Cosmology:
Lecture #1
Our Universe at present:
main ingredients and the expansion law

Dmitry Gorbunov

Institute for Nuclear Research of RAS, Moscow

21^{st} European School on High-Energy Physics,

**CERN-JINR,
Parádfürdő, Hungary, 06.06.2013**

Standard Model: Success and Problems

Gauge fields (interactions): γ, $W^{\pm}$, Z, g

Three generations of matter: $L = \binom{\nu_L}{e_L}$, e_R; $Q = \binom{u_L}{d_L}$, d_R, u_R

- Describes
  - all experiments dealing with electroweak and strong interactions
- Does not describe
  - Neutrino oscillations
  - Dark matter (Ω_{DM})
  - Baryon asymmetry (Ω_B)
  - Inflationary stage
  - Dark energy (Ω_Λ)
  - Strong CP: ? (boundary terms, new topology, ...)
  - Gauge hierarchy: ? (No new scales!)
  - Quantum gravity

Try to explain all above

Planck-scale physics saves the day

Outline

"Natural" units in particle physics

$$\hbar = c = k_{\rm B} = 1$$

measured in GeV: energy E, mass M, temperature T

$m_p = 0.938$ GeV, $\quad 1\ \text{K} = 8.6 \times 10^{-14}$ GeV

measured in GeV^{-1}: time t, length L

$1\ \text{s} = 1.5 \times 10^{24}\ \text{GeV}^{-1}, \quad 1\ \text{cm} = 5.1 \times 10^{13}\ \text{GeV}^{-1}$

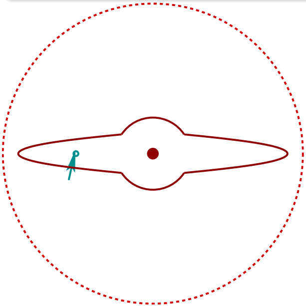
Gravity (General Relativity): $V(r) = -G\frac{m_1 m_2}{r}$ $\qquad [G] = M^{-2}$

$$M_{\rm Pl} = 1.2 \times 10^{19}\ \text{GeV} = 22\ \mu\text{g} \qquad G \equiv \frac{1}{M_{\rm Pl}^2}$$

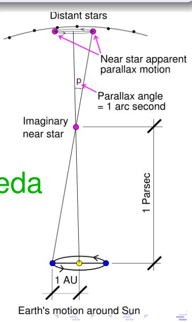
“Natural” units in cosmology

$$1\ \mathrm{Mpc} = 3.1 \times 10^{24}\ \mathrm{cm}$$

$1\ \mathrm{AU} = 1.5 \times 10^{13}\ \mathrm{cm}$	mean Earth-to-Sun distance
$1\ \mathrm{ly} = 0.95 \times 10^{18}\ \mathrm{cm}$	distance light travels in one year
	$1\ \mathrm{yr} = 3.16 \times 10^{7}\ \mathrm{s}$
$1\ \mathrm{pc} = 3.3\ \mathrm{ly} = 3.1 \times 10^{18}\ \mathrm{cm}$	distance to object which has a parallax angle of one arcsec



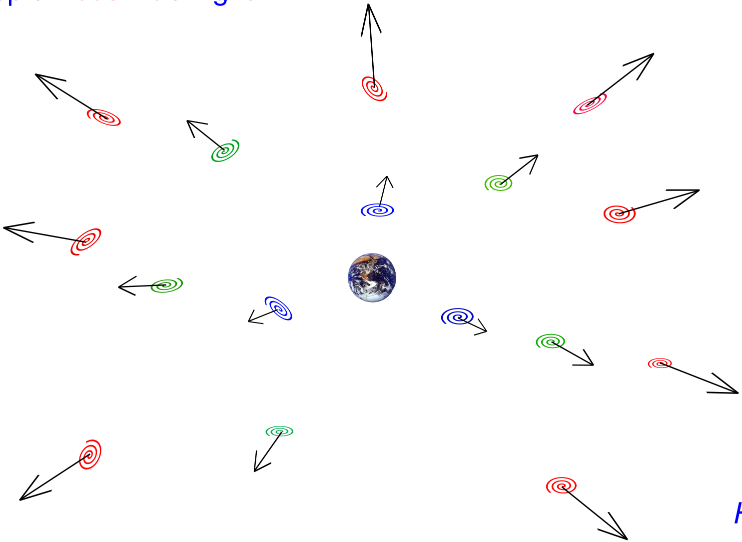
100 AU — Solar system size
1.3 pc — nearest-to-Sun stars
1 kpc — size of dwarf galaxies
50 kpc — distance to dwarves
0.8 Mpc — distance to Andromeda
1-3 Mpc — size of clusters
15 Mpc — distance to Virgo



Outline

Universe is expanding

Doppler redshift of light



$L \propto a(t)$

$n \propto a^{-3}(t)$

$H(t) = \frac{\dot{a}(t)}{a(t)}$

Hubble parameter

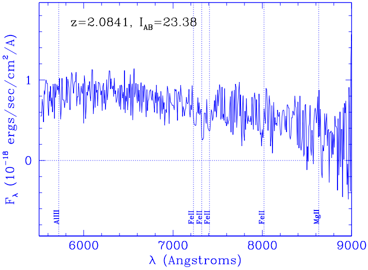
Hubble Law

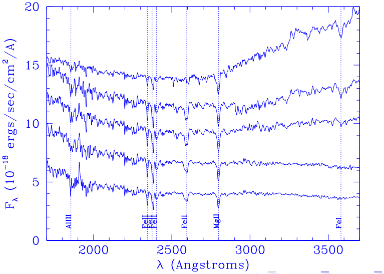
$H(t_0)\, r = v_r$

Expansion: redshift z $\qquad \lambda_{\text{abs.}}/\lambda_{\text{em.}} \equiv 1+z$

