad + \bar{Q}_i \, i D_\mu \gamma^\mu \, Q_i + \bar{u}_{Ri} \, i D_\mu \gamma^\mu \, u_{R_i} + \bar{d}_{Ri} \, i D_\mu \gamma^\mu \, d_{R_i}
\end{aligned}
$$

$+ L_H$

$$
\begin{aligned}
G^a_{\mu\nu} &= \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s \, f^{abc} G^b_\mu G^c_\nu \\
W^a_{\mu\nu} &= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g_2 \, \epsilon^{abc} W^b_\mu W^c_\nu \\
B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu
\end{aligned}
$$

$$
D_\mu \psi = \left(\partial_\mu - i g_s T_a G^a_\mu - i g_2 T_a W^a_\mu - i g_1 \frac{Y_q}{2} B_\mu \right) \psi
$$

$$
Y_f = 2Q_f - 2I^3_f \;\Rightarrow\; Y_{L_i} = -1, \; Y_{e_{R_i}} = -2, \; Y_{Q_i} = \frac{1}{3}, \; Y_{u_{R_i}} = \frac{4}{3}, \; Y_{d_{R_i}} = -\frac{2}{3}
$$

A very elegant theoretical construction!

Observation of the Higgs-like boson in 2012 at 7 TeV LHC energy

Physics Letters B 710 (2012) 26–48

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Physics Letters B

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Combined results of searches for the standard model Higgs boson in pp collisions at $\sqrt{s} = 7$ TeV ☆

CMS Collaboration *

CERN, Switzerland

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ABSTRACT

Combined results are reported from searches for the standard model Higgs boson in proton–proton collisions at $\sqrt{s} = 7$ TeV in five Higgs boson decay modes: $\gamma\gamma$, bb, $\tau\tau$, WW, and ZZ. The explored Higgs boson mass range is 110–600 GeV. The analysed data correspond to an integrated luminosity of 4.6–4.8 fb^{-1}. The expected excluded mass range in the absence of the standard model Higgs boson is 118–543 GeV at 95% CL. The observed results exclude the standard model Higgs boson in the mass range 127–600 GeV at 95% CL, and in the mass range 129–525 GeV at 99% CL. An excess of events above the expected standard model background is observed at the low end of the explored mass range making the observed limits weaker than expected in the absence of a signal. The largest excess, with a local significance of 3.1σ, is observed for a Higgs boson mass hypothesis of 124 GeV. The global significance of observing an excess with a local significance $\geqslant 3.1\sigma$ anywhere in the search range 110–600 (110–145) GeV is estimated to be 1.5σ (2.1σ). More data are required to ascertain the origin of the observed excess.

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Physics Letters B

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Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC ☆

ATLAS Collaboration *

This paper is dedicated to the memory of our ATLAS colleagues who did not live to see the full impact and significance of their contributions to the experiment.

ARTICLE INFO

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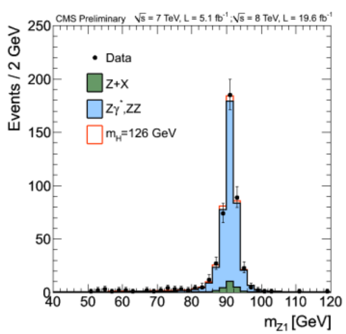
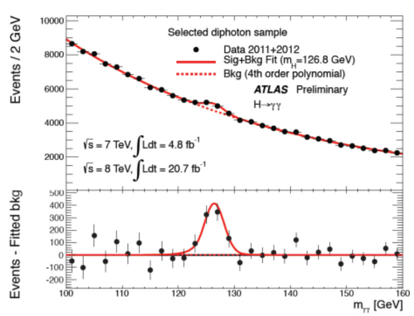
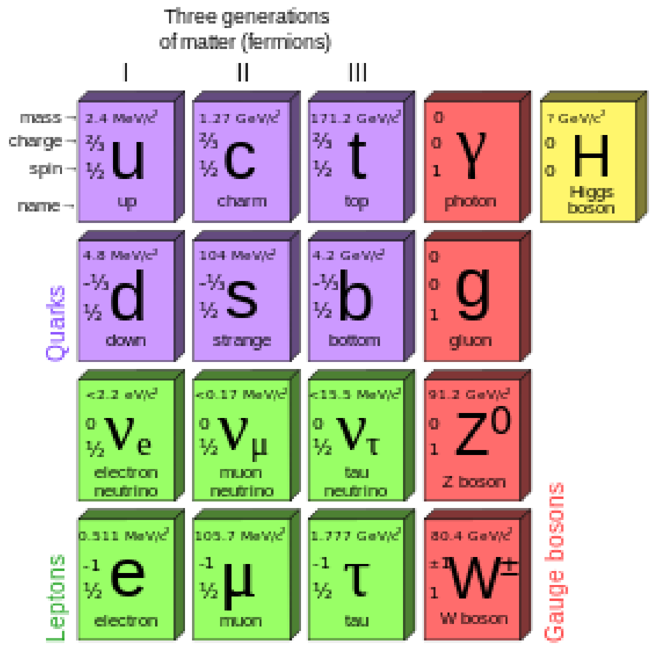
ABSTRACT

A search for the Standard Model Higgs boson in proton–proton collisions with the ATLAS detector at the LHC is presented. The datasets used correspond to integrated luminosities of approximately 4.8 fb^{-1} collected at $\sqrt{s} = 7$ TeV in 2011 and 5.8 fb^{-1} at $\sqrt{s} = 8$ TeV in 2012. Individual searches in the channels $H \to ZZ^{(*)} \to 4\ell$, $H \to \gamma\gamma$ and $H \to WW^{(*)} \to e\nu\mu\nu$ in the 8 TeV data are combined with previously published results of searches for $H \to ZZ^{(*)}$, $WW^{(*)}$, $b\bar{b}$ and $\tau^+\tau^-$ in the 7 TeV data and results from improved analyses of the $H \to ZZ^{(*)} \to 4\ell$ and $H \to \gamma\gamma$ channels in the 7 TeV data. Clear evidence for the production of a neutral boson with a measured mass of 126.0 ± 0.4 (stat) ± 0.4 (sys) GeV is presented. This observation, which has a significance of 5.9 standard deviations, corresponding to a background fluctuation probability of 1.7×10^{-9}, is compatible with the production and decay of the Standard Model Higgs boson.

Confirmation in 2013 with more statistics






Three generations of matter (fermions)
I
II
III
mass
charge
spin
name
2.4 MeV/c²
⅔
½
u
up
1.27 GeV/c²
⅔
½
c
charm
171.2 GeV/c²
⅔
½
t
top
0
0
1
γ
photon
7 GeV/c²
0
0
H
Higgs boson
4.8 MeV/c²
-⅓
½
d
down
104 MeV/c²
-⅓
½
s
strange
4.2 GeV/c²
-⅓
½
b
bottom
0
0
1
g
gluon
<2.2 eV/c²
0
½
νe
electron neutrino
<0.17 MeV/c²
0
½
νμ
muon neutrino
<15.5 MeV/c²
0
½
ντ
tau neutrino
91.2 GeV/c²
0
1
Z0
Z boson
0.511 MeV/c²
-1
½
e
electron
105.7 MeV/c²
-1
½
μ
muon
1.777 GeV/c²
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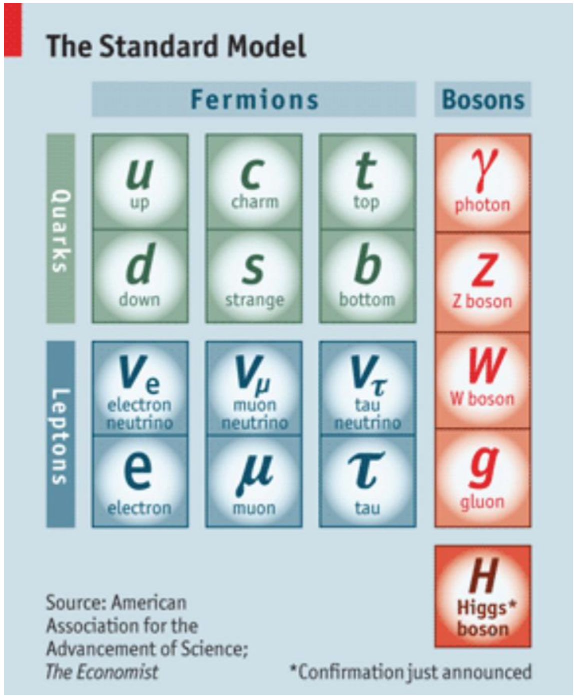
W±
W boson
Quarks
Leptons
Gauge bosons


