QCD for collider experiments
an introduction in four lectures

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When you measure what you are speaking about and express it in numbers, you know something about it, but when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind.

Lord Kelvin

Four Lectures

- ☛ Thursday:
  QCD as QFT of strong interactions
- ☛ Friday:
  QCD perturbation theory at fixed order:
  $\sigma(e^+e^- \to \text{hadrons})$
- ☛ Sunday:
  QCD perturbation theory at fixed order:
  event shapes and jets
- ☛ Monday:
  - ☞ QCD perturbation theory:
  partons in the initial state,

Assumptions

- ☛ You are
  - ☞ familiar with QFT & QED (previous talk)
  - ☞ eager to understand & learn QCD
  - ☞ thus willing to enjoy work (= compute)
- ☛ You accept that there are no stupid questions only stupid silence

  (and sometimes answers, but that's our fault ☺)

Lecture 1
QCD as QFT of strong interactions

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Today

- ☛ Symmetries of the QCD Lagrangian
  - ☞ exact
  - ☞ approximate
- ☛ Many amazing faces of QCD
- ☛ Property worth a Nobel prize: asymptotic freedom
  - ☞ renormalization group (RG) equation
  - ☞ β-function
  - ☞ running coupling and masses

Claims

- ☛ Today we have a satisfactory quantum field theory of strong interactions based on a non-abelian gauge symmetry: QCD
- ☛ 40 years of immense efforts lead to a lot of results and deep understanding, yet we are far from a complete and satisfactory solution
- ☛ indeed: Happy anninversary QCD
  H. Fritzsch, M. Gell-Mann, H. Leutwyler: *Advantages of the colour octet picture*
  PLB 47 (1973) 365

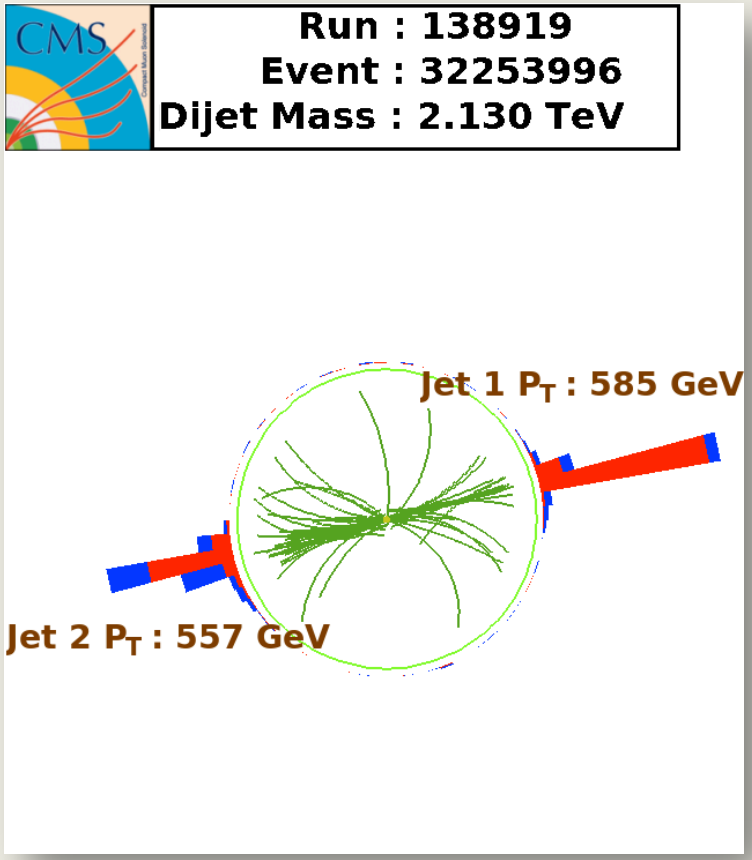
Goals

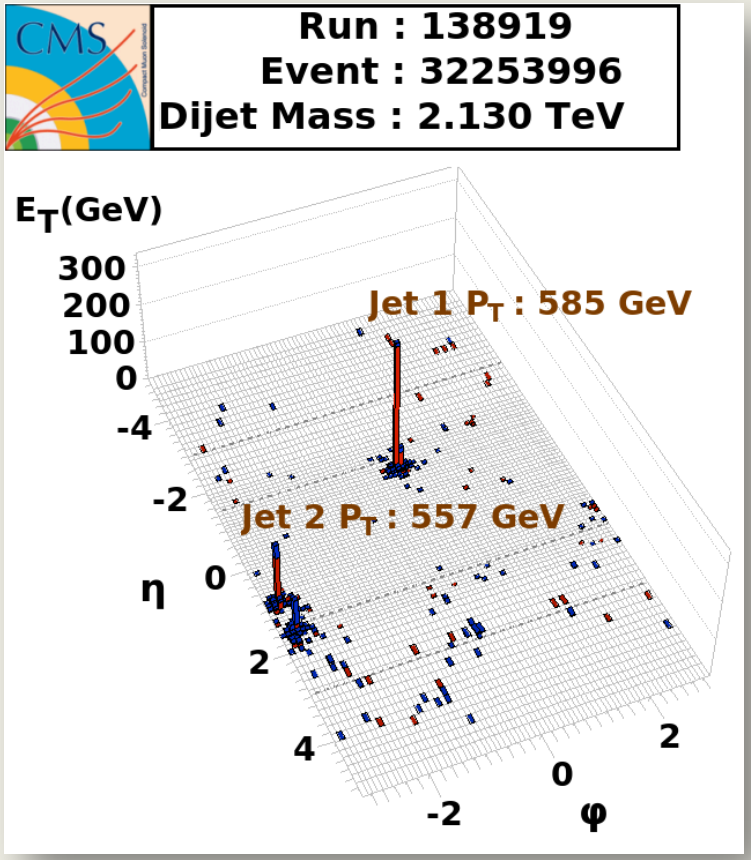
a) ambitious: solve QCD

b) pragmatic: develop tools for modeling particle interactions in high energy collider experiments

