QCD for collider experiments
an introduction in four lectures

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When you measure what you are speaking about and express it in numbers, you know something about it, but when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind.

Lord Kelvin
Four Lectures

☞ Thursday:
QCD as QFT of strong interactions

☞ Friday:
QCD perturbation theory at fixed order:
\( \sigma(e^+e^- \rightarrow \text{hadrons}) \)

☞ Sunday:
QCD perturbation theory at fixed order:
event shapes and jets

☞ Monday:
☞ QCD perturbation theory:
partons in the initial state,
Assumptions

☞ You are
☞ familiar with QFT & QED (previous talk)
☞ eager to understand & learn QCD
☞ thus willing to enjoy work (= compute)

☞ You accept that there are no stupid questions only stupid silence
   (and sometimes answers, but that’s our fault 😊)
Lecture 1

QCD as QFT of strong interactions

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Today

🔗 Symmetries of the QCD Lagrangian
   🔗 exact
   🔗 approximate

🔗 Many amazing faces of QCD

🔗 Property worth a Nobel prize: asymptotic freedom
   🔗 renormalization group (RG) equation
   🔗 $\beta$-function
   🔗 running coupling and masses
Today we have a satisfactory quantum field theory of strong interactions based on a non-abelian gauge symmetry: QCD.

40 years of immense efforts lead to a lot of results and deep understanding, yet we are far from a complete and satisfactory solution.

indeed: Happy anniversary QCD

H. Fritzsch, M. Gell-Mann, H. Leutwyler: *Advantages of the colour octet picture* PLB 47 (1973) 365
Goals

a) ambitious: solve QCD

b) pragmatic: develop tools for modeling particle interactions in high energy collider experiments

We pursue (b) here
Understand events quantitatively from first principles
Understand events quantitatively from first principles

**Photon**
- $p_T = 128 \text{ GeV/c}$
- $\eta = 0.5$
- $\phi = 2.0 \text{ rad}$

**Anti-k$_T$ 0.5 PFJet**
- $p_T = 139 \text{ GeV/c}$
- $\eta = -0.6$
- $\phi = -1.1 \text{ rad}$
There is a long way

...to distributions, full of pitfalls & difficulties
QCD Lagrangian

- QCD is a part of the Standard Model
- SM is a gauge theory with underlying $\text{SU}_c(3) \times \text{SU}_L(2) \times \text{U}_Y(1)$

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}^0_{\text{QCD}} + \mathcal{L}_{\text{sources}}$$

$$\mathcal{L}^0_{\text{QCD}} = \mathcal{L}_{\text{classical}} + \mathcal{L}_{\text{gauge fix}} + \mathcal{L}_{\text{ghost}}$$

in physical gauges
Classical Lagrangian

\[ \mathcal{L}_c = \sum_{f=1}^{6} \mathcal{L}_f(q_f, m_f) + \mathcal{L}_g(A) \]

\[ \mathcal{L}_f(q_f, m_f) = \sum_{a,b=1}^{N_c} \bar{q}_f^a (i\gamma_\mu D^\mu - m_f)_{ab} q_f^b \]

\[ \{\gamma_\mu, \gamma_5\} = 0 \quad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \]

many degrees of freedom:

- flavours (f), colours (a,b)
- spin and space-time position (not shown)

QCD computations are very cumbersome
Field content

* Quark fields dictated by the electroweak sector: $q_f \ f=1,\ldots,6 \ B=1/3$

<table>
<thead>
<tr>
<th>f</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_f$</td>
<td>u</td>
<td>d</td>
<td>s</td>
<td>c</td>
<td>b</td>
<td>t</td>
</tr>
<tr>
<td>$m_f$</td>
<td>$\sim 5\text{MeV}$</td>
<td>$\sim 7\text{MeV}$</td>
<td>$\sim 100\text{MeV}$</td>
<td>1.2GeV</td>
<td>4.2GeV</td>
<td>173GeV</td>
</tr>
</tbody>
</table>