$z \ll 1$ Hubble law : $z = H_0 r$

$$H_0 = h \cdot 100 \, \frac{\text{km}}{\text{s} \cdot \text{Mpc}} \qquad h = 0.705 \pm 0.015$$



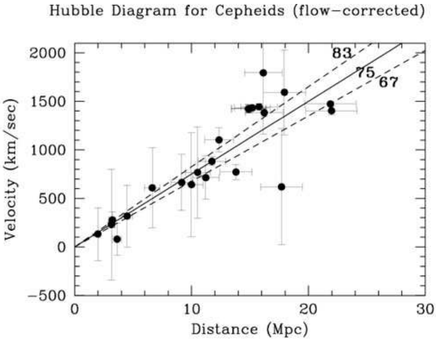


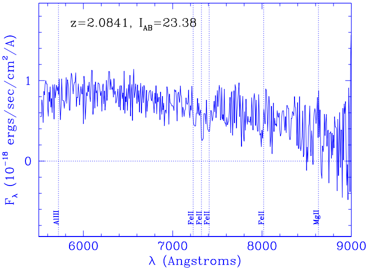
Expansion: redshift z $\qquad \lambda_{\text{abs.}}/\lambda_{\text{em.}} \equiv 1+z$

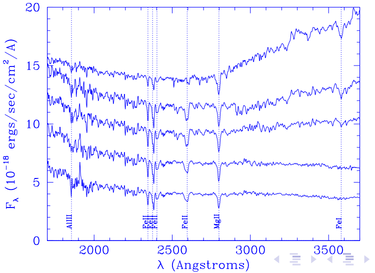
$z \ll 1$ Hubble law : $z = H_0 r$

$$H_0 = h \cdot 100\,\frac{\text{km}}{\text{s}\cdot\text{Mpc}} \qquad h = 0.705 \pm 0.015$$

standard candles



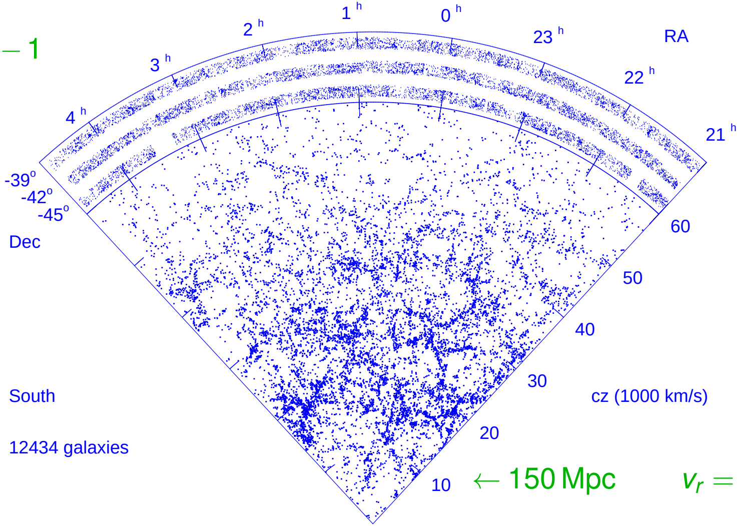




Universe is homogeneous and isotropic

redshift

$$z \equiv \frac{\lambda_{\text{detector}}}{\lambda_{\text{source}}} - 1$$



$\leftarrow$ 150 Mpc

$v_r = c\,z$

The Universe: age & geometry & energy density

$$[H_0] = L^{-1} = t^{-1}$$

time scale: $t_{H_0} = H_0^{-1} \approx 14 \times 10^9$ yr — age of our Universe

spatial scale: $l_{H_0} = H_0^{-1} \approx 4.3 \times 10^3$ Mpc — size of the visible Universe

t_{H_0} is in agreement with various observations

homogeneity and isotropy in 3d:

flat, spherical or hyperbolic

Observations: “very” flat $l_{H_0}/R_{curv} < 0.1$

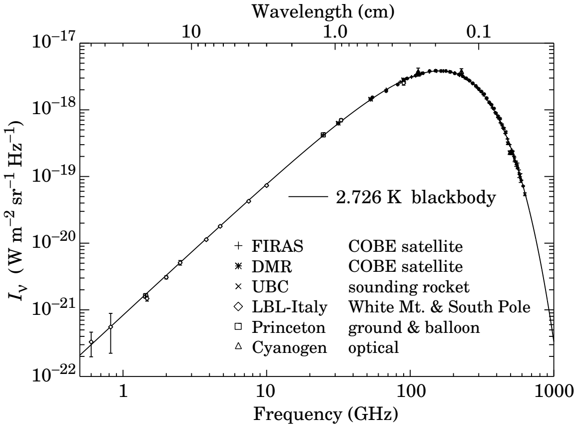
order-of-magnitude estimate: $GM_U/l_U \sim G\rho_0 l_{H_0}^3/l_{H_0} \sim 1$

flat Universe

$$\rho_c = \frac{3}{8\pi} H_0^2 M_{Pl}^2 \approx 0.53 \times 10^{-5} \frac{\text{GeV}}{\text{cm}^3}$$