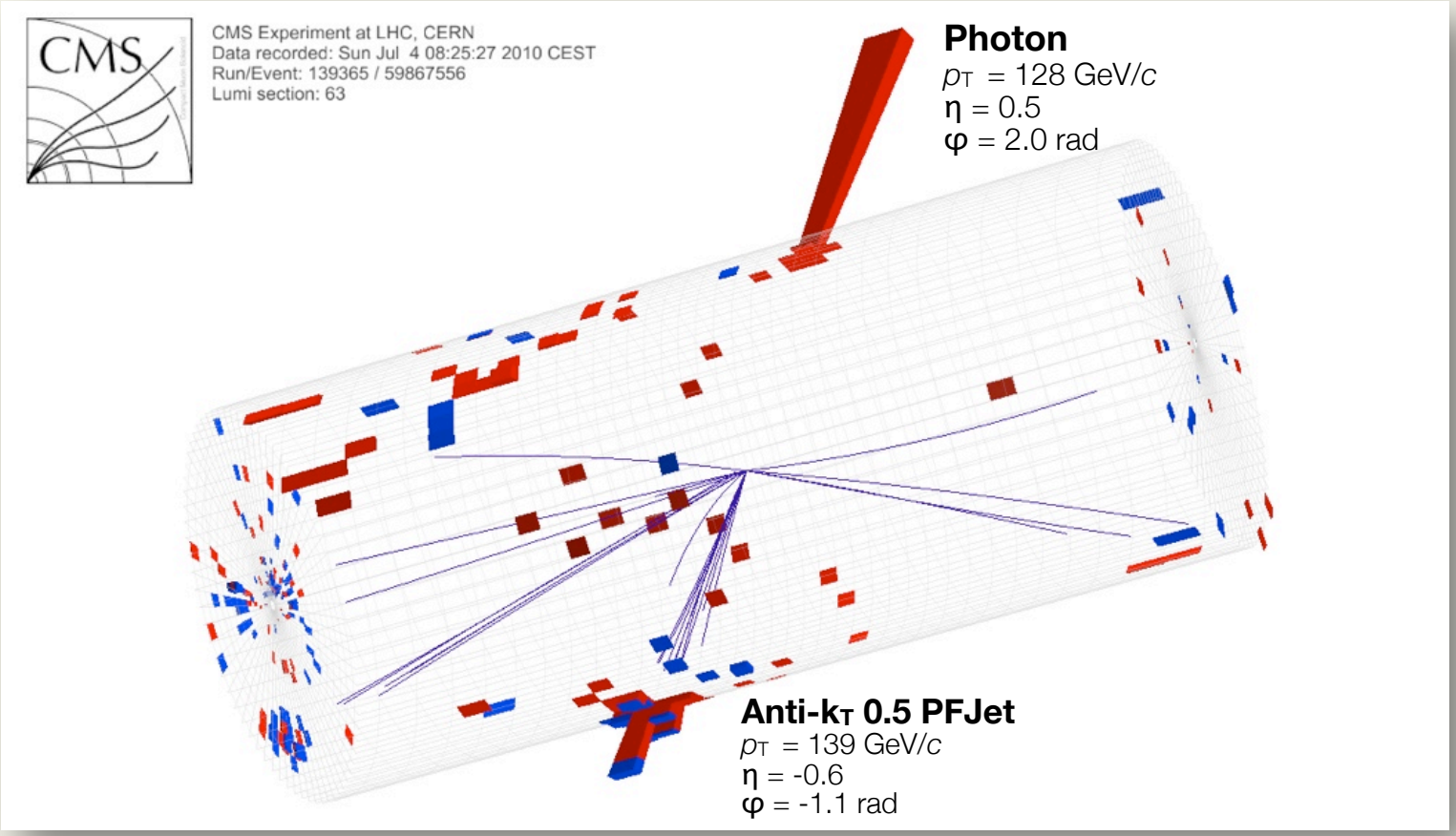
We pursue (b) here

Understand events quantitatively from first principles





Understand events quantitatively from first principles



There is a long way

…to distributions, full of pitfalls & difficulties

There is a long way from the basics

QCD Lagrangian

- QCD is a part of the Standard Model
- SM is a gauge theory with underlying

$$SU_c(3) \times SU_L(2) \times U_Y(1)$$

$$\mathcal{L}_{\rm QCD} = \mathcal{L}^0_{\rm QCD} + \mathcal{L}^{\rm sources}_{\rm QCD}$$

$$\mathcal{L}^0_{\rm QCD} = \mathcal{L}_{\rm classical} + \mathcal{L}_{\rm gauge\ fix} + \cancel{\mathcal{L}_{\rm ghost}}$$

in physical gauges

Classical Lagrangian

$$\mathcal{L}_{\mathrm{c}} = \sum_{f=1}^{6} \mathcal{L}_f(q_f,\, m_f) + \mathcal{L}_g(A)$$

$$\mathcal{L}_f(q_f,\, m_f) = \sum_{a,b=1}^{N_c} \bar{q}_f^a \left(\mathrm{i}\gamma_\mu D^\mu - m_f\right)_{ab} q_f^b$$

$$\{\gamma_\mu, \gamma_5\} = 0 \qquad\qquad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$$

- ☛ many degrees of freedom:
  - ☞ flavours (*f*), colours (*a,b*)
  - ☞ spin and space-time position (not shown)
  - ➔ QCD computations are very cumbersome

Field content

- Quark fields dictated by the electroweak sector: q_f f=1,…,6 B=1/3

f	1	2	3	4	5	6
q_f	u	d	s	c	b	t
m_f	~5MeV	~7MeV	~100MeV	1.2GeV	4.2GeV	173GeV

- for heavy ones m_f running masses (see later) at μ = 2GeV, approximate values
- Gluon fields prescribed by $SU_c(3)$ gauge symmetry: A^a, a = 1,…,8

Field content

$$(D_\mu)_{ab} = \left(\partial_\mu + \mathrm{i} g_\mathrm{s} \sum_c A_\mu^c T^c \right)_{ab}$$

$$\mathcal{L}_g(A) = -\frac{1}{4} \sum_{a=1}^{N_c^2-1} \mathcal{F}_{\mu\nu}^a(A) \mathcal{F}^{a\,\mu\nu}(A)$$

T^c are the SU(N_c) generators with algebra:

$$\left[T^a, T^b\right] = \mathrm{i} f^{abc} T^c \qquad Tr\left(T^a\, T^b\right) = T_R\, \delta^{ab} \qquad T_R = \frac{1}{2}$$

fundamental representation: $\left(T^a\right)_{ij} = \frac{1}{2}\lambda_{ij} = t_{ij}^a$

adjoint representation: $\left(T^a\right)_{bc} = -\mathrm{i} f^{abc} = F_{ij}^a$

Number of colours

$\sigma(e^+e^- \to q\bar{q})$ is leading order approximation to $\sigma(e^+e^- \to \text{hadrons})$