The Standard Model
Fermions
Bosons
Quarks
Leptons
u
up
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charm
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top
d
down
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strange
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νe
electron neutrino
νμ
muon neutrino
ντ
tau neutrino
e
electron
μ
muon
τ
tau
γ
photon
Z
Z boson
W
W boson
g
gluon
H
Higgs* boson
Source: American Association for the Advancement of Science; The Economist
*Confirmation just announced

Elements of quantum field theory

In classical mechanics a system evolution follows from the principle of least action:

$$\delta S = \delta \int_{t_i}^{t_f} dt L(q(t), \dot{q}(t)) = 0; \quad \int_{t_i}^{t_f} \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta(\dot{q}) \right] = 0; \quad \delta(\dot{q}) = \frac{d}{dt}\delta q$$

Because of arbitrary small variation one gets:

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$ **Lagrange equation of motion**

For the Lagrangian $L = \frac{m\dot{q}^2}{2} - V(q) \implies m\ddot{q} = -\frac{\partial V}{\partial q} = F$ **2nd Newton law**

In Hamilton formalism $H(p,q) = p\dot{q} - L(q,\dot{q})$ **- Hamiltonian, where**

$\dot{q}$ **is a solution of the equation** $p = \frac{\partial L}{\partial \dot{q}}$

In quantum mechanics $p, q \to \hat{p}, \hat{q}$

In Heisenberg picture:

$$\frac{\partial \hat{q}}{\partial t} = -i\hbar[\hat{H}, \hat{q}] \Longrightarrow \hat{q}(t) = e^{i\hat{H}t}\hat{q}(0)e^{-i\hat{H}t}$$

(Indeed $\frac{\partial \hat{q}}{\partial t} = i\hat{H}e^{i\hat{H}t}\hat{q}(0)e^{-i\hat{H}t} + e^{i\hat{H}t}\hat{q}(0)e^{-i\hat{H}t}\hat{H} = i\hat{H}\hat{q}(t) - i\hat{q}(t)\hat{H} = i[\hat{H}, \hat{q}]$)

One postulates: $[\hat{p}(0), \hat{q}(0)] = -i\hbar$

One gets the Heisenberg uncertainty principle $\triangle q \cdot \triangle p \geq 1/2$

Simple proof: Mid value of some operator and its dispersion

$$\langle\psi|\hat{A}|\psi\rangle = \overline{A} \qquad \langle\psi|(\hat{A} - \overline{A})^2|\psi\rangle$$

Let us take $\hat{A} = \hat{p} + i\gamma\hat{q} - (\overline{p} + i\gamma\overline{q}); \quad \hat{A}^\dagger = \hat{p} - i\gamma\hat{q} - (\overline{p} - i\gamma\overline{q})$

$$\langle\psi|\hat{A}^\dagger A|\psi\rangle \geq 0 \ \Rightarrow \ \langle\psi|\left[(\hat{p} - \overline{p}) - i\gamma(\hat{q} - \overline{q})\right]\left[(\hat{p} - \overline{p}) + i\gamma(\hat{q} - \overline{q})\right]|\psi\rangle =$$

$$= (\Delta p)^2 + \gamma^2\Delta q^2 - i\gamma(\hat{q}\hat{p} - \hat{p}\hat{q}) = (\Delta p)^2 + \gamma^2\Delta q^2 + \gamma\hbar \geq 0$$

This is true for any value of γ , the determinant is not positive

$$\Rightarrow \quad \frac{\hbar^2}{4} - \Delta q^2 \Delta p^2 \leq 0 \quad \Rightarrow \quad \Delta q \Delta p \geq \frac{1}{2}\hbar$$

Let us consider the simplest system – Harmonic oscillator

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{q}^2)$$

Let us introduce two operators: $\hat{a} = \sqrt{\frac{\omega}{2}}\hat{q} + i\frac{1}{\sqrt{2\omega}}\hat{p}; \quad \hat{a}^\dagger = \sqrt{\frac{\omega}{2}}\hat{q} - i\frac{1}{\sqrt{2\omega}}\hat{p}$

$$[\hat{p}, \hat{q}] = -i \quad \Longrightarrow \quad [\hat{a}, \hat{a}^\dagger] = 1 \quad \hat{H} = \frac{\omega}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})$$

It is easy to show: $[\hat{H}, \hat{a}] = -\omega\hat{a}$ and $[\hat{H}, \hat{a}^\dagger] = \omega\hat{a}^\dagger$

$$\frac{d\hat{a}}{dt} = i[\hat{H}\hat{a}] = -i\omega\hat{a} \quad \Longrightarrow \quad \hat{a}(t) = \hat{a}(0)e^{-i\omega t}; \quad \hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{i\omega t}$$

Let us consider states with definite energy and construct Hilbert space

$$\hat{H}|E\rangle = E|E\rangle \qquad \hat{H}\hat{a}|E\rangle = \hat{a}\hat{H}|E\rangle - \omega\hat{a}|E\rangle = (E-\omega)\hat{a}|E\rangle$$

$$\hat{H}\hat{a}^\dagger|E\rangle = (E+\omega)\hat{a}^\dagger|E\rangle$$

Vacuum state is defined as: $\hat{a}|0\rangle = 0 \qquad \hat{H}|0\rangle = \frac{\omega}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})|0\rangle = \frac{\omega}{2}|0\rangle$

States of Hilbert space $|n\rangle = (\hat{a}^\dagger)^n|0\rangle$ **have energies:** $\hat{H}(\hat{a}^\dagger)^n|0\rangle = \omega(n+1/2)|n\rangle$

Such a construction is very successful in describing non-relativistic quantum phenomena (spectra of atoms, molecules, nuclea….)

But there are problems

- Production of new particles, antiparticles, many particles

- Relativity and causality
  (If in 4-dimensional areas X and Y points are separated by the space like interval $(x-y)^2 < 0$ events in the points x and y are causally independent. But in QM we have the uncertainty principle

$$\Delta X * \Delta P \geq 1/2$$

  In the fuzzy domain $\Delta X \sim 1/m$ the causality may be violated)

Quantum Field Theory is needed

To describe a particle and antiparticle with mass m, momentum $\vec{k}$ and energy $\omega_k = k^0 = \sqrt{\vec{k}^2 + m^2}$

let us introduce two sets of oscillators and creation and annihilation operators $\hat{a}, \hat{a}^\dagger$ $\hat{b}, \hat{b}^\dagger$ for each momentum point $\vec{k}$

Vacuum is defined as: $\hat{a}|0\rangle = \hat{b}|0\rangle = 0$

The Hamiltonian for every $\vec{k}$

$$H_k = \frac{\omega_k}{2}(\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}) + \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \hat{b}(\vec{k})\hat{b}^\dagger(\vec{k}) + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k}))$$

One can use different normalization of an integral measure in order to get

$$\int \overline{dk}[\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k})] = 1; \quad \int \overline{dk}[\hat{b}(\vec{k})\hat{b}^\dagger(\vec{k})] = 1$$

We are using: $\overline{dk} = \dfrac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \Rightarrow [\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k'})] = (2\pi)^2 2\omega_k \delta(\vec{k} - \vec{k'})$