* for heavy ones $m_f$ running masses (see later) at $\mu = 2\text{GeV}$, approximate values

* Gluon fields prescribed by $SU_c(3)$ gauge symmetry: $A^\alpha, \ \alpha = 1,\ldots,8$
Field content

\[(D_{\mu})_{ab} = \left( \partial_{\mu} + ig_s \sum_c A_\mu^c T^c \right)_{ab} \]
\[
\mathcal{L}_g(A) = -\frac{1}{4} \sum_{a=1}^{N_c^2-1} \mathcal{F}_a^\mu \mathcal{F}_a^{\mu\nu}(A)
\]

\(T^c\) are the SU\((N_c)\) generators with algebra:

\[ [T^a, T^b] = if^{abc} T^c \quad Tr(T^a T^b) = TR \delta^{ab} \quad TR = \frac{1}{2} \]

fundamental representation: \((T^a)_{ij} = \frac{1}{2} \lambda_{ij} = t_{ij}^a\)

adjoint representation: \((T^a)_{bc} = -if^{abc} = F_{ij}^a\)
Number of colours

\[ \sigma(e^+ e^- \rightarrow q\bar{q}) \] is leading order approximation to \[ \sigma(e^+ e^- \rightarrow \text{hadrons}) \]

\[ \Rightarrow \text{hadronic } R \text{ ratio in LO perturbation theory} \]

\[ R = \frac{\sigma(e^+ e^- \rightarrow q\bar{q})}{\sigma(e^+ e^- \rightarrow \mu^+ \mu^-)} = \left( \sum_{q=1}^{N_f} e_q^2 \right) \frac{N_c}{3} = \]

\[ = 2\left( \frac{N_c}{3} \right), \quad N_f = 3 \]

\[ = \frac{10}{3}\left( \frac{N_c}{3} \right), \quad N_f = 4 \]

\[ = \frac{11}{3}\left( \frac{N_c}{3} \right), \quad N_f = 5 \]
Number of colours = 3
Colour factors

Eigenvalues of the quadratic Casimir operator (like eigenvalues of $J^2$)

☞ in the fundamental representation

$$\left( t^a t^a \right)_{bc} = C_F \delta_{bc} \quad C_F = T_R \frac{N_c^2 - 1}{N_c} = \frac{4}{3} \quad \text{in QCD}$$

☞ in the adjoint representation:

$$\left( F^a F^a \right)_{bc} = C_A \delta_{bc} \quad C_A = 2 T_R N_c = 3 \quad \text{in QCD}$$

⇒ measuring $C_F$ and $C_A$ determines $N_C$
Simultaneous measurement of the strong coupling and colour factors
Source of non-Abelian nature

\[ F^a_{\mu \nu}(A) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + i g_s F^a_{b\mu} A^b_\mu A^c_\nu \]

- Main difference between QCD and QED
- Makes it a ‘perfect theory’: among QFT theories in \( d = 4 \) only non-Abelian gauge theories are asymptotically free (below)
- Possible source of colour confinement (colour neutrality of hadrons)?
Feynman rules

can be read off the action

\[ S = i \int d^4x (\mathcal{L}_f + \mathcal{L}_g) \equiv S_0 + S_I \]

\[ S_0 = i \int d^4x \mathcal{L}_0 \quad S_I = i \int d^4x \mathcal{L}_I \]

☞ \( \mathcal{L}_0 \) contains bilinear terms of the fields

☞ \( \mathcal{L}_I \) contains the rest (called interactions)