➔ hadronic R ratio in LO perturbation theory

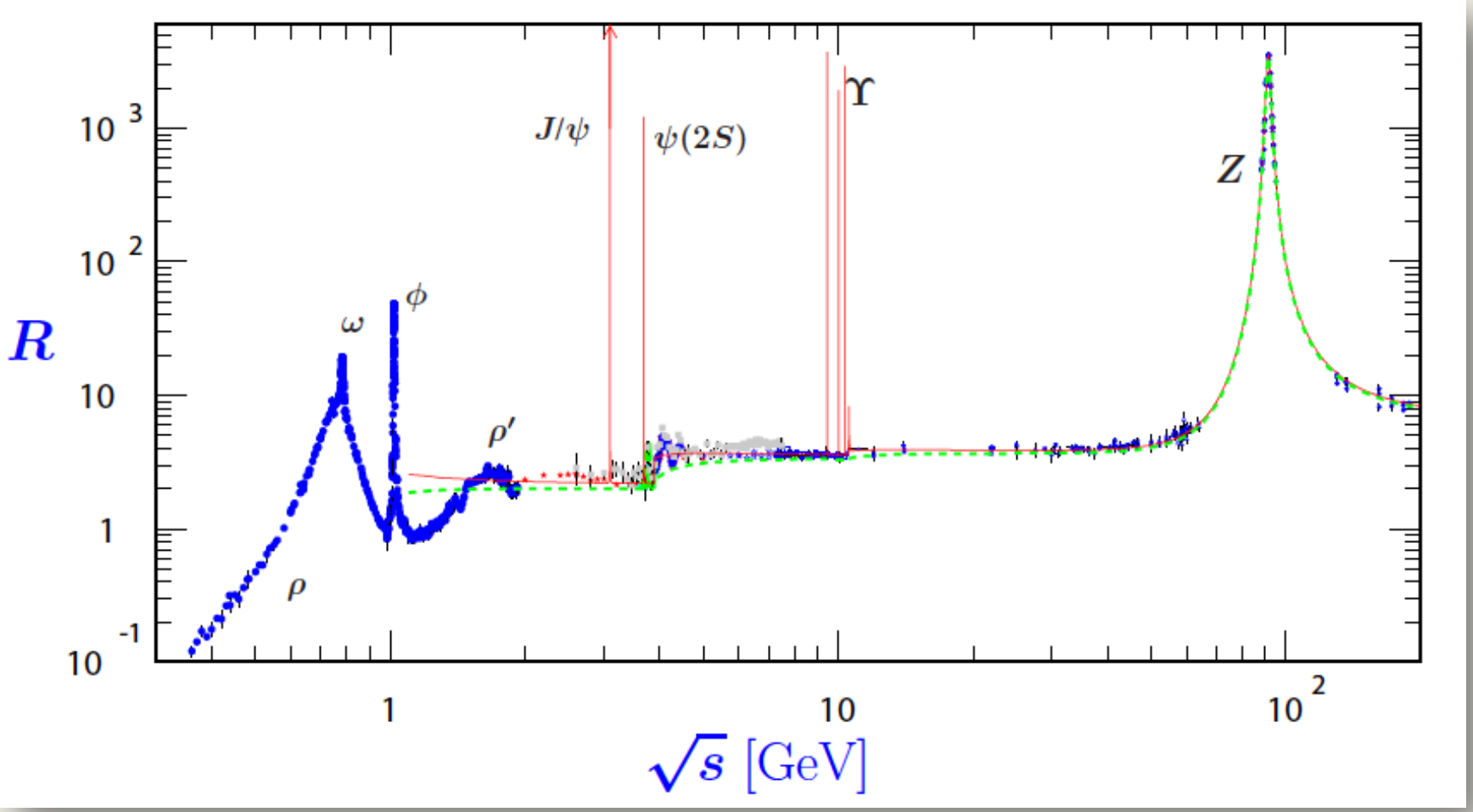
$$R = \frac{\sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \left(\sum_{q=1}^{N_f} e_q^2\right) N_c =$$

$$= 2(N_c/3)\,, \qquad N_f = 3$$

$$= \frac{10}{3}(N_c/3)\,, \qquad N_f = 4$$

$$= \frac{11}{3}(N_c/3)\,, \qquad N_f = 5$$

Number of colours = 3



Colour factors

Eigenvalues of the quadratic Casimir operator (like eigenvalues of J^2)

- ☞ in the fundamental representation

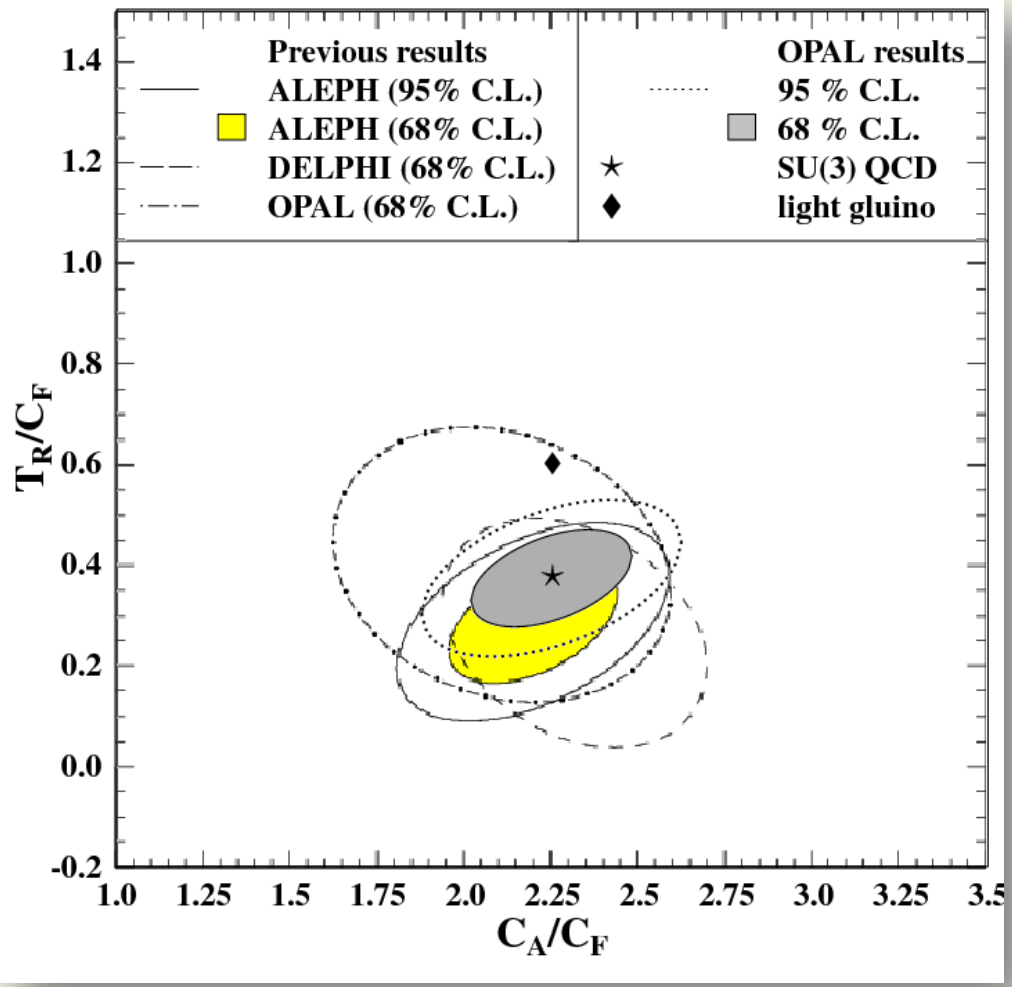
$$\left(t^a t^a\right)_{bc} = C_F \delta_{bc} \qquad C_F = T_R \frac{N_c^2 - 1}{N_c} = \frac{4}{3}$$ in QCD

