Quantum Field Theory and the Electroweak Standard Model

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Outline

1. Introduction
2. Elements of Quantum Field Theory
3. Construction of the Electroweak Standard Model Lagrangian
4. Phenomenology of the Electroweak Standard Model
5. Concluding remarks
Various experimental studies of weak decay and scattering processes lead to a number of observations:

1. Electron neutrino and antineutrino are different. Muon neutrino and antineutrino are different. Electron and muon are different.

   \[
   \nu_e n \rightarrow e^- p, \quad \bar{\nu}_e p \rightarrow e^+ n, \quad \nu_\mu n \rightarrow \mu^- p, \quad \bar{\nu}_\mu p \rightarrow \mu^+ n \\
   \bar{\nu}_e n \rightarrow e^- p, \quad \nu_e p \rightarrow e^+ n, \quad \bar{\nu}_\mu n \rightarrow \mu^- p, \quad \nu_\mu p \rightarrow \mu^+ n
   \]

2. The decay \( \mu \rightarrow e X \) has not been observed

3. Only left-handed leptons and right-handed antileptons participate in process with \( |\Delta Q| = 1 \) for leptons of the same flavor

   \[
   \ell \rightarrow p, \quad \bar{\ell} \rightarrow p \\
   \bar{\ell} \rightarrow \bar{p}, \quad \ell \rightarrow \bar{p}
   \]

4. Three generations have been observed
The observation lead to the assumption that the lepton interactions with $|\Delta Q| = 1$ occur via charged current in so-called $(V - A)$ form

$$J_\ell \sim \bar{\ell} \gamma_\mu (1 - \gamma_5) \nu_\ell$$

Corresponding 4-fermion interaction Lagrangian for muon and electron currents

$$L = \frac{G_F}{\sqrt{2}} \bar{\mu} \gamma_\sigma (1 - \gamma_5) \nu_\mu \bar{e} \gamma_\sigma (1 - \gamma_5) \nu_e + h.c.$$  

$G_F$ is well-known Fermi constant with dimension $[m]^{-2}$

For the muon decay

$$\Gamma \sim G_F^2 \cdot m_\mu^5$$

Result of direct computation:

$$\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3} \cdot f \left( \frac{m_e^2}{m_\mu^2} \right)$$

$$f = 1 + 0 \left( \frac{m_e^2}{m_\mu^2} \right)$$

$$G_F = (1.166371 \pm 6 \cdot 10^{-6}) \cdot 10^{-5} \text{ GeV}^{-2}$$

As we know any fermion field $\Psi(x)$ may be presented as

$$\Psi = \frac{1 - \gamma_5}{2} \Psi + \frac{1 + \gamma_5}{2} \Psi = \Psi_L + \Psi_R$$

Therefore the current involves the left component of the fermion field only

$$J_\ell = \bar{\Psi}_L \gamma_\mu \Psi_L$$

E. Boos Quantum Field Theory and the Electroweak SM
SM – the quantum field theory based on principles and requirements:

- gauge invariance with lowest dimension (dimension 4) operators;
  *SM gauge group: $SU(3)_c \times SU(2)_L \times U(1)_Y$*

- correct electromagnetic neutral currents and (V-A) charged currents (Fermi);

- 3 generations without chiral anomalies

- Higgs mechanism of spontaneous symmetry breaking
The electroweak Standard Model gauge group

\[ SU_L(2) \otimes U_Y(1) \]

\( SU_L(2) \) is called the weak isospin group and \( U_Y(1) \) is called the weak hypercharge group. The hypercharge group is needed to have electromagnetic \( U_{em}(1) \).

\[ \begin{pmatrix} \nu_e \\ e^R \end{pmatrix}_L \begin{pmatrix} \nu_\mu \\ \mu^R \end{pmatrix}_L \begin{pmatrix} \nu_\tau \\ \tau^R \end{pmatrix}_L \begin{pmatrix} u \\ d^R \end{pmatrix}_L \begin{pmatrix} c \\ s^R \end{pmatrix}_L \begin{pmatrix} t \\ t^R \end{pmatrix}_L. \]