Total momentum and charge operators ("normal ordering"):

$$\widehat{P}^\mu = \int \overline{dk} k^\mu \left[\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})\right] \qquad \widehat{Q} = \int \overline{dk} \left[\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) - \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})\right]$$

Prove: $\left[\widehat{P}^\mu, \hat{a}^\dagger(\vec{k})\right] = k^\mu \hat{a}^\dagger(\vec{k}); \quad \left[\widehat{P}^\mu, \hat{a}(\vec{k})\right] = -k^\mu \hat{a}(\vec{k}) \quad \left[\widehat{Q}, \hat{a}^\dagger(\vec{k})\right] = \hat{a}^\dagger(\vec{k}); \quad \left[\widehat{Q}, \hat{b}^\dagger(\vec{k})\right] = -\hat{b}^\dagger(\vec{k})$

Now one can construct the Field operator:

$$\widehat{\Phi}(x) = \int \overline{dk} \left[e^{-ikx} \hat{a}(k) + e^{ikx} \hat{b}^{\dagger}(k) \right]$$

Very important to note that such a field operator obeys the Klein-Gordoon equation

$$[\Box^2 + m^2] \widehat{\Phi}(x) = \int \overline{dk} \left[e^{-ikx}(-k^2 + m^2)\hat{a}(k) + e^{ikx}(-k^2 + m^2)\hat{b}(k) \right] = 0$$

Check that the field operator creates the state with the negative charge and conjugated operator – with the positive charge

$$\widehat{Q}\widehat{\Phi}(x)|0> = -\widehat{\Phi}(x)|0> \qquad \widehat{Q}\widehat{\Phi}^{\dagger}(x)|0> = \widehat{\Phi}^{\dagger}(x)|0>$$

$t_2 > t_1$

$< 0|\widehat{\Phi}^{\dagger}(x_2)\widehat{\Phi}(x_1)|0 >$: $\widehat{\Phi}(x_1)|0 >$ creates the charge -1 in the point x_1

$\widehat{\Phi}^{\dagger}(x_2)\widehat{\Phi}(x_1)|0 >$ annihilates the charge -1 in the point x_2

Charge -1 propagates from x_1 to x_2

$t_2 < t_1$

$< 0|\widehat{\Phi}(x_1)\widehat{\Phi}^{\dagger}(x_2)|0 >$: $\widehat{\Phi}^{\dagger}(x_2)|0 >$ creates the charge +1 in the point x_2

$\widehat{\Phi}(x_1)\widehat{\Phi}^{\dagger}(x_2)|0 >$ annihilates the charge -1 in the point x_1

Charge +1 propagates from x_2 to x_1

We should take both possibilities into account:

$$D_C = <0|\widehat{\Phi}^\dagger(x_2)\widehat{\Phi}(x_1)|0> \Theta(t_2 - t_1) + <0|\widehat{\Phi}(x_1)\widehat{\Phi}^\dagger(x_2)|0> \Theta(t_1 - t_2)$$

$$= i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x_2 - x_1)}}{k^2 - m^2 + i0}$$

This function is called the Feynman propagator.
Obviously, it is a Green function of the Klein-Gordon equastion

One can check that all the commutators between the Field operators in the points x and y separated by the space like interval $(x-y)^2 < 0$ are equal to ZERO.

So the causality takes place!

$$\hat{H}\left|\vec{k}_1, \ldots \vec{k}_n\right\rangle = \left(\sum_{i=1}^{n} k_i^0\right)\left|\vec{k}_1, \ldots \vec{k}_n\right\rangle$$

Multiparticle states $\left|\vec{k}_1, \ldots \vec{k}_n\right\rangle = \prod_{i=1}^{n} \hat{a}^\dagger(\vec{k}_i)\left|0\right\rangle$

$$\hat{\vec{P}}\left|\vec{k}_1, \ldots \vec{k}_n\right\rangle = \left(\sum_{i=1}^{n} \vec{k}_i\right)\left|\vec{k}_1, \ldots \vec{k}_n\right\rangle$$

Of course, the same result may be obtained from the canonical quantization

$$L = \partial_\mu \varphi^\dagger \partial^\mu \varphi - m^2 \varphi^\dagger \varphi,$$

$$q(t) \to \varphi(x); \quad \dot{q}(t) \to \partial_\mu \varphi(x)$$

$$\pi(x) = \dot{\varphi}^\dagger(x); \quad \pi^\dagger(x) = \dot{\varphi}(x)$$

$$[\hat{\pi}(\vec{x}, t), \hat{\varphi}(\vec{x}', t')]|_{t=t'} = -i\delta(\vec{x} - \vec{x}')$$

Functional integral approach

It allows to quantize non-abilean gauge field theories, to clarify better boundary conditions and renormalization procedure, to get reduction formula (connection between S-matrix elements and Green functions)

Once more we begin with the quantum mechanics as a simple example

$$L(q_i, \dot{q}_i) \quad \to \quad p_i = \frac{\partial L}{\partial q_i}, \quad H(q_i, p_i) = \dot{q}_i p_i - L(q_i, \dot{q}_i)$$

$$\left[\hat{q}^i(t), \hat{p}^j(t)\right] = i\hbar\delta^{ij}\hat{1}$$

The formal solution of the Schrödinger equation: $i\frac{\partial}{\partial x}|\Psi\rangle = \hat{H}|\Psi\rangle$

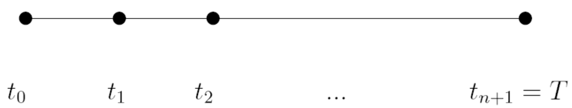
$$|\Psi(t)\rangle = e^{-i\hat{H}t}|\Psi(0)\rangle$$

One can define states: $|q, t\rangle : \quad \hat{q}\,|q, t\rangle = q\,|q, t\rangle \;\; \langle q|q'\rangle = \delta(q - q')$

Wave function in coordinate representation: $\Psi(q, t) = \langle q|\Psi(t)\rangle = \left\langle q \left| e^{-i\hat{H}t} \right| \Psi(0) \right\rangle$

$$\hat{1} = \int dq_0\,|q_0\rangle\,\langle q_0| \qquad \Psi(q, t) = \int dq_0\,\langle q|\,e^{-i\hat{H}t}\,|q_o\rangle\,\langle q_0|\Psi(0)\rangle = \int dq_0 K(q, q_0, t)\,\langle q_0|\Psi(0)\rangle$$

Kernel of the Schrödinger equation



$$\hat{1} = \int dq_i \, |q_i\rangle \langle q_i|$$

Introducing the unity operator in each time point one gets:

$$K(q, q_0; t) = N \int dq_i \prod_i e^{i\left[\frac{m}{2}\left(\frac{q_{i+1}-q_i}{\delta t^2}\right)^2 - V(q_i)\right]\delta t}$$ **for the system** $$H(\hat{p}, \hat{q}) = \frac{\hat{p}^2}{2m} + V(\hat{q})$$

More general:

$$\boxed{K(q, q_0; t) = \int D(q) e^{i\int_0^t dt L(q, \dot{q}, t)}}$$

$$D(q) = \lim_{n \to \inf, \delta t = \frac{t}{n} \to 0} \cdot \sqrt{\frac{m}{2\pi i \delta t}} \prod_{i=1}^{n} \left(\sqrt{\frac{m}{2\pi i \delta t}} dq_i\right)$$