- ☞ in the adjoint representation:

$$\left(F^a F^a\right)_{bc} = C_A \delta_{bc} \qquad C_A = 2 T_R N_c = 3$$ in QCD

➔measuring C_F and C_A determines N_C

Simultaneous measurement of the strong coupling and colour factors



Source of non-Abelian nature

$$\mathcal{F}^a_{\mu\nu}(A) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \mathrm{i} g_\mathrm{s} F^a_{bc} A^b_\mu A^c_\nu$$

- ☛ Main difference between QCD and QED
- ☛ Makes it a 'perfect theory': among QFT theories in d = 4 only non-Abelian gauge theories are asymptotically free (below)
- ☛ Possible source of colour confinement (colour neutrality of hadrons)?

Feynman rules

can be read off the action

$$S = \mathrm{i} \int \mathrm{d}^4 x (\mathcal{L}_f + \mathcal{L}_g) \equiv S_0 + S_I$$

$$S_0 = \mathrm{i} \int \mathrm{d}^4 x \mathcal{L}_0 \qquad S_I = \mathrm{i} \int \mathrm{d}^4 x \mathcal{L}_I$$

- ☞ $\mathcal{L}_0$ contains bilinear terms of the fields
- ☞ $\mathcal{L}_I$ contains the rest (called interactions)
- ☞ gluon propagator $\Delta_{g,\mu\nu}$ is the inverse of the bilinear term in A_μ – in momentum space:

$$\Delta_{g,\mu\nu}(p)\, \mathrm{i}\left[p^2 g^{\nu\rho} - p^\nu p^\rho\right] = \delta^\rho_\mu$$

Feynman rules

but $\mathrm{i}\left[p^2 g^{\nu\rho} - p^\nu p^\rho\right] p_\rho = 0$, hence $\left[p^2 g^{\nu\rho} - p^\nu p^\rho\right]^{-1}$ does not exist

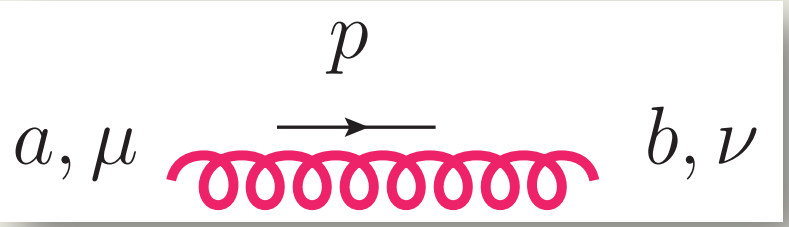
➔Exploit gauge invariance to rewrite $\mathcal{L}$ in a physically equivalent form − gauge fixing = imposing a constraint on A_μ by adding a term with a Lagrange multiplicator (colour implicit)

covariant: $\partial_\mu A^\mu(x) = 0$ ➔$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\lambda}\left(\partial_\mu A^\mu\right)^2$

axial $n^\mu \neq p^\mu$: $n_\mu A^\mu(x) = 0$ ➔$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\lambda}\left(n_\mu A^\mu\right)^2$

Feynman rules: propagators

- **gluon**



  - in covariant gauges:

$$-\frac{\mathrm{i}\delta^{\mathrm{ab}}}{p^2}\left[g_{\mu\nu}-(1-\lambda)\frac{p_\mu p_\nu}{p^2}\right]$$

with usual choice $\lambda = 1$ (Feynman gauge)

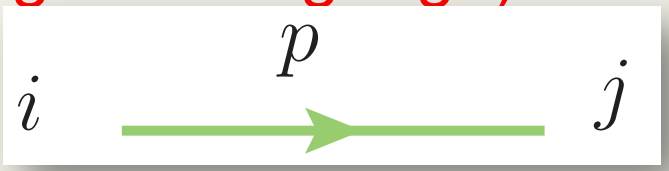
  - in axial gauges: $d_{\mu\nu}(p,n) = -g_{\mu\nu} + \frac{p_\mu n_\nu + n_\mu p_\nu}{p\cdot n}$

$$\frac{\mathrm{i}\delta^{\mathrm{ab}}}{p^2}\left(d_{\mu\nu}(p,n)-\left(n^2+\lambda\, p^2\right)\frac{p_\mu p_\nu}{(p\cdot n)^2}\right)$$

with usual choice $n^2 = 0$, $\lambda = 0$ (light-cone gauge)

- **quark**

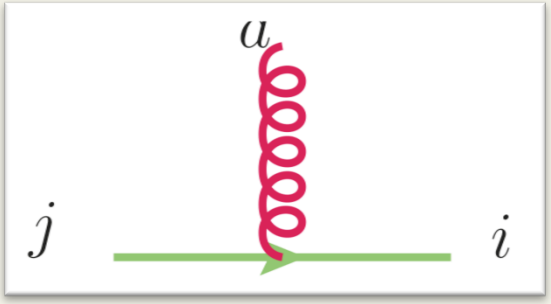
$$\mathrm{i}\,\delta_{ij}\,\frac{\not{p}+m}{p^2-m^2}$$



Feynman rules: vertices

☛ **quark-gluon**

$$-\mathrm{i}g_\mathrm{s}\, t^a_{ij}\, \gamma^\mu$$



So far same as QED if $g_s \to e$ and $(t^a)_{ij} \to \delta_{ij}$.