☞ gluon propagator \( \Delta_{g,\mu\nu} \) is the inverse of the bilinear term in \( A_\mu \) – in momentum space:

\[ \Delta_{g,\mu\nu}(p) i \left[ p^2 g^{\nu\rho} - p^\nu p^\rho \right] = \delta^\rho_\mu \]
Feynman rules

but \[ i \left[ p^2 g^{\nu \rho} - p^\nu p^\rho \right] p^\rho = 0 \], hence \[ \left[ p^2 g^{\nu \rho} - p^\nu p^\rho \right]^{-1} \] does not exist

\[ \Rightarrow \] Exploit gauge invariance to rewrite \( L \) in a physically equivalent form – gauge fixing = imposing a constraint on \( A_\mu \) by adding a term with a Lagrange multiplicator (colour implicit)

covariant: \[ \partial_\mu A^\mu (x) = 0 \Rightarrow \mathcal{L}_{gf} = -\frac{1}{2\lambda} (\partial_\mu A^\mu)^2 \]

axial \( n^\mu \neq p^\mu \): \[ n_\mu A^\mu (x) = 0 \Rightarrow \mathcal{L}_{gf} = -\frac{1}{2\lambda} (n_\mu A^\mu)^2 \]
**Feynman rules: propagators**

** gluon  

in covariant gauges:

\[
\frac{i\delta^{ab}}{p^2} \left[ g_{\mu\nu} - (1 - \lambda) \frac{p_\mu p_\nu}{p^2} \right]
\]

with usual choice \( \lambda = 1 \) (Feynman gauge)

in axial gauges:

\[
d_{\mu\nu}(p, n) = -g_{\mu\nu} + \frac{p_\mu n_\nu + n_\mu p_\nu}{p \cdot n}
\]

\[
\frac{i\delta^{ab}}{p^2} \left( d_{\mu\nu}(p, n) - (n^2 + \lambda p^2) \frac{p_\mu p_\nu}{(p \cdot n)^2} \right)
\]

with usual choice \( n^2 = 0, \lambda = 0 \) (light-cone gauge)

** quark  

\[
i \delta_{ij} \frac{p^2 + m}{p^2 - m^2}
\]
Feynman rules: vertices

\[ -ig_s t^a_{ij} \gamma^\mu \]

So far same as QED if \( g_s \to e \) and \( (t^a)_{jj} \to \delta_{ij} \).

With rules for external states can compute QED cross sections:

\[ \text{outgoing (anti-)fermion:} \quad (\nu(p)) \bar{u}(p) \]
\[ \text{incoming (anti-)fermion:} \quad (\bar{\nu}(p)) u(p) \]
\[ \text{outgoing/incoming vector boson:} \quad \sum_{\lambda=1,2} \epsilon^{(\lambda)}_\mu(p) \epsilon^{(\lambda)}_\nu(p)^* = d_{\mu\nu}(p, n) \]
Feynman rules: vertices

**gluon self couplings**

- **triple:**
  \[-i g_s F_{bc}^\alpha \left[ + g^{\alpha \beta} (p - q)^\gamma 
  + g^{\beta \gamma} (q - r)^\alpha 
  + g^{\gamma \alpha} (r - p)^\beta \right] \]

- **quartic:**
  \[-i g_s^2 \left[ + f^{x ac} f^{x bd} \left( g^{\alpha \beta} g^{\gamma \delta} - g^{\alpha \delta} g^{\beta \gamma} \right) 
  + f^{x ad} f^{x bc} \left( g^{\alpha \beta} g^{\gamma \delta} - g^{\alpha \gamma} g^{\beta \delta} \right) 
  + f^{x ad} f^{x bc} \left( g^{\alpha \gamma} g^{\beta \delta} - g^{\alpha \delta} g^{\beta \gamma} \right) \]
Feynman rules: vertices

Four-gluon vertex is not in a factorized form of a colour and a tensor factor

Introduce a fake field, without dynamics

Propagator:

\[ \frac{i}{2} \delta^{ab} (g^{\alpha\beta} g_{\gamma\delta} - g^{\alpha\delta} g_{\beta\gamma}) \]