The electroweak part with the \( SU_c(3) \) strong interactions are combine into the gauge-fermion part of the Standard Model Lagrangian
\[ L = \left(-\frac{1}{4}W_{\mu\nu}^i (W^{\mu\nu})^i - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}G_{\mu\nu}^a (G^{\mu\nu})^a + \sum_{f=\ell,q} \bar{\Psi}_L^f (iD^L_\mu \gamma^\mu) \Psi_L^+ + \sum_{f=\ell,q} \bar{\Psi}_R^f (iD^R_\mu \gamma^\mu) \Psi_R^+ \right) \]

\[ W_{\mu\nu}^i = \partial_\mu W_\nu^i - \partial_\nu W_\mu^i + g_2 \varepsilon^{ijk} W_\mu^j W_\nu^k \]

\[ B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \]

\[ G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_S f^{abc} A_\mu^b A_\nu^c \]

\[ D^L_\mu = \partial_\mu - ig_2 W_\mu^i \tau^i - ig_1 B_\mu \left( \frac{Y^f_L}{2} \right) - ig_S A_\mu^a t^a \]

\[ D^R_\mu = \partial_\mu - ig_1 B_\mu \left( \frac{Y^f_R}{2} \right) - ig_S A_\mu^a t^a \]

\[ i = 1, 2, 3; a = 1, \ldots, 8, \]

For the leptons $g_S$ should be put to zero

The gauge fields are taken in the adjoint representations and leptons and quark fields are in the fundamental representation of $SU_L(2)$ and $SU_C(3)$ groups.
We do not assume from the beginning the Gelmann-Nishijima relations for weak hypercharges

\[
Q = (T_3)_L + \frac{Y_L}{2} \\
Q = (T_3)_R + \frac{Y_R}{2}
\]

Let us take weak hypercharges as free parameters for a moment, and try to fix them from two physics requirements:

1. Correct electromagnetic interactions
2. (V-A) weak charge currents

Let us consider for simplicity the only fermions from the first generation
Lagrangian for interactions of the leptons of the first generation  
(from covariant derivatives)

\[ L^\ell = -i^2 (\bar{\nu}_e L \bar{e}_L) \gamma_\mu \begin{pmatrix} \frac{1}{2} g_2 W_\mu^3 + g_1 \frac{Y_\mu^e}{2} B_\mu \\ g_2 \frac{W_\mu^-}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{W^+_\mu}{\sqrt{2}} \\ -\frac{1}{2} g_2 W_\mu^3 + g_1 \frac{Y_\mu^e}{2} B_\mu \end{pmatrix} \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} + \bar{e}_R \gamma_\mu g_1 \frac{Y_\mu^R}{2} B_\mu e_R \]

\[ \tau^i W^i = \frac{\sigma^i}{2} W^i = \frac{1}{2} \begin{pmatrix} W_\mu^3 & \sqrt{2} W_\mu^+ \\ \sqrt{2} W_\mu^- & -W_\mu^3 \end{pmatrix} \]

\[ W^\pm_\mu = (W_\mu^1 \pm iW_\mu^2) / \sqrt{2} \]

Products of non-diagonal elements give the form of the charge current with needed \((V - A)\) structure

\[ L^\ell_{CC} = \frac{g_2}{\sqrt{2}} \bar{\nu}_e \gamma_\mu W^+_\mu e_L + h.c. = \frac{g_2}{2\sqrt{2}} \bar{\nu}_e \gamma_\mu (1 - \gamma_5) W^+_\mu e + h.c. \]