$\longrightarrow$ 5 protons in each 1 m^3

Universe is occupied by “thermal” photons

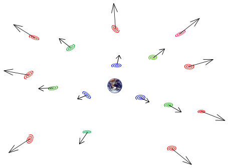


$T_0 = 2.726$ K

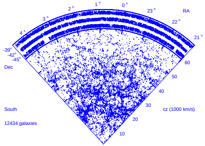
the spectrum (shape and normalization!) is thermal

$n_\gamma = 411$ cm^{-3}

Conclusions from observations



The Universe is homogeneous, isotropic, hot and expanding...



Conclusions

- interval between events gets modified

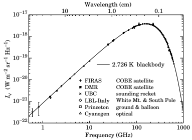
$$\Delta s^2 = c^2\,\Delta t^2 - a^2(t)\,\Delta\mathbf{x}^2$$

in GR expansion is described by the Friedmann equation

$$\left(\frac{\dot a}{a}\right)^2 = H^2(t) = \frac{8\pi}{3}\,G\rho^{\text{energy}}_{\text{density}}$$

$$\rho^{\text{energy}}_{\text{density}} = \rho_{\text{radiation}} + \rho_{\text{matter}} + \dots$$

- in the past the matter density was higher, our Universe was “hotter” filled with electromagnetic plasma



$$\rho_{\text{matter}} \propto 1/a^3(t)\,,\quad \rho_{\text{radiation}} \propto 1/a^4(t)\,,\quad \rho_{\text{curvature}} \propto 1/a^2(t)$$

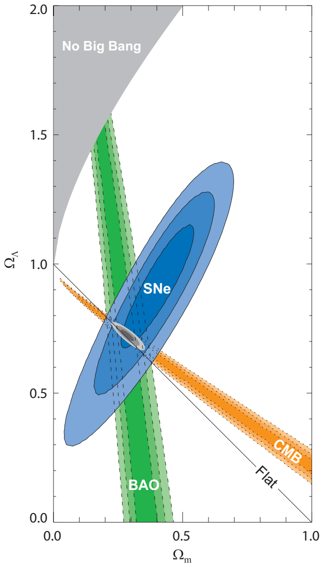
certainly known up to $T \sim 1\,\text{MeV} \sim 10^{10}\,\text{K}$

Why do we need dark components (within GR)?

- Astrophysical data favor Dark Matter
  - Observations in galaxies
  - Observations in galaxy clusters
- Cosmological data favor Dark Matter and Dark Energy
  - Observation of objects at cosmological distances (far=early)
  - Baryonic Aciustic (Sakharov) Oscillations (BAO) in two-point galaxy correlation function
  - Evolution of galaxy clasters in the Universe
  - Anisotropy of Cosmic Microwave Background (CMB)

Outline

Astrophysical and cosmological data are in agreement



$$\left(\frac{\dot a}{a}\right)^2 = H^2(t) = \frac{8\pi}{3} G \rho^{\text{energy}}_{\text{density}}$$

$$\rho^{\text{energy}}_{\text{density}} = \rho_{\text{radiation}} + \rho^{\text{ordinary}}_{\text{matter}} + \rho^{\text{dark}}_{\text{matter}} + \rho_\Lambda$$

$$\rho_{\text{radiation}} \propto 1/a^4(t) \propto T^4(t)\,, \quad \rho_{\text{matter}} \propto 1/a^3(t)$$

$$\rho_\Lambda = \text{const}$$

$$\frac{3H_0^2}{8\pi G} = \rho^{\text{energy}}_{\text{density}}(t_0) \equiv \rho_c \approx 0.53 \times 10^{-5}\,\frac{\text{GeV}}{\text{cm}^3}$$

radiation: $\Omega_\gamma \equiv \frac{\rho_\gamma}{\rho_c} = 0.5 \times 10^{-4}$

Baryons (H, He): $\Omega_B \equiv \frac{\rho_B}{\rho_c} = 0.046$

Neutrino: $\Omega_\nu \equiv \frac{\sum \rho_{\nu_i}}{\rho_c} < 0.01$

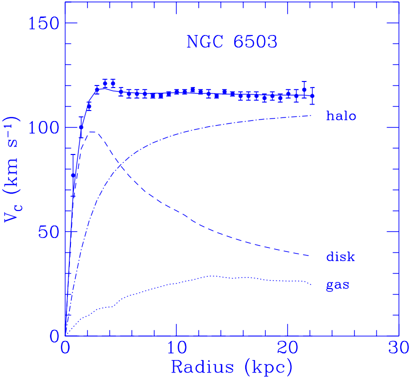
Dark matter: $\Omega_{DM} \equiv \frac{\rho_{DM}}{\rho_c} = 0.23$