With rules for external states can compute QED cross sections:

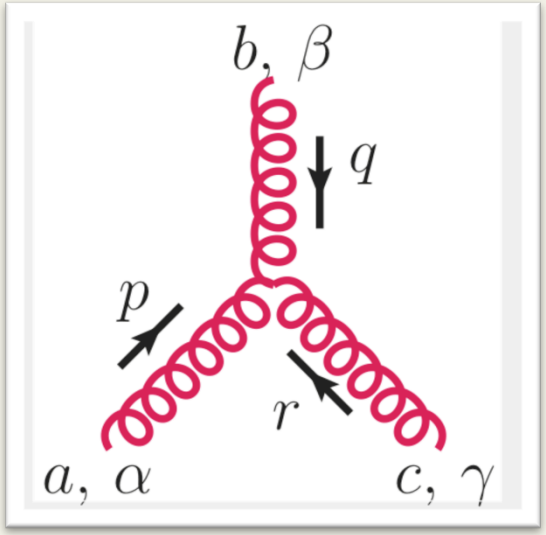
- ☞ outgoing (anti-)fermion: $\big(v(p)\big)\bar{u}(p)$
- ☞ incoming (anti-)fermion: $\big(\bar{v}(p)\big)u(p)$
- ☞ outgoing/incoming vector boson: $\epsilon^{(\lambda)}_\mu(p)^* \Big/ \epsilon^{(\lambda)}_\mu(p)$

$$\sum_{\lambda=1,2} \epsilon^{(\lambda)}_\mu(p)\epsilon^{(\lambda)}_\nu(p)^* = d_{\mu\nu}(p,n)$$

Feynman rules: vertices

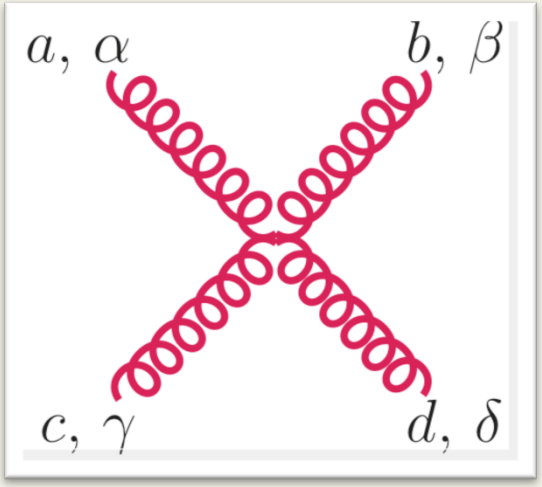
- **gluon** self couplings
  - triple:

$$-\mathrm{i}g_\mathrm{s}\, F^a_{bc}\,[+g^{\alpha\beta}(p-q)^\gamma + g^{\beta\gamma}(q-r)^\alpha + g^{\gamma\alpha}(r-p)^\beta]$$



  - quartic:

$$-\mathrm{i}g_\mathrm{s}^2\,[+f^{xac}\,f^{xbd}\,(g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma}) + f^{xad}\,f^{xbc}\,(g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}) + f^{xad}\,f^{xbc}\,(g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma})]$$



Feynman rules: vertices

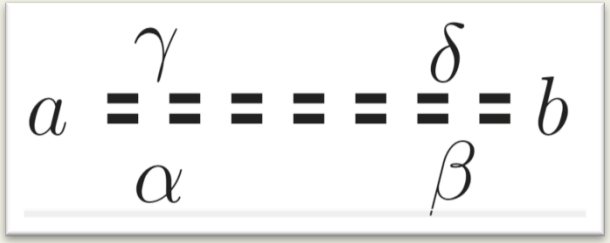
☛ four-gluon vertex is not in a factorized form of a colour and a tensor factor

➔ introduce a fake field, without dynamics

☞ propagator:

$$\frac{\mathrm{i}}{2}\delta^{ab}(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\beta\gamma})$$

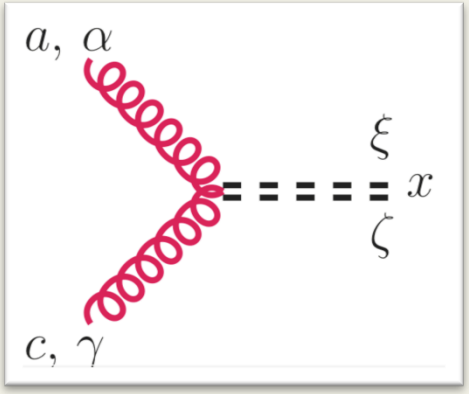
that couples only to gluon



☞ with vertex:

$$\mathrm{i}\sqrt{2}g_{\mathrm{s}}\, f^{xac} g^{\alpha\xi} g^{\gamma\zeta}$$

helps automation



Colour algebra

☛ Factorization of colour in Feynman rules allows for separate computation of colour sums, implemented in programs, e.g. type

```
In[1]:= Import["http://www.feyncalc.org/install.m"]
```

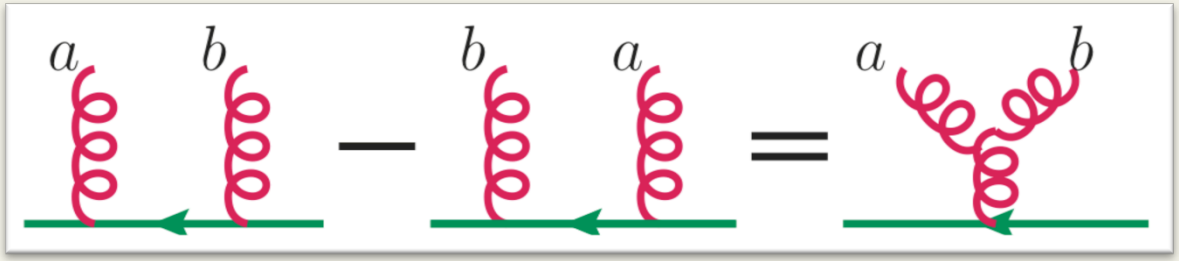
in your Mathematica

☛ but it is also easy&fun graphically

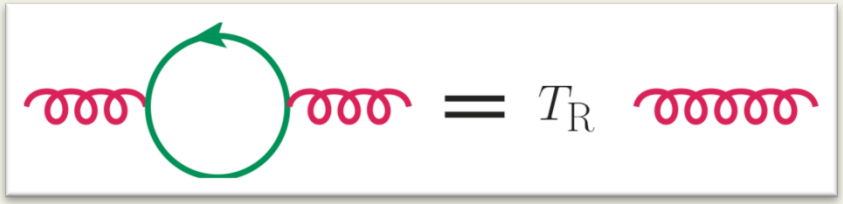
(the following graphs are meant in colour space only − try to write the corresponding algebraic expressions)