Products of diagonal elements lead to neutral currents

\[ L^\ell_{NC} = \bar{\nu}_e \gamma_\mu \left( \frac{1}{2} g_2 W_\mu^3 + g_1 \frac{Y_\mu^e}{2} B_\mu \right) \nu_{eL} + \]
\[ + \bar{e}_L \gamma_\mu \left( -\frac{1}{2} g_2 W_\mu^3 + g_1 \frac{Y_\mu^e}{2} B_\mu \right) e_L + \]
\[ + \bar{e}_R \gamma_\mu g_1 \frac{Y_\mu^R}{2} B_\mu e_R. \]
The neutral component of the $W$ field and $B$ field can mix

$$W^3_\mu = Z_\mu \cos \theta_W + A_\mu \sin \theta_W$$

$$B_\mu = -Z_\mu \sin \theta_W + A_\mu \cos \theta_W$$

One of this fields, say $A$, we try to identify with the photon - it should not interact with the neutrino and should have well-known Dirac interaction with the electron field. From these physics requirements we get:

$$e - A \gamma_\mu: \quad \frac{1}{2} \left[ \left( -\frac{g_2}{2} \sin \theta_W + \frac{g_1}{2} Y^\ell_L \cos \theta_W \right) + \frac{g_1}{2} Y^\ell_R \cos \theta_W \right] = Q_e e$$

$$e - A \gamma_\mu \gamma_5: \quad \frac{1}{2} \left[ \left( \frac{g_2}{2} \sin \theta_W - \frac{g_1}{2} Y^\ell_L \cos \theta_W \right) + \frac{g_1}{2} Y^\ell_R \cos \theta_W \right] = 0$$

$$\nu - A: \quad \frac{g_2}{2} \sin \theta_W + \frac{g_1}{2} Y^\ell_L \cos \theta_W = 0, \quad (Q_e = -1)$$
\[-\frac{g_2}{2} \sin \theta_W + \frac{g_1}{2} Y_L^\ell \cos \theta_W = \frac{g_1}{2} Y_R^\ell \cos \theta_W = Q_e e\]

\[\frac{g_2}{2} \sin \theta_W + \frac{g_1}{2} Y_L^\ell \cos \theta_W = 0, \quad (Q_e = -1)\]

\[g_2 \sin \theta_W = e\]

\[g_1 Y_L^\ell \cos \theta_W = -e\]

\[Y_R^\ell = 2 Y_L^\ell\]

The hypercharges of the left and right chiral leptons are proportional but they are not fully fixed.
In the quark sector there are both left and right chirality components for up and down quarks

\[
\begin{pmatrix}
    u \\
    d
\end{pmatrix}_L, \quad u_R, \quad d_R
\]

The EW interaction Lagrangian for quarks as follows from the covariant derivatives

\[
(\bar{u}d)_L \gamma_\mu \left( \begin{array}{cc}
    \frac{1}{2}g_2 W^3_\mu + g_1 \frac{Y^q}{2} B_\mu & g_2 \frac{W^+_\mu}{\sqrt{2}} \\
    g_2 \frac{W^-_\mu}{\sqrt{2}} & -\frac{1}{2}g_2 W^3_\mu + g_1 \frac{Y^q}{2} B_\mu
\end{array} \right) \begin{pmatrix}
    u \\
    d
\end{pmatrix}_L + \bar{u}_R \gamma_\mu g_1 \frac{Y^u}{2} B_\mu u_R + \bar{d}_R \gamma_\mu g_1 \frac{Y^d}{2} B_\mu d_R
\]

Charge currents have needed (V-A) structure

\[
L^q_{CC} = \frac{g_2}{2\sqrt{2}} \bar{u} \gamma_\mu (1 - \gamma_5) W^+_\mu d + \frac{g_2}{2\sqrt{2}} \bar{d} \gamma_\mu (1 - \gamma_5) W^-_\mu u
\]
Requirements for up and down quarks to have Dirac electromagnetic interactions with charges $+2/3e$ and $-1/3e$ lead to:

\[
\frac{1}{2} g_2 \sin \theta_W + \frac{1}{2} g_1 Y_L^q \cos \theta_W = \frac{1}{2} Y_R^u g_1 \cos \theta_W = \frac{2}{3} e \\
- \frac{1}{2} g_2 \sin \theta_W + \frac{1}{2} g_1 Y_L^q \cos \theta_W = \frac{1}{2} Y_R^d g_1 \cos \theta_W = -\frac{1}{3} e
\]