Dark energy: $\Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_c} = 0.73$

Galactic dark halos: flat rotation curves

$$v(R) = \sqrt{G\frac{M(R)}{R}}$$

$$M(R) = 4\pi \int_0^R \rho(r) r^2 dr$$



observations: $v(R) \simeq$ const

visible matter:

internal regions $v(R) \propto \sqrt{R}$

external (“empty”) regions $v(R) \propto 1/\sqrt{R}$

Dark Matter in clusters

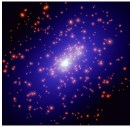
X-rays from hot gas in clusters

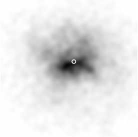
$$\frac{dP}{dR} = -\mu n_e(R) m_p \frac{GM(R)}{R^2}\,, \quad M(R) = 4\pi \int_0^R \rho(r) r^2 dr\,, \quad P(R) = n_e(R) T_e(R)$$

galaxies in clusters — virial theorem

$$U + 2E_k = 0$$

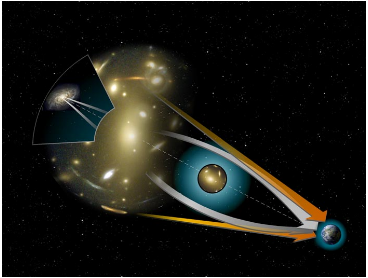
$$3M\langle v_r^2 \rangle = G\frac{M^2}{R}$$

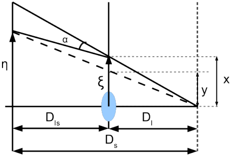




Milky Way: Virgo infall

Gravitational lensing in GR: $\alpha = 4GM/(c^2 b)$



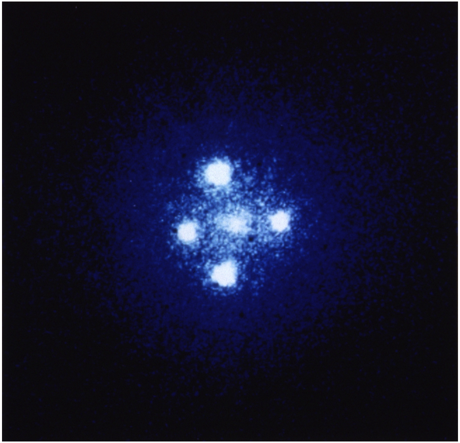


$$\vec{\eta} = \frac{D_s}{D_l}\vec{\xi} - D_{ls}\vec{\alpha}\left(\vec{\xi}\right)$$

common lens with specific refraction coefficient

$$\vec{\alpha}\left(\vec{\xi}\right) = \frac{4G}{c}\int \frac{\vec{\xi}-\vec{\xi}'}{\left|\xi-\vec{\xi}'\right|^2} d^2\xi' \int \rho\left(\vec{\xi}', z\right) dz$$

Einstein Cross

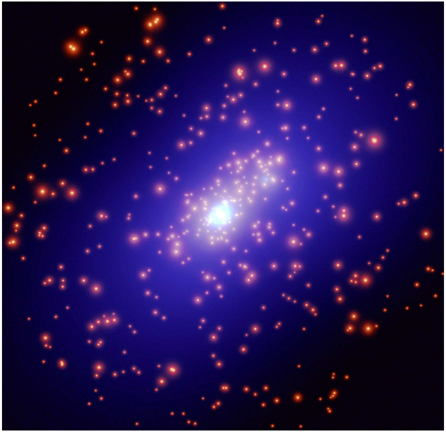


source: quasar $D_s = 2.4$ Gpc

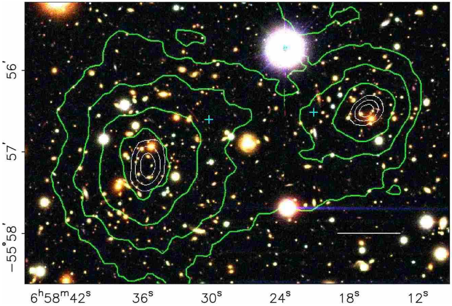
lens: galaxy $D_l = 120$ Mpc

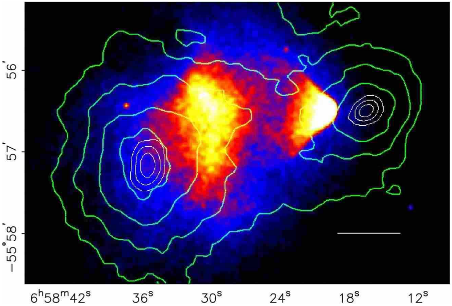
Dark Matter in clusters

gravitational lensing $\rho_{\mathrm{B}} \approx 0.25\rho_{DM}$



Colliding clusters (Bullet clusters 1E0657-558)





gravitational lensing

Observations in X-rays

$M \simeq 10 \times m$

scale is 200 kpc

clusters are at 1.5 Gpc

Dark Matter Properties $p = 0$

(If) particles:

1. stable on cosmological time-scale
2. nonrelativistic long before RD/MD-transition (either Cold or Warm, $v_{RD/MD} \lesssim 10^{-3}$)
3. (almost) collisionless
4. (almost) electrically neutral