Some details:

$$\left\langle q_{i+1} \left| e^{-i\hat{H}\delta t} \right| q_i \right\rangle = \left\langle q_{i+1} \left| e^{-i\frac{\hat{p}^2}{2m}\delta t} \cdot e^{-i\hat{V}(q)\delta t} \right| q_i \right\rangle$$

$$e^{\varepsilon(\hat{A}+\hat{B})} = e^{\varepsilon\hat{A}} * e^{\varepsilon\hat{B}}(1 + o(\varepsilon^2))$$

$$\left\langle q_{i+1} \left| e^{-i\hat{H}\delta t} \right| q_i \right\rangle \approx e^{-i\hat{V}(q_i)\delta t} \cdot \left\langle q_{i+1} \left| e^{-i\frac{\hat{p}^2}{2m}\delta t} \right| q_i \right\rangle$$

$$\left\langle q_{i+1} \left| e^{-i\frac{\hat{p}^2}{2m}\delta t} \right| q_i \right\rangle = \int \frac{dp}{2\pi} \left\langle q_{i+1} \left| e^{-i\frac{\hat{p}^2}{2m}\delta t} \right| p \right\rangle \langle p | q_i \rangle = \int \frac{dp}{2\pi} e^{-ip(q_{i+1}-q_i)\delta t} e^{-i\frac{\hat{p}^2}{2m}\delta t}$$

$$\langle q|p\rangle = e^{ipq}, \ \langle p|q\rangle = e^{-ipq}, \ \hat{1} = \int \frac{dp}{2\pi} |p\rangle \langle p|$$

Few useful well-known Gaussian integrals:

$$Z = \int dx_1 \dots dx_n \cdot e^{-\frac{1}{2}x^T A x} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}$$

$$Z_C = \int \prod_{k=1}^{n} dz_k^* dz_k e^{-z^\dagger B z} = \frac{\pi^n}{\det B}$$

$$z = x + iy$$

$$z^* = x - iy$$

$$Z[J] = \int dx_1 \dots dx_n \cdot e^{-\frac{1}{2}x^T A x + J^T x} = e^{\frac{1}{2}J^T A^{-1} J} \cdot \frac{(2\pi)^{n/2}}{\sqrt{\det A}}$$

$$Z_C[J] = \int \prod_{k=1}^{n} dz_k^* dz_k e^{-z^\dagger B z + J^\dagger z + z^\dagger J} = e^{J^\dagger B^{-1} J} \cdot \frac{\pi^n}{\det B}$$

$$Z_{int}[J] = \frac{1}{Z_{int}} \int \prod dx e^{-\frac{1}{2}xAx + Jx - V(x)}$$

$$Z_{int} = Z_{int}[0]$$

$$= e^{-V[\frac{\partial}{\partial J}]} \cdot e^{\frac{1}{2}JA^{-1}J}$$

A transition from a quantum mechanics to a field theory

$$q(t) \to \varphi(x); \quad \dot{q}(t) \to \partial_\mu \varphi(x)$$

From the correspondence =>

The evolution kernel in case of quantum field theory:

$$Z[J] = \int D(\varphi) e^{i\int d^4x L(\varphi, \partial_\mu \varphi) + i\int d^4x J(x)\varphi(x)}$$

Integration over all possible trajectories (field configurations)

$$D(\varphi) = \prod_x d\varphi(x) \cdot \mathbf{N} \quad \Longrightarrow \quad Z[0] = 1$$

Now all the formulas we have shown for Gaussian integrals one can apply here.

Functional derivative instead of the usual one $\quad \frac{\delta J(y)}{\delta J(x)} = \delta^{(4)}(x-y)$

$Z[J]$ **is called generating functional for the Green functions**

Consider the Lagrangian for the free scalar field

$$L = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2}m^2\varphi^2 = -\frac{1}{2}\varphi\left(\Box^2 + m^2\right)\varphi$$

The integral is Gaussian and can be taken only with +i0 prescription:

$$Z[J] = \exp\frac{1}{2}\int d^4x d^4y J(x) D_c(x-y) J(y) \qquad \int D(\varphi)\exp -\varepsilon \int dx \varphi^2(x)$$

with a normalization $Z[0] = 1$

We obtain the same function – the Feynman propagator

$$D_c = \; i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x_2-x_1)}}{k^2-m^2+i0}$$

Propagator is a Green function or in other words the inverse quadratic form in the action:

$$D_c^{-1}\cdot D_c = 1 \qquad L = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 = -\frac{1}{2}\varphi\left(\Box^2+m^2\right)\varphi \equiv -\frac{1}{2}\varphi D_c^{-1}\varphi$$

In the case of interacting theory with the potential $V(\varphi)$ we get the following general expression for the generating functional

$$Z_V[J] = \exp\left(-i\int d^4x V\left(\frac{\delta}{i\delta J(x)}\right)\right)\cdot\exp\left(\frac{1}{2}\int dydz J(y) D_c(y-z) J(z)\right)$$

How Green functions are related to physics amplitudes?

L.D.Faddeev, A.A.Slavnov

Functional integral in holomorphic representation

Let us consider once more the Harmonic oscillator

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{q}^2)$$

$$\hat{a} = \sqrt{\frac{\omega}{2}}\hat{q} + i\frac{1}{\sqrt{2\omega}}\hat{p}; \quad \hat{a}^\dagger = \sqrt{\frac{\omega}{2}}\hat{q} - i\frac{1}{\sqrt{2\omega}}\hat{p} \qquad [\hat{a}, \hat{a}^\dagger] = 1 \quad \hat{H} = \frac{\omega}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})$$

The commutation relation has very nice representation in terms of holomorphic functions which are introduced by giving the following scalar product

$$\hat{a}^\dagger \cdot f(a^*) = a^* f(a^*), \quad \hat{a} f(a^*) = \frac{d}{da^*} f(a^*) \quad \langle f_1 | f_2 \rangle = \int (f_1(a^*))^* f_2(a^*) e^{-a^* a} \frac{da^* da}{2\pi i}$$

By direct substitution one can prove the following relation $\langle f_1 | \hat{a}^\dagger f_2 \rangle = \langle \hat{a} f_1 | f_2 \rangle$

which means that operators $\hat{a}$ and $\hat{a}^\dagger$ are conjugated to each other

Few more relations

The set of functions form a complete orthonormalized basis:

$$\Psi_n(a^*) = \frac{(a^*)^n}{\sqrt{n!}},\ n \geq 0 \quad \langle\Psi_n|\Psi_m\rangle = \frac{1}{\sqrt{n!m!}} \int a^n (a^*)^m e^{-a^* a} \frac{da^* da}{2\pi i} = \delta_{nm} \quad \sum_n |\Psi_n\rangle\langle|\Psi_n| = 1$$

The function $A(a^*, a) = \sum_{nm} A_{nm} \frac{(a^*)^n}{\sqrt{n!}} \frac{a^m}{\sqrt{m!}}$ is called the kernel of the operator $\hat{A}$

where $A_{nm} = \left\langle \Psi_n | \hat{A} | \Psi_m \right\rangle$ Normal ordering: $\hat{A} = \sum_{nm} K_{nm} (\hat{a}^\dagger)^n (\hat{a})^m$

The normal symbol of the operator: $K(a^*, a) = \sum_{nm} K_{nm} (a^*)^n a^m$ $\boxed{A(a^*, a) = e^{a^* a} K(a^*, a)}$

The kernel of the evolution operator $\hat{U} = e^{-i\hat{H}\cdot\triangle t}$ for small time interval:

$$U(a^*, a) = e^{[a^* a - ih(a^* a)]\triangle t}$$

In case of finite interval we can split it on small pieces $t'' - t' = N \cdot \triangle t$

$$U(a^*, a; t'', t') = \int \exp\left([a^* \alpha_{N-1} - \alpha^*_{N-1}\alpha_{N-1} + \ldots - \alpha^*_1 \alpha_1 + \alpha^*_1 \alpha_0] - \right.$$

$$\left. i\triangle t[h(a^*, \alpha_{N-1}) + \ldots + h(\alpha^*, \alpha_0)]\right) \cdot \prod_{k=1}^{N-1} \frac{d\alpha^*_k d\alpha_k}{2\pi i}$$