Colour algebra

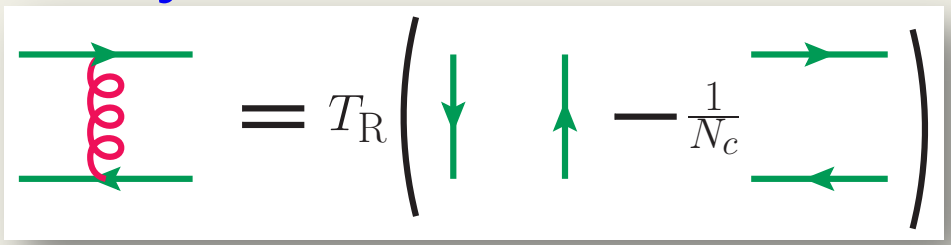
- commutation relation:



- normalization:



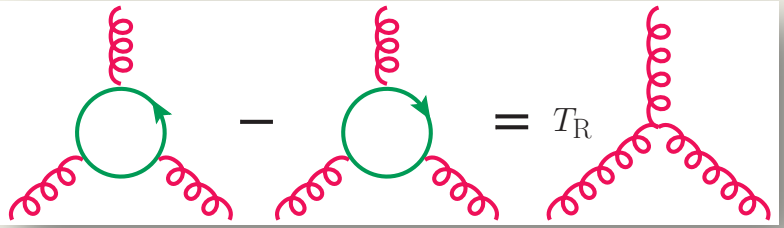
- Fierz-identity:



Colour algebra

- use the Fierz-id. to evaluate C_F from

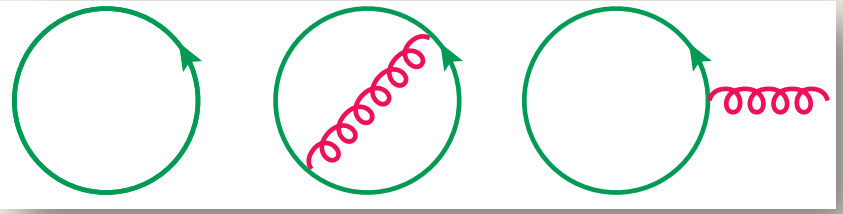
- multiply the commutation relation with the quark gluon vertex (in colour) and prove

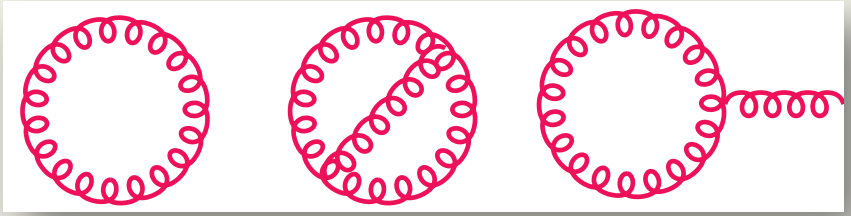


- use the above resolution to compute C_A from

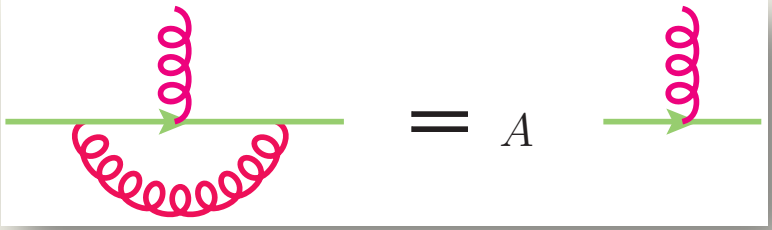
Colour algebra

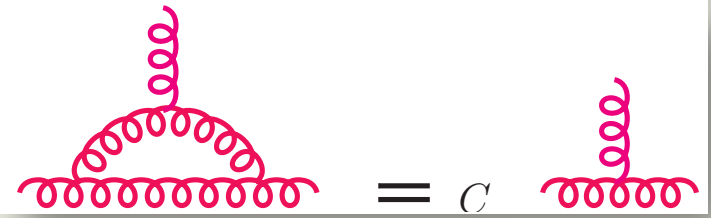
What is the value of the following numbers?





Compute the factors *A*, *B*, *C*





Are we done?

- ☛ Seemingly yes: we can compute the cross section of any process to any accuracy in PT, like in QED

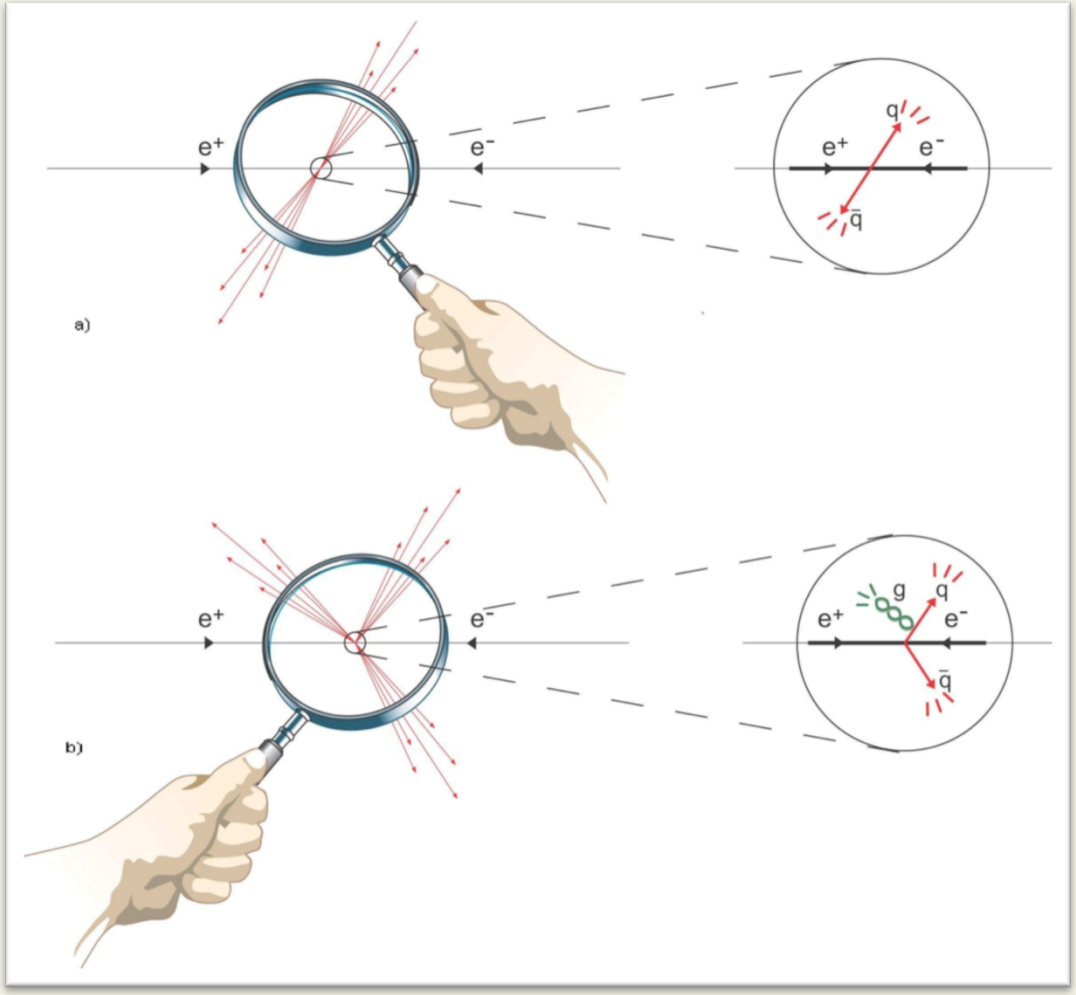
…but rather not: you will find big surprises due to gluon self coupling!