If were in thermal equilibrium: $M_X \gtrsim 1$ keV

If not: for bosons

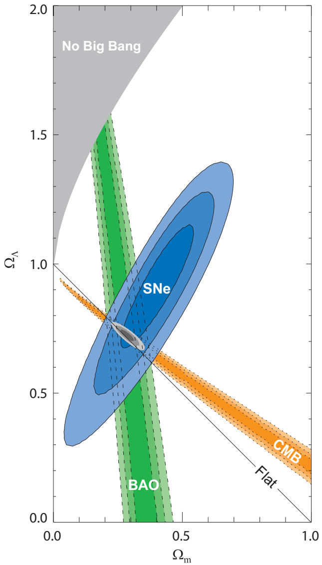
$\lambda = 2\pi/(M_X v_X)$, in a galaxy $v_X \sim 0.5 \cdot 10^{-3} \longrightarrow M_X \gtrsim 3 \cdot 10^{-22}$ eV

for fermions

Pauli blocking: $M_X \gtrsim 750$ eV

$$f(\mathbf{p}, \mathbf{x}) = \frac{\rho_X(\mathbf{x})}{M_X} \cdot \frac{1}{\left(\sqrt{2\pi} M_X v_X\right)^3} \cdot e^{-\frac{\mathbf{p}^2}{2M_X^2 v_X^2}} \Big|_{\mathbf{p}=0} \leq \frac{g_X}{(2\pi)^3}$$

Astrophysical and cosmological data are in agreement



$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(t) = \frac{8\pi}{3} G \rho_{\text{density}}^{\text{energy}}$$

$$\rho_{\text{density}}^{\text{energy}} = \rho_{\text{radiation}} + \rho_{\text{matter}}^{\text{ordinary}} + \rho_{\text{matter}}^{\text{dark}} + \rho_\Lambda$$

$$\rho_{\text{radiation}} \propto 1/a^4(t) \propto T^4(t)\,, \quad \rho_{\text{matter}} \propto 1/a^3(t)$$

$$\rho_\Lambda = \text{const}$$

$$\frac{3H_0^2}{8\pi G} = \rho_{\text{density}}^{\text{energy}}(t_0) \equiv \rho_c \approx 0.53 \times 10^{-5}\,\frac{\text{GeV}}{\text{cm}^3}$$

radiation: $\Omega_\gamma \equiv \frac{\rho_\gamma}{\rho_c} = 0.5 \times 10^{-4}$

Baryons (H, He): $\Omega_B \equiv \frac{\rho_B}{\rho_c} = 0.046$

Neutrino: $\Omega_\nu \equiv \frac{\sum \rho_{\nu_i}}{\rho_c} < 0.01$

Dark matter: $\Omega_{DM} \equiv \frac{\rho_{DM}}{\rho_c} = 0.23$

Dark energy: $\Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_c} = 0.73$

Determination of $a(t)$ reveals the composition of the present Universe

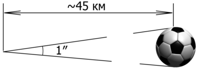
$\Delta s^2 = c^2\,\Delta t^2 - a^2(t)\,\Delta\vec{x}^2 \;\rightarrow\; ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu$ Light propagation changes...

How do we check it? by measuring distance L to an object!

- Measuring angular size θ of an object of known size d

single-type galaxies

$$\theta = \frac{d}{L}$$



- Measuring angular size $\theta(t)$ corresponding to physical size $d(t)$ with known evolution

$$\theta(t) = \frac{d(t)}{L}$$

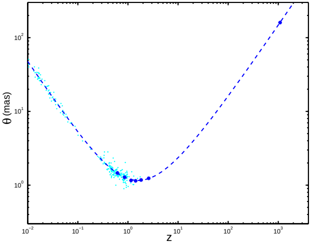
- Measuring brightness J of an object of known luminosity F

"standard candles"

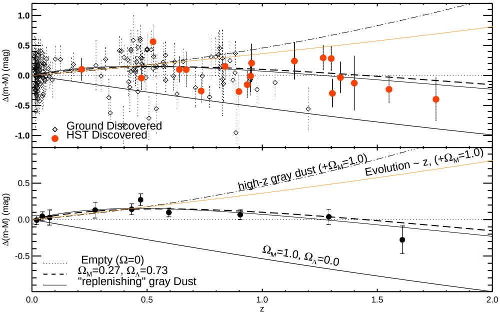
$$J = \frac{F}{4\pi L^2}$$

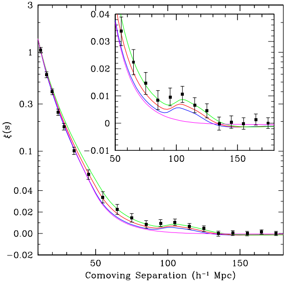
In the expanding Universe all these laws get modified

Results of distance measurements



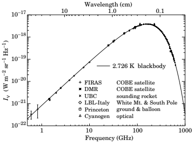
$$\Delta(m-M) = 5\log\frac{r_{ph}}{r_{ph}(\Omega_C = 0.8, \Omega_M = 0.2)}$$