In the limit $\triangle t \to 0,\, N \to \infty,\, \triangle N = t'' - t'$ one gets

the following functional integral for the kernel of the evolution operator

$$U(a^*, a; t'', t') = \int e^{a^* \alpha(t")} \cdot \exp\left(\int_{t'}^{t"} [-\alpha^* \alpha - i h(\alpha^*, \alpha)] dt\right) \cdot \prod_t \frac{d\alpha^* d\alpha}{2\pi i}$$

with the boundary conditions $\alpha^*(t'') = a^*,\, \alpha(t') = a$

In the field theory $\hat{\Phi}(x) = \int \frac{d\vec{k}}{(2\pi)^3 2k^0} [e^{-ikx} \hat{a} + e^{ikx} \hat{a}^\dagger(k)]$

$$\begin{matrix} a \\ a^* \end{matrix} \longrightarrow \begin{matrix} a(\vec{k}) \\ a^*(\vec{k}) \end{matrix}$$

The S-matrix:

$$\hat{S} = \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} e^{i\hat{H}_0 t''} \hat{U}(t'', t') e^{-i\hat{H}_0 t'}$$

$\hat{S} = 1$ if $\hat{H} = \hat{H}^0$

$$h^0(a^*, a) = \omega a^* a$$

The kernel for the S-matrix:

$$S(a^*, a) = \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} e^{\int \frac{d\vec{k}}{(2\pi)^3 2k^0} \left(\alpha^*(\vec{k}, t") \alpha(\vec{k}, t'') - \int_{t'}^{t''} [\alpha^*(\vec{k}, t) \dot{\alpha}(\vec{k}, t) + h(\alpha^*, \alpha)] dt\right)} \cdot \prod_{t, \vec{k}} \frac{d\alpha^* d\alpha}{2\pi i}$$

with Feynman boundary conditions

$$\alpha^*(\vec{k}, t'') = a^*(\vec{k}) e^{i\omega t''}, \quad \alpha(\vec{k}, t') = a(\vec{k}) e^{-i\omega t'}$$

System with an external source J(x):

$$L = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + J(x)\varphi$$

The kernel of the interaction operator

$$V = -J(x)\hat{\varphi}(x)$$

$$V(a^*, a) = \int \frac{d\vec{k}}{(2\pi)^3 2k^0}\left[\gamma(\vec{k},t)a^*(\vec{k}) + \gamma^*(\vec{k},t)a(\vec{k})\right] \qquad \gamma(\vec{k},t) = -\int J(\vec{x},t)e^{-ikx}d\vec{x}$$

In order to take the integral one should solve the extremum conditions

$$\dot{\alpha}(\vec{k},t) + i\omega(\vec{k})\alpha(\vec{k},t) + i\gamma(\vec{k},t) = 0$$
$$\dot{\alpha}^*(\vec{k},t) - i\omega(\vec{k})\alpha^*(\vec{k},t) - i\gamma^*(\vec{k},t) = 0$$

with the boundary conditions

$$\alpha^*(\vec{k},t'') = a^*(\vec{k})e^{i\omega t''}, \quad \alpha(\vec{k},t') = a(\vec{k})e^{-i\omega t'}$$

$$\Rightarrow$$

$$\alpha(\vec{k},t) = a(\vec{k})e^{-i\omega t} - ie^{-i\omega t}\int_{t'}^{t''} e^{i\omega s}\gamma(\vec{k},s)ds$$

$$\alpha^*(\vec{k},t) = a^*(\vec{k})e^{i\omega t} + ie^{i\omega t}\int_{t'}^{t''} e^{-i\omega s}\gamma^*(\vec{k},s)ds$$

Kernel of the S-matrix:

$$S_J(a^*, a) = \exp\left(\int \frac{d\vec{k}}{(2\pi)^3 2k^0}\left[a^*(\vec{k})a(\vec{k}) + \right.\right.$$
$$+\int dt \int d\vec{x} J(\vec{x},t)(a^*(\vec{k})e^{i\omega t - i\vec{k}\vec{x}} + a(\vec{k})e^{-i\omega t + i\vec{k}\vec{x}})/(2\omega) -$$
$$\left.\left. -\frac{1}{2}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} ds \int d\vec{x}d\vec{y} J(\vec{x},t)J(\vec{x},t)/(2\omega)e^{-i\omega|t-s|}e^{i\vec{k}\vec{x}-i\vec{k}\vec{y}}\right]\right)$$

Solution of free Klein-Gordon equation:

$$\varphi_0(x) = \int \frac{d\vec{k}}{(2\pi)^3 2k^0}\left[a(\vec{k})e^{i\omega t} + a^*(\vec{k})e^{-i\omega t}\right]$$

Feynman propagator:

$$D_c(x) = i\int \frac{d\vec{k}}{(2\pi)^3 2\omega}e^{i\vec{k}\vec{x}}e^{-i\omega_k|t|} = i\int \frac{d^4k e^{-ikx}}{k^2 - m^2 + i0}$$

In terms of fields one gets for the normal symbol of S-matrix:

$$S_V(\varphi_0) = \exp\left(-i\int V\left(\frac{\delta}{i\delta J}\right) d^4x\right) \cdot$$
$$\cdot \exp\left(i\int J(y)\varphi_0(y)d^4y + \frac{1}{2}\int d^4z d^4y J(y)D_c(y-z)J(z)\right)\Bigg|_{J=0}$$

$\varphi_0(x)$ **is the solution of free Klein-Gordon equation and**
$D_c(x)$ **is the Feynman propagator**

Now we can compare the functionals for the S-matrix and for the Green functions

$$S_V(\varphi_0, J) = \exp\left(-i\int V\left(\frac{\delta}{i\delta J}\right) d^4x\right) \cdot$$
$$\cdot \exp\left(i\int J(y)\varphi_0(y)d^4y + \frac{1}{2}\int d^4z d^4y J(y)D_c(y-z)J(z)\right)$$

$$Z_V(J) = \exp\left(-i\int V\left(\frac{\delta}{i\delta J}\right) d^4x\right) \cdot \exp\left(\frac{1}{2}\int d^4z d^4y J(y)D_c(y-z)J(z)\right)$$

One finds from the comparison the basic relation between S-matrix elements and Green functions known as

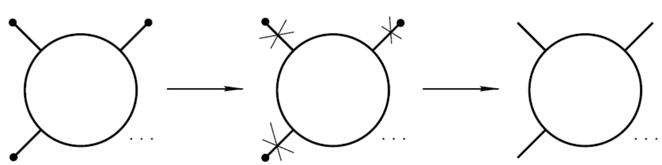
Leman-Simanchik-Zimmermann reduction formula

$$S_V(\varphi_0) = \sum_n \frac{1}{n} \int dx_1 \ldots dx_n \varphi_0(x_1) \ldots \varphi_0(x_n) S_n(x_1 \ldots x_n)$$

$$Z_V(J) = \sum_n \frac{1}{n!} \int dx_1 \ldots dx_n J(x_1) \ldots J(x_n) \frac{\delta Z}{i\delta J(x_1) \ldots i\delta J(x_n)}$$

$$\int \prod_i dx_i \varphi_0(x_i) \left[\frac{1}{i} \frac{\delta}{\delta \varphi_1} \ldots \frac{\delta}{\delta \varphi_n} S_V(\varphi_0, J)\Big|_{\varphi, J=0} - \right.$$
$$\left. - \frac{1}{i} \frac{\delta}{\delta \tilde{J}_1(x_1)} \ldots \frac{\delta}{\delta \tilde{J}_n(x_n)} Z(\tilde{J})\Big|_{J=0} \right] = 0 \qquad \tilde{J}(x) = \int D_c(x-y) J(y)$$

1. one should compute the Green function
2. multiply all legs to inverse propagators
3. multiply the result to the product of corresponding free fields



How to compute Green functions?