That couples only to gluon with vertex:

\[ i \sqrt{2} g_s f^{x \alpha c} g^{\alpha\xi} g_{\gamma\zeta} \]

Helps automation
Colour algebra

Factorization of colour in Feynman rules allows for separate computation of colour sums, implemented in programs, e.g. type

\[
\text{In}[1]:= \text{Import}[^\text{http://www.feyncalc.org/install.m}^] \\
\]

in your Mathematica

but it is also easy&fun graphically

(the following graphs are meant in colour space only – try to write the corresponding algebraic expressions)
Colour algebra

- commutation relation:

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.6\textwidth]{commutation_relation.png}
\end{array}
\end{array}
\]

- normalization:

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.6\textwidth]{normalization.png}
\end{array}
\end{array}
\]

- Fierz-identity:

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.6\textwidth]{fierz_identity.png}
\end{array}
\end{array}
\]
Colour algebra

 gebruik de Fierz-id. om $C_F$ te bepaLEN

 multiply de commutatie-relatie met de quark gluon vertex (in kleur) en bewijs

 gebruik de bovenstaande oplossing om $C_A$ te berekenen
Colour algebra

What is the value of the following numbers?

Compute the factors $A$, $B$, $C$
Are we done?

Seemingly yes: we can compute the cross section of any process to any accuracy in PT, like in QED

...but rather not: you will find big surprises due to gluon self coupling!

in QCD complexity becomes prohibitive at higher orders – OK, compute at one-loop

Oops! – the coupling decreases with increasing scattering energies – LO enough?
Parton-hadron duality

- Electrons and photons exist as free particles.
- Quarks and gluons have never been observed in detectors, only hadrons.
- Assumption: a low-order perturbative computation in QCD is an approximation to sufficiently inclusive hadronic cross-section if:
  - total cm energy $Q$ of partons is much larger than the mass of quarks and hadrons, $Q \gg m$.
  - $Q$ is far from hadronic resonances and thresholds.
Parton-hadron duality
Exact symmetries of the classical Lagrangian

Predictions should reflect the symmetries of $\mathcal{L}_c$, hence it is useful to study those first in space-time: against transformations of the conformal group

- translations
- Lorentz transformations (rotation and boost)
- scale if quarks are massless

$$x^\mu \rightarrow \lambda x^\mu \quad A_\mu(x) \rightarrow \lambda^{-1} A_\mu(\lambda x) \quad q(x) \rightarrow \lambda^{-3/2} q(\lambda x)$$

conformal transformations
Exact symmetries of the classical Lagrangian

- **in colour space**: local gauge invariance

\[ U : q_i(x) \rightarrow q'_i(x) = U(x)_{ij} q_j(x), \quad U(x) = \exp\left( i \sum_{a=1}^{N_c^2-1} \alpha^a(x) T^a \right) \in SU(N_c) \]

- **the covariant derivative transforms as the field itself** (prove if you have not yet done):

\[ U : \left( D_\mu q(x) \right)_i \rightarrow \left( D'_\mu q'(x) \right)_i = U(x)_{ij} \left( D_\mu q(x) \right)_j \]

- **quark mass term** SU\(_c(3)\) gauge invariant
Exact symmetries of the classical Lagrangian

- discrete: C, P and T in agreement with observed properties of strong interactions (C, P and T violating strong decays are not observed)

- there exists additional gauge invariant dimension-four operator, the $\Theta$–term:

\[
L_\Theta = \frac{\Theta g_s}{32\pi^2} F_{\mu\nu} \cdot \tilde{F}^{\mu\nu}, \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}
\]

violates P and T, $\Theta$ is small ($<10^{-9}$ experimentally), set $\Theta=0$ in perturbative QCD
almost supersymmetric

Massless QCD for $N_f = 1$

$$
\mathcal{L}^{QCD}_C = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \overline{q} \, i\gamma_\mu D^\mu \, q
$$