- ☛ in QCD complexity becomes prohibitive at higher orders − OK, compute at one-loop
- ☛ Oops! − the coupling decreases with increasing scattering energies − LO enough?

Parton-hadron duality

- Electrons and photons exist as free particles
- Quarks and gluons have never been observed in detectors, only hadrons
- Assumption: a low-order perturbative computation in QCD is an approximation to sufficiently inclusive hadronic xsection if
  - total cm energy Q of partons is much larger than the mass of quarks and hadrons, $Q \gg m$
  - Q is far from hadronic resonances and thresholds

Parton-hadron duality



Exact symmetries of the classical Lagrangian

- Predictions should reflect the symmetries of $\mathcal{L}_{\mathrm{C}}$, hence it is useful to study those first
- in space-time: against transformations of the conformal group
  - translations
  - Lorentz transformations (rotation and boost)
  - scale if quarks are massless

$$x^\mu \to \lambda x^\mu \quad A_\mu(x) \to \lambda^{-1} A_\mu(\lambda x) \quad q(x) \to \lambda^{-3/2} q(\lambda x)$$

  - conformal transformations

Exact symmetries of the classical Lagrangian

- in colour space: local gauge invariance

$$U : q_i(x) \to q'_i(x) = U(x)_{ij} q_j(x)\,,\; U(x) = \exp\left(\mathrm{i} \sum_{a=1}^{\mathcal{N}_c^2-1} \alpha^a(x) T^a \right) \in SU(N_c)$$

- the covariant derivative transforms as the field itself (prove if you have not yet done):

$$U : \left(D_\mu q(x)\right)_i \to \left(D'_\mu q'(x)\right)_i = U(x)_{ij} \left(D_\mu q(x)\right)_j$$

- quark mass term $SU_c(3)$ gauge invariant

Exact symmetries of the classical Lagrangian

- discrete: C, P and T in agreement with observed properties of strong interactions (C, P and T violating strong decays are not observed)
- there exists additional gauge invariant dimension-four operator, the Θ-term:

$$\mathcal{L}_\Theta = \frac{\Theta g_s}{32\pi^2} F_{\mu\nu} \cdot \tilde{F}^{\mu\nu}, \qquad \tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

violates P and T, Θ is small ($<10^{-9}$ experimentally), set Θ=0 in perturbative QCD

Symmetries of the classical Lagrangian

- almost supersymmetric
  - Massless QCD for $N_f = 1$

$$\mathcal{L}_C^{QCD} = -\frac{1}{4}\mathcal{F}^a_{\mu\nu}\mathcal{F}^{a\mu\nu} + \bar{q}\,\mathrm{i}\gamma_\mu D^\mu\, q$$

  - SUSY Yang-Mills:

$$\mathcal{L}_C^{SYM} = -\frac{1}{4}\mathcal{F}^a_{\mu\nu}\mathcal{F}^{a\mu\nu} + \bar{\lambda}\,\mathrm{i}\gamma_\mu D^\mu\, \lambda$$

quarks transform under fundamental, gluinos under adjoint representation of SU(N)

Approximate symmetries of the classical Lagrangian

Related to the quark mass matrix

- ☛ introduce

$$\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \qquad P_\pm = \frac{1}{2}(1 \pm \gamma_5)$$

- ☛ $P_\pm$ are pojections

$$P_+ P_- = P_- P_+ = 0\,, \quad P_\pm^2 = P_\pm\,, \quad P_+ + P_- = 1$$

- ☛ from Dirac algebra: $\gamma_\mu P_\pm = P_\mp \gamma_\mu$
- ☛ let $\psi_\pm = P_\pm \psi\,, \quad \overline{\psi} = \psi^\dagger \gamma_0 \;\Rightarrow\; \overline{\psi_\pm} = \overline{\psi} P_\mp$

$$\gamma_5^2 = 1 \;\Rightarrow\; \gamma_5 \psi_\pm = \pm \psi_\pm$$

Approximate symmetries of the classical Lagrangian

- the quark sector of the Lagrangian can be rewritten:

$$\mathcal{L}_{Chir} = \bar{\psi}\, \mathrm{i}\gamma_\mu D^\mu \psi = \bar{\psi}\left(P_+ + P_-\right)\mathrm{i}\gamma_\mu D^\mu \left(P_+ + P_-\right)\psi$$
$$= \bar{\psi}\, P_+\, \mathrm{i}\gamma_\mu D^\mu\, P_-\, \psi + \bar{\psi}\, P_-\, \mathrm{i}\gamma_\mu D^\mu\, P_+\, \psi$$
$$= \bar{\psi}_-\, \mathrm{i}\gamma_\mu D^\mu \psi_- + \bar{\psi}_+\, \mathrm{i}\gamma_\mu D^\mu \psi_+ \equiv \mathcal{L}_- + \mathcal{L}_+ \equiv \mathcal{L}_L + \mathcal{L}_R$$

- would not work if gluons were not vectors (in D^μ)
- the left- and right-handed fields are not coupled