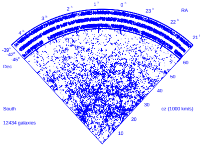


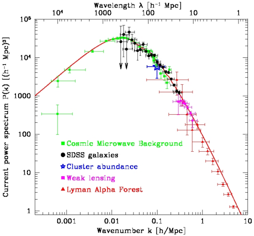


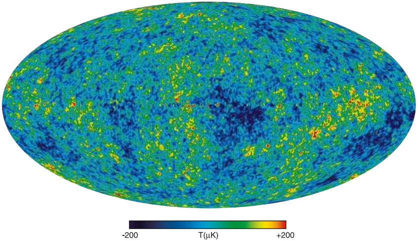
Key observable: matter perturbations

- CMB is isotropic, but "up to corrections, of course..."
  1. Earth movement with respect to CMB
  $$\frac{\Delta T_{\text{dipole}}}{T} \sim 10^{-3}$$
  2. More complex anisotropy!
  $$\frac{\Delta T}{T} \sim 10^{-4} - 10^{-5}$$
- There were matter inhomogenities $\Delta\rho/\rho \sim \Delta T/T$ at the stage of recombination ($e + p \to \gamma + H^*$)
- Jeans instability in the system of gravitating particles at rest $\Longrightarrow$ $\Delta\rho/\rho \nearrow$ $\Longrightarrow$ galaxies (CDM halos)





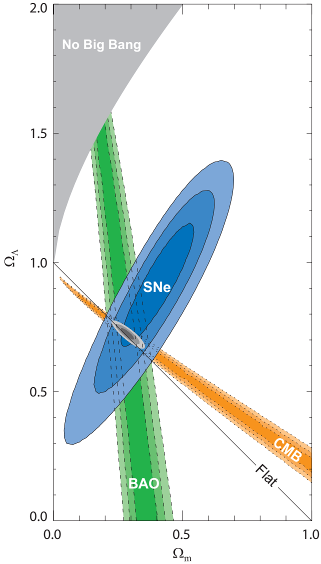




-

Outline

Dark Energy: nonclumping matter?



- estimates of Matter contribution confined in galaxies and clusters
  $\rho_c - \rho_M \neq 0$ but the Universe is flat, so $\rho_{curv} \simeq 0$
- corrections to the Hubble law : red shift – brightness curves for standard candles (SN Ia)
- The age of the Universe
- CMB anisotropy, large scale structures (galaxy clusters formation), etc

$$\rho_\Lambda = 0.73\rho_c$$

$$\rho_\Lambda \sim 10^{-5}\ \text{GeV/cm}^3 \sim \left(10^{-11.5}\ \text{GeV}\right)^4$$

Dark Energy: all evidences are from cosmology

Working hypothesis is cosmological constant $\Lambda \approx \left(2.5\times 10^{-3}\,\text{eV}\right)^4$:
$p = w(t)\,\rho$, $w = \text{const} = -1$, $\rho = \Lambda$

$$S_\Lambda = -\Lambda \int d^4x\,\sqrt{-\det g_{\mu\nu}}$$

both parts contribute

$$S_{\text{grav}} = -\frac{1}{16\pi G}\int d^4x\,\sqrt{-\det g_{\mu\nu}}\,R\,,$$
$$S_{\text{matter}} = \int d^4x\,\sqrt{-\det g_{\mu\nu}}\left(\frac{1}{2}g^{\lambda\rho}\partial_\lambda\phi\,\partial_\rho\phi - V(\phi)\right)$$

natural values

$$\Lambda_{\text{grav}} \sim 1/G^2 \sim \left(10^{19}\,\text{GeV}\right)^4\,, \quad \Lambda_{\text{matter}} \sim V(\phi_{vac}) \sim (100\,\text{GeV})^4\,, (100\,\text{MeV})^4\,, \ldots$$

Why Λ is small? Why $\Lambda \sim \rho$? Why $\rho_B \sim \rho_{DM} \sim \rho_\Lambda$ today?

The future Universe

We will live for a while. . .

the same period with the same expansion rate

$t \sim 10\,$Billion years

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(t) = \frac{8\pi}{3} G \rho^{\text{energy}}_{\text{density}}$$

$$\rho^{\text{energy}}_{\text{density}} = \rho_{\text{radiation}} + \rho^{\text{ordinary}}_{\text{matter}} + \rho^{\text{dark}}_{\text{matter}} + \rho_{\Lambda}$$

$$\rho_{\text{radiation}} \propto 1/a^4(t) \propto T^4(t)\,, \quad \rho_{\text{matter}} \propto 1/a^3(t)$$

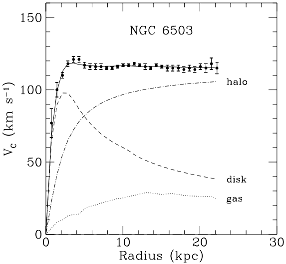
$$\rho_{\Lambda} = \text{const}$$

Why do we think it is most probably new particle physics
(new gravity if any is not enough) ?

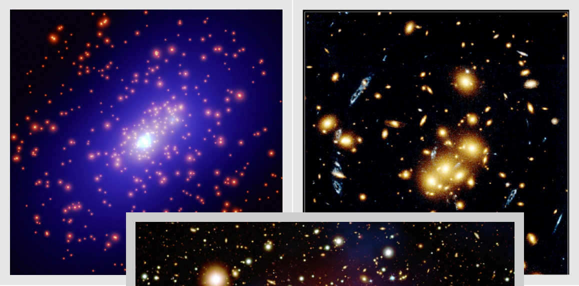
DM at various spatial scales, BAU requires baryon number violation