Perturbation theory and Feynman diagrams

The generating functional:

$$Z[J] = \int D(\varphi) \cdot \exp\left(i \int d^4x L(\varphi, \partial_\mu \varphi) + i \int d^4x J(x)\varphi(x) \right)$$

where $L = \frac{1}{2}\partial_\mu\varphi \cdot \partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - V(\varphi)$

As we know:

$$Z[J] = \exp\left(-i \int dx V\left(\frac{\delta}{\delta J}\right)\right) \cdot \exp\left(\frac{1}{2} \int dy dz J(y) D_c(y-z) J(z)\right)$$

$$\left.\frac{\delta^{(2)} Z}{i\delta J(x_1) i\delta J(x_2)}\right|_{J=0} = \langle \varphi_1(x_1)\varphi_2(x_2)\rangle$$

$$D_c(x_1 - x_2) = \langle 0|\, T\{\hat{\Phi}(x_1)\hat{\Phi}(x_2)\}\, |0\rangle = \langle \varphi_1(x_1)\varphi_2(x_2)\rangle$$

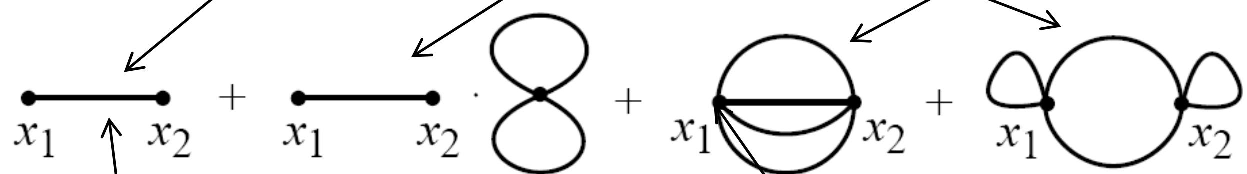
This is always the case. Derivatives of the generating functional automatically give T-products of the corresponding number of field operators

$$\frac{\delta^{(n)} Z}{i\delta J(x_1)\dots i\delta J(x_n)} \equiv \langle \varphi_1(x_1)\dots\varphi_n(x_n)\rangle = \langle 0|\, T\{\hat{\Phi}(x_1)\dots\hat{\Phi}(x_n)\}\, |0\rangle$$

Let us consider as an example a theory with $V(\varphi) = \frac{\lambda}{4!}\varphi^4$

If we take two derivatives and expand the exponent on λ

$$\begin{aligned}\langle \varphi_1(x_1)\varphi_2(x_2)\rangle \;&=\; D_c(x_1-x_2) + \lambda\left[D_c(x_1-x_2)D_c^2(0) + \ldots\right] + \\ &+\frac{1}{2}\left(\frac{\lambda}{4!}\right)^2\left[72D_c^2(0)D_c^2(x_1-x_2) + 24D_c^4(x_1-x_2) + \ldots\right] + \ldots\end{aligned}$$



$D_c(x-y)$ **- propagator** $\qquad -i\lambda = -i\left.\dfrac{d^4V(\varphi)}{d\varphi^4}\right|_{\varphi=0}$ **- vertex**

In the momentum space the integral $\displaystyle\int \frac{dp}{(2\pi)^4}$ **corresponds to each loop**

The generating functional W gives connected Creen functions

$$iW[J] = \ln Z[J]$$

$$\frac{\delta W}{\delta J} = \frac{1}{Z}\frac{\delta Z}{i\delta J}, \qquad \frac{\delta^2 W}{\delta J_1 \delta J_2} = i\frac{1}{Z}\frac{\delta^2 Z}{i\delta J_1 i\delta J_2} - \frac{1}{Z^2}\frac{\delta Z}{i\delta J_1}\frac{\delta Z}{i\delta J_2} \cdots$$

additional terms exactly cancel out disconnected pieces in the Green functions

One-particle irreducible Green functions are given by the functional Legandre transformation:

$$\Gamma[\varphi_{cl}] = W[J] - \int d^4x J(x)\varphi_{cl}(x) \qquad \begin{aligned} \varphi_{cl} &= \delta W/\delta J \\ J &= J(\varphi_{cl}) \end{aligned}$$

$$\frac{\delta \Gamma}{\delta \varphi_{cl}} = \frac{\delta W}{\delta J}\frac{\delta J}{\delta \varphi_{cl}} - J(x) - \varphi_{cl}\frac{\delta J}{\delta \varphi_{cl}} = -J(x)$$

$$\Gamma_n(x_1, \ldots x_n) = -i\frac{\delta^{(n)}\Gamma[\varphi_{cl}]}{\delta\varphi_{cl}(x_1)\ldots\delta\varphi_{cl}(x_n)}\bigg|_{\varphi_{cl}=0}$$

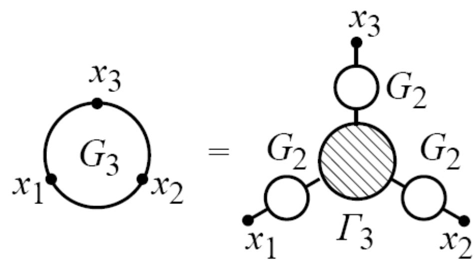
Connected Green functions:

$$G_n(x_1, \ldots x_n) = -i \frac{\delta W[J]}{i\delta J(x_1) \ldots i\delta J(x_2)}\bigg|_{J=0}$$

One-particle irreducible Green functions:

$$\Gamma_n(x_1, \ldots x_n) = -i \frac{\delta^{(n)} \Gamma[\varphi_{cl}]}{\delta\varphi_{cl}(x_1) \ldots \delta\varphi_{cl}(x_n)}\bigg|_{\varphi_{cl}=0}$$

$$\Gamma_2 = G_2^{-1}$$



At tree level or quasi-classical limit $(\hbar \to 0)$ the functional integral is dominated by the stationary trajectory

$$Z[J] = \int D(\varphi) \exp\left(\frac{i}{\hbar} S[\varphi] + i \int dx J(x)\varphi(x)\right) \qquad \frac{\delta S[\varphi]}{\delta\varphi(x)}\bigg|_{\varphi=\varphi_{cl}} + J = 0$$

The irreducible Green functions are effective vertexes of the theory

$$Z[J] \sim \exp\left(\frac{i}{\hbar} S[\varphi_{cl}] + i \int dx J(x)\varphi_{cl}(x)\right) \qquad W[J] = S[\varphi_{cl}] + \int dx J(x)\varphi_{cl}$$

The effective action:

$$\Gamma[\varphi_{cl}] = S[\varphi_{cl}]$$

$$\Gamma_n(x_1, \ldots x_n) = (-i) \frac{\delta^{(n)} S[\varphi_{cl}]}{\delta\varphi_{cl}(x_1) \ldots \delta\varphi_{cl}(x_n)}$$

Very useful formula to get Feynman rules for complicated vertexes!

Fermion fields

Spin-1/2 particles with mass m are described by the 4-component field which obeys the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\,\Psi = 0$$

The equation follows from the well-known Lagrangian

$$L = \bar{\Psi} i \partial_\mu \gamma^\mu \Psi - m\bar{\Psi}\Psi$$

4 matrices (4×4) γ^μ $(\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ $\{\gamma_\mu \gamma_\nu\} = 2\eta_{\mu\nu}$

There are several representations for gamma-matrices. In Standard Model chiral or Weyl spinors are of particular importance.