SUSY Yang-Mills:

$$
\mathcal{L}^{SYM}_C = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \overline{\lambda} \, i\gamma_\mu D^\mu \, \lambda
$$

quarks transform under fundamental, gluinos under adjoint representation of SU(N)
Approximate symmetries of the classical Lagrangian

Related to the quark mass matrix

introduce

\[ \psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \]

\[ P_{\pm} = \frac{1}{2} (1 \pm \gamma_5) \]

\( P_{\pm} \) are projections

\[ P_+ P_- = P_- P_+ = 0, \quad P_{\pm}^2 = P_{\pm}, \quad P_+ + P_- = 1 \]

from Dirac algebra:

\[ \gamma_\mu P_{\pm} = P_\mp \gamma_\mu \]

let

\[ \psi_{\pm} = P_{\pm} \psi, \quad \overline{\psi} = \psi^\dagger \gamma_0 \Rightarrow \overline{\psi}_{\pm} = \overline{\psi} P_\mp \]

\[ \gamma_5^2 = 1 \Rightarrow \gamma_5 \psi_{\pm} = \pm \psi_{\pm} \]
Approximate symmetries of the classical Lagrangian

_books the quark sector of the Lagrangian can be rewritten:\_

\[
\mathcal{L}_{\text{Chir}} = \overline{\psi} i\gamma_\mu D^\mu \psi = \overline{\psi} \left( P_+ + P_- \right) i\gamma_\mu D^\mu \left( P_+ + P_- \right) \psi
= \overline{\psi} P_+ i\gamma_\mu D^\mu P_- \psi + \overline{\psi} P_- i\gamma_\mu D^\mu P_+ \psi
= \overline{\psi}_- i\gamma_\mu D^\mu \psi_- + \overline{\psi}_+ i\gamma_\mu D^\mu \psi_+ \equiv \mathcal{L}_- + \mathcal{L}_+ \equiv \mathcal{L}_L + \mathcal{L}_R
\]

_books would not work if gluons were not vectors (in \(D^\mu\))

_books the left- and right-handed fields are not coupled

\(
\mathcal{L}_{\text{Chir}} \) is invariant under \(U_L(N_f) \times U_R(N_f)\)

This symmetry acts separately on left- and right-handed fields: chiral symmetry
Approximate symmetries of the classical Lagrangian

the group elements can be parametrised in terms of $2 N_f^2$ real numbers:

$$\left(g_L, g_R\right) = \left(e^{i\alpha_a T^a} e^{i\beta_a T^a}, e^{i\alpha_a T^a} e^{-i\beta_a T^a}\right)$$

$$\in U_V(1) \otimes U_A(1) \otimes SU_L(N_f) \otimes SU_R(N_f)$$

has vector subgroups $SU_V(N_f) \times U_V(1)$

$$\left(g, g\right) = \left(e^{i\alpha_a T^a}, e^{i\alpha_a T^a}\right) = e^{i\alpha_a T^a I} \in SU_V(N_f)$$

axial transformations do not form a subgroup

$$\left(h, h^+\right) = \left(e^{i\beta_a T^a}, e^{-i\beta_a T^a}\right) = e^{i\beta_a T^a \gamma_5} \quad \left[T^a \gamma_5, T^b \gamma_5\right] = i f^{abc} T^c I$$
Chiral perturbation theory

- Chiral symmetry is not observed in the hadron spectrum.

- In QCD it is believed that the vacuum has a non-zero VEV of the light-quark operator:

\[
\langle 0 | \bar{q} q | 0 \rangle = \langle 0 | \bar{u}u + \bar{d}d | 0 \rangle \approx (250 \text{ MeV})^3
\]

This quark condensate breaks chiral symmetry spontaneously to \( SU_V(N_f) \times U_V(1) \) \( \Rightarrow \) isospin & conserved baryon number, „massless“ mesons.