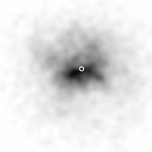
Universe content from astrophysics

Rotational curves



Gravitational lensing



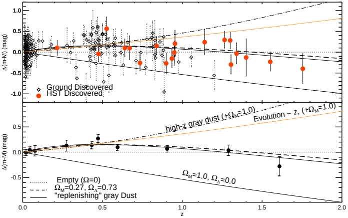


X-rays from centers of galaxy clusters

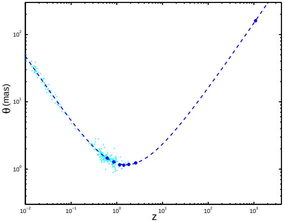
“Bullet” cluster

Universe content from cosmology

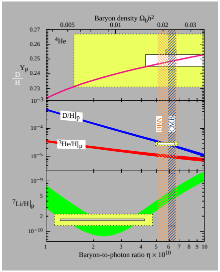
Standard candles

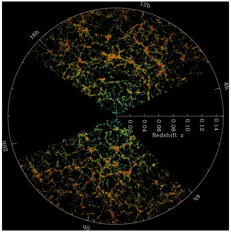


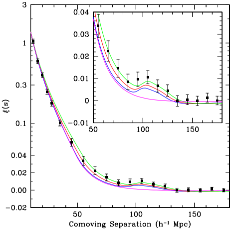
Angular distance

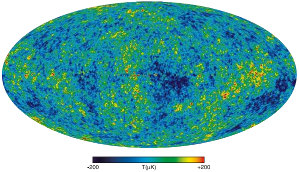


Nucleosynthesis

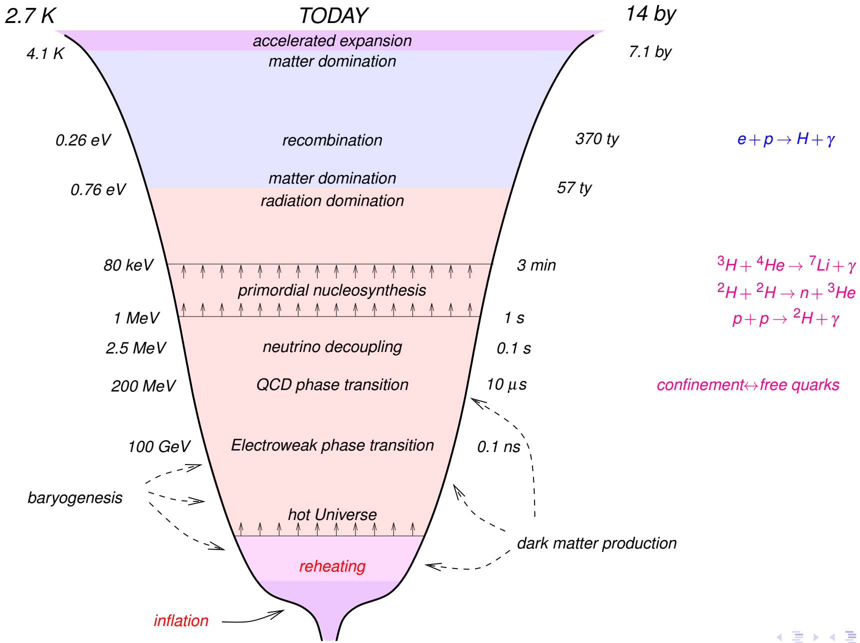








Large Scale Structures Baryon acoustic oscillations CMB anisotropy



2.7 K
TODAY
14 by
accelerated expansion
4.1 K
matter domination
7.1 by
0.26 eV
recombination
370 ty
$e + p \to H + \gamma$
matter domination
0.76 eV
radiation domination
57 ty
80 keV
3 min
$^3H + {}^4He \to {}^7Li + \gamma$
primordial nucleosynthesis
$^2H + {}^2H \to n + {}^3He$
1 MeV
1 s
$p + p \to {}^2H + \gamma$
2.5 MeV
neutrino decoupling
0.1 s
200 MeV
QCD phase transition
10 μs
confinement↔free quarks
100 GeV
Electroweak phase transition
0.1 ns
baryogenesis
hot Universe
dark matter production
reheating
inflation

Friedmann equation for the present Universe

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\left(\rho_{\mathrm{M}} + \rho_{rad} + \rho_{\Lambda} + \rho_{\mathrm{curv}}\right)$$

$$\frac{8\pi}{3} G\rho_{\mathrm{curv}} = -\frac{\varkappa}{a^2}\,, \quad \rho_c \equiv \frac{3}{8\pi G} H_0^2$$

$$\rho_c = \rho_{\mathrm{M},0} + \rho_{rad,0} + \rho_{\Lambda,0} = \rho_c = 0.53 \cdot 10^{-5} \frac{\mathrm{GeV}}{\mathrm{cm}^3}\,,$$

$$\Omega_X \equiv \frac{\rho_{X,0}}{\rho_c}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho_c \left[\Omega_{\mathrm{M}} \left(\frac{a_0}{a}\right)^3 + \Omega_{rad} \left(\frac{a_0}{a}\right)^4 + \Omega_{\Lambda}\right]$$

Simple tasks to be solved ✍

1. Refine the estimate of the age of our Universe

$$t_0 = \frac{1}{H_0} = 14 \times 10^9 \,\text{years}$$

2. When (z_{acc}, t_{acc}, T_γ – ?) did deceleration-acceleration transition happen?
3. When (z_{EQ}, t_{EQ}, T_γ – ?) did matter-radiation transition (Equality) happen?