In Weyl representation:

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad \sigma^0 = I, \sigma^i\,, \bar{\sigma}^0 = I, \bar{\sigma}^i = -\sigma^i$$

(2×2) Pauli matrices

$$\gamma^5 = i\gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

$$\Psi_{L,R} = \frac{1 \mp \gamma_5}{2}\Psi \qquad \Psi_L = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ 0 \\ 0 \end{pmatrix}, \quad \Psi_R = \begin{pmatrix} 0 \\ 0 \\ \Psi_3 \\ \Psi_4 \end{pmatrix}$$

In momentum space the function $\Psi(x)$ is decomposed into positive and negative energy parts

$$u_\lambda(p)e^{-ipx} \text{ and } v_\lambda(p)e^{ipx}$$

Dirac equations:
$$(p_\mu\gamma^\mu - m)\,u_\lambda(p) = 0$$
$$(p_\mu\gamma^\mu + m)\,v_\lambda(p) = 0$$

Solutions:

$$u_\lambda = \begin{pmatrix} \sqrt{p^0 + \vec{p}\vec{\sigma}}\xi_\lambda \\ \sqrt{p_0 - \vec{p}\vec{\sigma}}\xi_\lambda \end{pmatrix}$$

$$v_\lambda = \begin{pmatrix} \sqrt{p^0 + \vec{p}\vec{\sigma}}\eta_\lambda \\ -\sqrt{p_0 - \vec{p}\vec{\sigma}}\eta_\lambda \end{pmatrix}$$

ξ and η

2-component spinors determined by fixing some quantization axis

Left and right chiral spinors $u_{L,R} = \dfrac{1 \mp \gamma_5}{2} u_\lambda$ $\quad v_{L,R} = \dfrac{1 \mp \gamma_5}{2} v_\lambda$

Normalization conditions $\bar{u}_\lambda u_{\lambda'} = 2m\delta_{\lambda\lambda'}, \quad \bar{v}_\lambda v_{\lambda'} = -2m\delta_{\lambda\lambda'}$

Summation over indexes $\sum_\lambda u_\lambda \bar{u}_\lambda = p_\mu\gamma^\mu + m, \quad \sum_\lambda v_\lambda \bar{v}_\lambda = p^\mu\gamma_\mu - m$

Quantization of Dirac field: in order to have correct Fermi statistics and obey Pauli principle the commutation relations in scalar case should be replaced by corresponding anti-commutation relations

$$\{\hat{\pi}_\alpha(t,\vec{x}), \Psi_\beta(t,\vec{x}')\} = -i\delta_{\alpha\beta}\delta(\vec{x}-\vec{x}') \quad \pi_\alpha(t,\vec{x}) = \frac{\partial L}{\partial \dot{\Psi}_\alpha} = i\Psi^\dagger_\alpha \quad \alpha = 1,2,3,4$$

The fermionic field operators

$$\Psi(x) = \int \frac{d\vec{p}}{(2\pi)^3 p^0} \sum_{\lambda=1,2} \left[\hat{b}_\lambda(p) u_\lambda(p) e^{-ipx} + \hat{d}^\dagger_\lambda(p) v_\lambda(p) e^{ipx}\right]$$

$$\bar{\Psi}(x) = \Psi^\dagger\gamma^0 = \int \frac{d\vec{p}}{(2\pi)^3 p^0} \sum_{\lambda=1,2} \left[\hat{b}^\dagger_\lambda(p) \bar{u}_\lambda(p) e^{ipx} + \hat{d}_\lambda(p) \bar{v}_\lambda(p) e^{-ipx}\right]$$

It is easy to check from anti-commutators the creation and annihilation operators satisfy the following anti-commutation relations

$$\left\{\hat{b}_\lambda(\vec{p}), \hat{b}^\dagger_{\lambda'}(\vec{p}')\right\} = \left\{\hat{d}_\lambda(\vec{p}), \hat{d}^\dagger_{\lambda'}(\vec{p}')\right\} = (2\pi)^3 2p^0 \delta(\vec{p}-\vec{p}')\delta_{\lambda\lambda'}$$

Feynman propagator (T-ordered correlator):

$$\langle 0| T\left(\bar{\Psi}(x_1)\Psi(x_2)\right)|0\rangle = \frac{-1}{i}\int \frac{dp}{(2\pi)^4} \frac{p_\mu\gamma^\mu + m}{p^2 - m^2 + i0}$$

Going to the path integral method for the fermionic field one can construct holomorphic representation similar to the scalar case. However now we have to deal with anti-commuting numbers called Grassman numbers which form the Grassman algebra:

$$(a_\alpha)^* = a_\alpha^*, \quad (a_\alpha^*)^* = a_\alpha$$

$$\{a_\alpha a_\beta\} = \{a_\alpha^* a_\beta^*\} = \{a_\alpha a_\beta^*\} = 0$$

$$c a_\alpha = a_\alpha c; \quad c a_\alpha^* = a_\alpha^* c$$

a = 1, . . . n, *c - is a usual number*

One can proof:

$$\int da^* da \exp\left(\sum a_\alpha^* A_{\alpha\beta} a_\beta + \sum \eta_\alpha^* a_\alpha + i \sum \eta_\alpha a_\alpha^*\right) = \det A \exp\left(\eta_\alpha^* A_{\alpha\beta}^{-1} \eta_\beta\right)$$

In contrast to the scalar case the determinant appears in the numerator for fermions!

The generating functional for Green functions

$$Z(\bar\eta, \eta) = N^{-1} \int \exp\left(i \int d^4x (L(x) + \bar\eta \Psi + \bar\Psi \eta)\right) \prod_x d\bar\Psi(x) d\Psi(x)$$

Theories with gauge fields

QED is invariant under U(1) gauge transformations being formulated in terms of vector potential

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\hat{D} - m)\Psi, \quad \Psi \to e^{ie\alpha}\Psi, \quad A_\mu \to A_\mu + \partial_\mu \alpha$$

$$D_\mu = \partial_\mu - ieA_\mu$$

In the Standard Model we deal not only with the abelian U(1) group but also with non-abelian SU(N) groups
(SU(2) for the electroweak and SU(3) for the strong forces)

SU(N) is a group of unitary matrices U ($U^+U = 1$) with the determinant equal to 1 (detU=1). Elements of the group U(x) may depend on space-time point x.

$$L = \bar{\Psi}(i\hat{D} - m)\Psi \qquad D_\mu\Psi \to (D_\mu\Psi)^U = UD_\mu\Psi, \qquad A_\mu^U = UA_\mu U^{-1} + \frac{i}{g}U\partial_\mu U^{-1}$$

$$\left(\partial_\mu - igA_\mu^U\right)U\Psi = U\left(\partial_\mu - igA_\mu\right)\Psi$$

In SM all gauge fields are taken in the adjoint representation

$$L_A = -\frac{1}{2}Tr(F_{\mu\nu}F^{\mu\nu}) \qquad A_\mu(x) = A_\mu^a(x)t^a \qquad L = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu}$$

$$[t^a, t^a] = f^{abc}t^c, \quad Tr(t^a) = 0 \qquad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$$

In construction of a quantum gauge theory there is a problem:

unphysical degrees of freedom

The kinetic form $A^\mu D^{-1}_{\mu\nu} A^\nu = A^\mu \left(\Box g_{\mu\nu} - \partial_\mu \partial_\nu\right) A^\nu$ does not have the inverse form. The propagator can not be defined from the functional integral

$$\int \prod_{\mu,x} dA_\mu(x) \exp\left\{ i \int dx \left(-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \right) \right\} \qquad \left(k^2 g_{\mu\nu} - k_\mu k_\nu\right)$$

Infinite number of gauge cofigurations which differ only by the gauge transformation give identical results due to gauge invariance of any physics observable. One should integrate only taking 1 representative from all that configurations.