➔ $\mathcal{L}_{Chir}$ is invariant under $U_L(N_f)\times U_R(N_f)$

This symmetry acts separately on left- and right-handed fields: chiral symmetry

Approximate symmetries of the classical Lagrangian

☛ the group elements can be parametrised in terms of 2 N_f^2 real numbers:

$$\left(g_L, g_R\right) = \left(\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\beta}\mathrm{e}^{\mathrm{i}\alpha_a T^a}\mathrm{e}^{\mathrm{i}\beta_a T^a}, \mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{-\mathrm{i}\beta}\mathrm{e}^{\mathrm{i}\alpha_a T^a}\mathrm{e}^{-\mathrm{i}\beta_a T^a}\right)$$
$$\in U_V(1) \otimes U_A(1) \otimes SU_L(N_f) \otimes SU_R(N_f)$$

☞ has vector subgroups $SU_V(N_f) \times U_V(1)$

$$\left(g, g\right) = \left(\mathrm{e}^{\mathrm{i}\alpha_a T^a}, \mathrm{e}^{\mathrm{i}\alpha_a T^a}\right) = \mathrm{e}^{\mathrm{i}\alpha_a T^a \mathrm{I}} \in \mathrm{SU_V}\left(N_f\right)$$

☞ axial transformations do not form a subgroup

$$\left(h, h^+\right) = \left(\mathrm{e}^{\mathrm{i}\beta_a T^a}, \mathrm{e}^{-\mathrm{i}\beta_a T^a}\right) = \mathrm{e}^{\mathrm{i}\beta_a T^a \gamma_5} \qquad \left[T^a\gamma_5, T^b\gamma_5\right] = \mathrm{i}f^{abc}T^c\mathrm{I}$$

Chiral perturbation theory

- chiral symmetry is not observed in the hadron spectrum
- in QCD it is believed that the vacuum has a non-zero VEV of the light-quark operator

$$\langle 0 | \bar{q} q | 0 \rangle = \langle 0 | \bar{u} u + \bar{d} d | 0 \rangle \approx (250\,\mathrm{MeV})^3$$

this quark condensate breaks chiral symmetry spontaneously to $SU_V(N_f) \times U_V(1)$ ➔ isospin & conserved barion number, „massless" mesons

because it connects left- and right-handed fields

$$\langle 0 | \bar{q} q | 0 \rangle = \langle 0 | \bar{q}_L q_R + \bar{q}_R q_L | 0 \rangle$$

Chiral perturbation theory

- ☛ SSB of chiral symmetry implies the existence of N_f^2-1massless Goldstone bosons,
- ☛ but light quarks are not exactly massless ➔ chiral symmetry is not exact, the Goldstone bosons are not massless ➔ pseudoscalar meson octet!
- ☛ m_f are treated as perturbation ➔ χPT ➔ masses of light quarks, scattering properties of pions!

QCD topics (at T=0)

Low-energy properties (<GeV)

χPT (➔light quark masses)

Our focus

High energy collisions (>GeV)

Jet physics

Perturbative

Sum rules, lattice QCD

Non-perturbative

Approximate symmetries of the classical Lagrangian

☛ choose Weyl representation:

$$\gamma_0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\psi_+ = \begin{pmatrix} \tilde{\psi}_+ \\ 0 \end{pmatrix}, \quad \psi_- = \begin{pmatrix} 0 \\ \tilde{\psi}_- \end{pmatrix} \qquad \tilde{\psi}_\pm(x) = \mathrm{e}^{-\mathrm{i}\, p\cdot x}\, \tilde{\psi}_\pm(p)$$

two-component Weyl spinors, satisfying

$$\mathrm{i}\gamma\cdot\partial\, \tilde{\psi}_\pm(x) = 0 \Rightarrow \gamma\cdot p\, \tilde{\psi}_\pm(p) = 0 \qquad \text{if } m=0$$

$$-E\tilde{\psi}_\pm(p) \pm \sigma_i p_i \tilde{\psi}_\pm(p) = 0$$

helicity eigenstates: $\sigma\cdot\hat{\mathrm{p}}\tilde{\psi}_\pm(p) = \pm\tilde{\psi}_\pm(p)$

Symmetries of the classical Lagrangian

- ☛ Some are useful for
  - ☞ easing computations
  - ☞ checking computations
  - ☞ hinting on solving QCD
- ☛ Some are violated by quantum corrections with important physical consequences
  - ☞ scaling violations
  - ☞ axial anomaly (not discussed here)

What is scaling?

☛ Consider a dimensionless physical observable $R = R(Q^2)$, with Q being a large energy scale,

$Q \gg$ any other dimensionful parameter (e.g. m_f)

➔ set $m_f = 0$ (check later if $R\,(m_f = 0)$ is OK)

Classically $\dim R = 0$ & $\dim Q = 1$, so

$$\frac{\mathrm{d}R}{\mathrm{d}Q} = 0 \quad \Rightarrow \quad \lim_{Q^2 \to \infty} R = R_0$$ constant

What is scaling violation?

- ☛ We'll see in a renormalized QFT we need an additional scale: μ renormalization scale, thus

$R = R(Q^2/\mu^2)$ is not a constant: scaling violation

- ☛ the „small" parameter in the perturbative expansion of R, $\alpha_s(\mu)$ also depends on the scale choice
- ☛ but μ is an arbitrary, non-physical parameter ($\mathcal{L}_C$ does not depend on it) ➔ physical quantites cannot depend on μ

Renormalization group equation

$$0 = \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} R\left(\frac{Q^2}{\mu^2}, \alpha_\mathrm{s}(\mu^2)\right) = \left(\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_\mathrm{s}}{\partial \mu^2} \frac{\partial}{\partial \alpha_\mathrm{s}}\right) R$$

☛ let $t = \ln(Q^2/\mu^2)$, $\beta(\alpha_s) = \mu^2 \left.\frac{\partial \alpha_\mathrm{s}}{\partial \mu^2}\right|_{\alpha_\mathrm{s}^{(0)}\,\mathrm{fixed}}$

$$\left(-\frac{\partial}{\partial t} + \beta(\alpha_\mathrm{s}) \frac{\partial}{\partial \alpha_\mathrm{s}}\right) R(e^t, \alpha_\mathrm{s}) = 0$$

How can we solve this?

Summary

- ☛ Established Feynman rules of pQCD
  - ☞ similar to QED with some complications
  - ☞ those of colour algebra can be factorized
  - ☞ those of gluon self coupling are tremendous (you have not yet seen it)
- ☛ Understood the origin of the RGE

Let us try to solve it