   Hint: neutrino contribution to radiation energy density is 70% of photon contribution
4. Find the time of Electroweak phase transition, $T \sim 100\,\text{GeV}$

   The early Universe to be tested at LHC ...

   Hint: all SM particles contribute to energy density as about 50 photon species

Outline

FLRW metric $g_{\mu\nu}$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)dl^2 = dt^2 - a^2(t)\gamma_{ij}dx^i dx^j \; ,$$

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

Special frame: different parts look similar
Also this is comoving frame: world lines of particles at rest are geodesics,

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda = 0$$

$$\gamma_{ij} \approx \delta_{ij}$$

Photons in the expanding Universe

$$S = -\frac{1}{4}\int d^4x\sqrt{-g}g^{\mu\nu}g^{\lambda\rho}F_{\mu\lambda}F_{\nu\rho}$$

$dt = ad\eta$ conformally flat metric

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j \longrightarrow ds^2 = a^2(\eta)[d\eta^2 - \delta_{ij}dx^i dx^j]$$

$$S = -\frac{1}{4}\int d^4x\, \eta^{\mu\nu}\eta^{\lambda\rho}F_{\mu\lambda}F_{\nu\rho}\,, \qquad A_\mu^{(\alpha)} = e_\mu^{(\alpha)}\mathrm{e}^{ik\eta - i\mathbf{kx}}\,, \quad k = |\mathbf{k}|$$

$$\Delta x = 2\pi/k\,, \quad \Delta\eta = 2\pi/k$$

$$\lambda(t) = a(t)\Delta x = 2\pi\frac{a(t)}{k}\,, \quad T = a(t)\Delta\eta = 2\pi\frac{a(t)}{k}$$

Redshift and the Hubble law $\lambda_0 = \lambda_i \frac{a_0}{a(t_i)} \equiv \lambda_i (1 + z(t_i))$

$$\mathbf{p}(t) = \frac{\mathbf{k}}{a(t)} \,, \quad \omega(t) = \frac{k}{a(t)}$$

for not very distant objects $\qquad 1\,\text{pc} \approx 3\,\text{ly}$

$$a(t_i) = a_0 - \dot{a}(t_0)(t_0 - t_i) \quad \longrightarrow \quad a(t_i) = a_0[1 - H_0(t_0 - t_i)]$$

$$z(t_i) = H_0(t_0 - t_i) = H_0 r \,, \qquad z \ll 1$$

$$H_0 = h \cdot 100 \, \frac{\text{km}}{\text{s} \cdot \text{Mpc}} \,, \qquad h = 0.705 \pm 0.013$$

similar reddening for other relativistic particles (small H, $\dot{H}$, etc.)

$\mathbf{p} = \frac{\mathbf{k}}{a(t)}$ is true for massive particles as well

Gas of free particles in the expanding Universe

homogeneous gas $dN = f(\mathbf{p},t) d^3\mathbf{X} d^3\mathbf{p}$

in comoving coordinates:

$$d^3\mathbf{x} = \text{const}\,, \quad d^3\mathbf{k} = \text{const}\,, \quad f(k) = \text{const}$$

$$f(k) d^3\mathbf{x} d^3\mathbf{k} = \text{const}$$

comoving volume equals physical volume

$$d^3\mathbf{x} d^3\mathbf{k} = d^3(a\mathbf{x}) d^3\left(\frac{\mathbf{k}}{a}\right) = d^3\mathbf{X} d^3\mathbf{p}$$

$$f(\mathbf{p},t) = f(\mathbf{k}) = f[a(t)\cdot\mathbf{p}]\,.$$

$$t = t_i\ :\quad f_i(\mathbf{p}) \ \longrightarrow\ f(\mathbf{p},t) = f_i\left(\frac{a(t)}{a(t_i)}\mathbf{p}\right)$$

Massless bosons (photons) fermions

$$f_i(\mathbf{p}) = f_{\mathrm{Pl}}\left(\frac{|\mathbf{p}|}{T_i}\right) = \frac{1}{(2\pi)^3}\frac{1}{e^{|\mathbf{p}|/T_i} - 1}$$

$$f(\mathbf{p},t) = f\left(\frac{a(t)|\mathbf{p}|}{a_i T_i}\right) = f\left(\frac{|\mathbf{p}|}{T_{eff}(t)}\right)$$

$$T_{eff}(t) = \frac{a_i}{a(t)} T_i$$

decoupling at $T \gg m$: neutrinos, hot(warm) dark matter

decoupling at $T \ll m$: $f(\mathbf{p}) = \frac{1}{(2\pi)^3}\exp\left(-\frac{m-\mu_i}{T_i}\right)\exp\left(-\frac{a^2(t)\mathbf{p}^2}{2ma_i^2 T_i}\right)$

$$f(\mathbf{p},t) = \frac{1}{(2\pi)^3}\exp\left(-\frac{m-\mu_{eff}}{T_{eff}}\right)\exp\left(-\frac{\mathbf{p}^2}{2mT_{eff}}\right)$$

$$T_{eff}(t) = \left(\frac{a_i}{a(t)}\right)^2 T_i\,, \qquad \frac{m-\mu_{eff}(t)}{T_{eff}} = \frac{m-\mu_i}{T_i}$$