The method proposed by Faddeev and Popov is based on quantization of constrained systems

$$\int DA\delta\left(F(A)\right) \det\left(\Delta^F_{gh}\right) e^{iS(A)}$$

Faddeev and Popov method in short

Let us substitute unity into the functional integral

$$1 = \int D\alpha \cdot \delta\left(F(A^\alpha)\right) \det\left(\frac{\delta F}{\delta\alpha}\right) \longrightarrow \int \prod_{\mu,x} dA_\mu(x) \exp\left\{i \int dx \left(-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}\right)\right\}$$

After the gauge transformation the (infinite) group integral is factorised and can be included into the overall renormalization

$$\int D\alpha \int DA e^{iS[A]}\, \delta\left(F(A^\alpha)\right) \cdot \det\left(\frac{\delta F}{\delta\alpha}\right) \longrightarrow \int D\alpha \int DA e^{iS[A]} \cdot \delta\left(F(A)\right) \cdot \det\left(\frac{\delta F}{\delta\alpha}\right)$$

$$A^a_\mu \to (A^\alpha)^a_\mu = A^a_\mu + \partial_\mu \alpha^a + g f^{abc} A^b_\mu \alpha^c = A^a_\mu + D^{ac}_\mu \alpha^c \qquad D^{ac}_\mu = \partial_\mu \delta^{ac} + g f^{abc} A^b_\mu$$

Illustrative example

$$I = \int \int_{-\infty}^{\infty} dx_1 dx_2 e^{-x_1^2 - x_2^2 + 2x_1 x_2} = \int dx_1 dx_2 e^{-x_i A_{ij} x_j} \qquad A_{ij} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$I = c \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{-x^2}$$

$$x = x_1 - x_2, \quad y = x_1 + x_2$$

The "action" is invariant under translation

$$x_1 \to x_1 + a, \quad x_2 \to x_2 + a$$

$$\int_{-\infty}^{\infty} d\omega \delta\left(F(x_i + \omega)\right) \det\left(\frac{\delta F}{\delta\omega}\right) = 1$$

$$\int d\omega \int dx_1 dx_2 e^{-x_i A_{ij} x_j} \delta\left(F(x_i + \omega)\right) \det\left(\frac{\delta F}{\delta\omega}\right)$$

$$x_i \to x_i - \omega$$

$$I_G = \int dx_1 \int dx_2 e^{-x_i A_{ij} x_j} \delta\left(F(x_i)\right) \left|\frac{\partial F}{\partial\omega}\right|$$

$$\det\left(\frac{\delta F}{\delta\alpha}\right)$$ the Faddeev-Popov ghost determinant

$$\Delta_{ch} = \frac{\delta F}{\delta\alpha}$$ so-called ghost operator

A determinant in the numerator appears when one integrates over anticommuting fields – the Faddeev-Popov gohsts

$$\det(\Delta_{ch}) = \int \prod d\bar{c}dc e^{i\int \bar{c}\Delta_{ch}c}$$

in case of the Lorentz gauge condition $F = \partial_\mu A_\mu$

$$\Delta_{ch} = \partial_\mu \cdot \partial^\mu$$ does not depend on the field A in abelian case (ghosts do not contribute)

$$\Delta_{ch} = \partial_\mu \cdot D_\mu$$ depend on the field A in non- abelian case (ghosts contribute!)

Let us consider the gauge condition in a covariant form

$$F(A) = \partial_\mu A^\mu - a(x)$$

The functional integral takes the form

$$e^{iW[J]} = \int \prod_{\mu,x} dA^a_\mu(x) d\bar{c}^a(x) dc^a(x) \cdot \delta\left(\partial_\mu A^{a\mu} - a^a(x)\right)$$

$$\exp\left(i\int d^4x \left[-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{c}^a\left(\Box c^a - f^{abc}\partial_\mu(A^c_\mu c^b)\right) + J^a_\mu A^{a\mu}\right]\right)$$

One can integrate with, for example, Gaussian normalization

$$\int da \exp\left(-\frac{a^2}{2\xi}\right)\delta\left(\partial_\mu A^\mu - a\right) = \exp\left(-\frac{(\partial_\mu A^\mu)^2}{2\xi}\right)$$

As the result we get the following quadratic part of the action

$$S = \int dx \left[-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) - \frac{1}{2\xi}\partial_\mu A^{a\mu}\partial_\nu A^{a\nu} + \bar{c}\Box c\right]$$

Now there is the invert form.

Propagators for non-abelian gauge and ghost fields

$$D^{ab}_{\mu\nu}(k) = -i\frac{\delta^{ab}}{k^2+i0}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]$$

$\xi = 1$ – 't Hooft-Feynman gauge,
$\xi = 0$ – Landau gauge,
$\xi = 3$ – Frautschi-Yenni gauge.

$$\Delta^{ab}_{ch}(k) = i\frac{\delta^{ab}}{k^2+i0}$$

Ghosts cancel a dependence on the gauge parameter ξ in physics quantities

$$k^\mu D^{ab}_{\mu\nu} = -\xi k_\nu i\frac{\delta^{ab}}{k^2+i0} = -\xi k_\nu D^{ab}_{ch}(k)$$

Feynman rules in momentum space for the vertexes one can get from the formula:

$$\Gamma^{a_1\ldots a_n}_{\mu_1\ldots\mu_n}(p_1\ldots p_n)\cdot(2\pi)^4\delta(p_1+\ldots+p_n) = -i\frac{\delta^{(n)}S}{\delta A^{a_1}_{\mu_1}(p_1)\ldots\delta A^{a_n}_{\mu_n}(p_n)}$$

$\frac{-i}{k^2+i0}\delta^{ab}[g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}]$ for massless gauge field

$\frac{i}{k^2+i0}\delta^{ab}$ for the ghost field

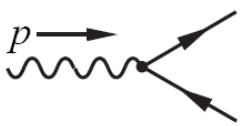
$u(p)$ for incoming fermion

$\bar{v}(p)$ for incoming antifermion

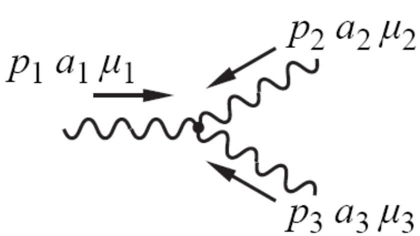
$u(p)$ for outgoing fermion

$\bar{v}(p)$ for outgoing antifermion

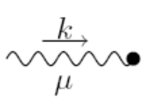
$i\frac{\hat{p}+m}{p^2-m^2+i0}$ for fermion propagator



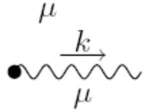
$ig\gamma_\mu(t^a)$ for fermion-gauge boson vertex



$gf^{a_1a_2a_3}\,[g_{\mu_1\mu_2}(p_1-p_2)_{\mu_3} + g_{\mu_2\mu_3}(p_2-p_3)_{\mu_1} + g_{\mu_3\mu_1}(p_3-p_1)_{\mu_2}]$



$e^{\mu}_{\lambda}(k)$ for incoming gauge boson



$e^{*\nu}_{\lambda}(k)$ for outgoing gauge boson

In notations of the Particle Data Group

Cross section

$$d\sigma_{ab} = \frac{|M|^2}{4\sqrt{(p_a p_b)^2 - m_a^2 m_b^2}} d\Phi_n$$

Decay width

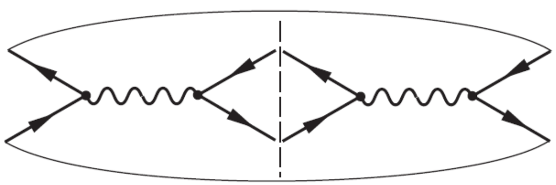
$$d\Gamma = \frac{|M|^2}{2m_a} d\Phi_n$$

$$d\Phi_n = (2\pi)^4 \delta(p_i - p_f) \cdot \frac{d^3\vec{p}_1}{(2\pi)^3 2p_1^0} \cdots \frac{d^3\vec{p}_n}{(2\pi)^3 2p_n^0}$$

One can formulate Feynman rules to compute squared matrix elements

Cuted lines – numerators of propagators:

Example: e+e- -> μ+μ-



1 for scalar particle

$$\begin{cases} \sum_\lambda u_\lambda(p) \times \bar{u}_\lambda(p) = p_\mu \gamma^\mu + m \\ \sum_\lambda v_\lambda(p) \times \bar{v}_\lambda(p) = -p_\mu \gamma^\mu + m \end{cases}$$ for spin 1/2 Dirac particles

$\sum_\lambda e_\lambda^\mu(k) e_\lambda^{*\nu}(k) = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$ for massless gauge fields in Landau gauge

$\sum_\lambda e_\lambda^\mu(k) e_\lambda^{*\nu}(k) = g_{\mu\nu}$ in 't Hooft-Feynman gauge

$\sum_\lambda e_\lambda^\mu(k) e_\lambda^{*\nu}(k) = g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2}$ for vector fields in unitary gauge