- Because it connects left- and right-handed fields:

\[
\langle 0 | \bar{q} q | 0 \rangle = \langle 0 | \bar{q}_L q_R + \bar{q}_R q_L | 0 \rangle
\]
Chiral perturbation theory

- SSB of chiral symmetry implies the existence of $N_f^2 - 1$ massless Goldstone bosons,

- but light quarks are not exactly massless
  - chiral symmetry is not exact, the Goldstone bosons are not massless
  - pseudoscalar meson octet!

- $m_f$ are treated as perturbation
  - $\chi$PT
  - masses of light quarks, scattering properties of pions!
QCD topics (at T=0)

Low-energy properties (<GeV)

High energy collisions (>GeV)

Perturbative

Non-perturbative

χPT (light quark masses)

Jet physics

Sum rules, lattice QCD

Our focus
Approximate symmetries of the classical Lagrangian

Choose Weyl representation:

$$
\gamma_0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

$$
\psi_+ = \begin{pmatrix} \tilde{\psi}_+ \\ 0 \end{pmatrix}, \quad \psi_- = \begin{pmatrix} 0 \\ \tilde{\psi}_- \end{pmatrix}, \quad \tilde{\psi}_\pm (x) = e^{-i p \cdot x} \tilde{\psi}_\pm (p)
$$

two-component Weyl spinors, satisfying

$$
i \gamma \cdot \partial \tilde{\psi}_\pm (x) = 0 \implies \gamma \cdot p \tilde{\psi}_\pm (p) = 0 \quad \text{if } m=0$$

$$
- E \tilde{\psi}_\pm (p) \pm \sigma_i p_i \tilde{\psi}_\pm (p) = 0
$$

Helicity eigenstates:

$$
\sigma \cdot \hat{p} \tilde{\psi}_\pm (p) = \pm \tilde{\psi}_\pm (p)
$$
Symmetries of the classical Lagrangian

- Some are useful for:
  - easing computations
  - checking computations
  - hinting on solving QCD

- Some are violated by quantum corrections with important physical consequences:
  - scaling violations
  - axial anomaly (not discussed here)
What is scaling?

Consider a dimensionless physical observable \( R = R(Q^2) \), with \( Q \) being a large energy scale,

\[ Q \gg \text{any other dimensionful parameter (e.g. } m_f) \]

\[ \Rightarrow \text{set } m_f = 0 \text{ (check later if } R(m_f = 0) \text{ is OK)} \]

Classically \( \text{dim} R = 0 \) & \( \text{dim} Q = 1 \), so

\[ \frac{dR}{dQ} = 0 \Rightarrow \lim_{Q^2 \to \infty} R = R_0 \text{ constant} \]
What is scaling violation?

- We’ll see in a renormalized QFT we need an additional scale: $\mu$ renormalization scale, thus $R = R(Q^2/\mu^2)$ is not a constant: scaling violation.
- The “small” parameter in the perturbative expansion of $R$, $\alpha_s(\mu)$ also depends on the scale choice.
- But $\mu$ is an arbitrary, non-physical parameter ($\mathcal{L}_C$ does not depend on it) $\Rightarrow$ physical quantities cannot depend on $\mu$. 
Renormalization group equation

\[ 0 = \mu^2 \frac{d}{d\mu^2} R \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) = \left( \mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s} \right) R \]

let \( t = \ln(\frac{Q^2}{\mu^2}) \), \( \beta(\alpha_s) = \mu^2 \left. \frac{\partial \alpha_s}{\partial \mu^2} \right|_{\alpha_s^{(0)} \text{ fixed}} \)

\[ \left( -\frac{\partial}{\partial t} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) R(e^t, \alpha_s) = 0 \]

How can we solve this?
Established Feynman rules of pQCD
- similar to QED with some complications
- those of colour algebra can be factorized
- those of gluon self coupling are tremendous (you have not yet seen it)

Understood the origin of the RGE

Let us try to solve it