To be compared with what we have from the lepton sector:

\[
\begin{align*}
\begin{cases}
g_2 \sin \theta_W &= e \\
g_1 Y_L^q \cos \theta_W &= \frac{1}{3} e \\
Y_R^u + Y_R^d &= 2Y_L^q \\
Y_R^u &= -2Y_R^d
\end{cases}
\end{align*}
\]

\[
\begin{align*}
Y_R^\ell &= 2Y_L^\ell \\
Y_L^\ell &= -3Y_L^q \\
Y_R^u &= -\frac{4}{3} Y_L^\ell \\
Y_R^d &= -Y_L^\ell
\end{align*}
\]
Simple algebraic computations lead to the interaction Lagrangian of the second W and B combination - Z vector field with leptons and quarks.

The neutral current Lagrangian takes the form:

\[
L_{NC} = e \sum_f Q_f J_{f\mu}^{em} A^\mu + \frac{e}{4 \sin \theta_W \cos \theta_W} \cdot \sum_f J_{f\mu}^Z Z^\mu
\]

\[
J_{f\mu}^{em} = \bar{f} \gamma_\mu f, \quad Q_\nu = 0, \quad Q_e = -1, \quad Q_u = 2/3, \quad Q_d = -1/3, \quad J_{f\mu}^Z = \bar{f} \gamma_\mu [v_f - a_f \gamma_5] f
\]

\[
v_{ui} = 1 - \frac{8}{3} s_W^2, \quad a_{ui} = 1; \quad v_{di} = -1 + \frac{4}{3} s_W^2, \quad a_{di} = -1;
\]

\[
v_\ell = -1 + 4 s_W^2, \quad a_\ell = -1; \quad v_\nu = 1, \quad a_\nu = 1.
\]

\[
v_f = 2T_3^f - 4Q_f s_W^2, \quad a_f = 2T_3^f
\]

Lepton and quark weak hypercharges do not present in the interactions!

One hypercharge, say, \( Y_L^\ell \) remains to be free.
We have constructed the theory:

\[ L_{CC}^e = \frac{g_2}{\sqrt{2}} \bar{\nu}_e \gamma_\mu W_\mu^+ e_L + h.c. = \frac{g_2}{2\sqrt{2}} \bar{\nu}_e \gamma_\mu (1 - \gamma_5) W_\mu^+ e + h.c. \]

\[ L_{CC}^q = \frac{g_2}{2\sqrt{2}} \bar{u} \gamma_\mu (1 - \gamma_5) W_\mu^+ d + \frac{g_2}{2\sqrt{2}} \bar{d} \gamma_\mu (1 - \gamma_5) W_\mu^- u \]

\[ L_{NC} = e \sum_f Q_f J_{f\mu}^{em} A_\mu + \frac{e}{4 \sin \theta_W \cos \theta_W} \cdot \sum_f J_{f\mu}^Z Z^\mu \]

How about chiral anomalies?

Generically, anomalies correspond to a situation in the field theory when some symmetry takes place at the level of a classical Lagrangian but it is violated at quantum level.

\[ L = \bar{\Psi}_L i \gamma_\mu \left( \partial_\mu - igA_\mu^a t^a \right) \Psi_L = \bar{\Psi} i \gamma_\mu \left( \partial_\mu - igA_\mu^a t^a \right) \frac{1 - \gamma_5}{2} \Psi \]

is gauge invariant for massless fermions at classical level

This invariance according to Noether theorem leads to conserving current

\[ j_\rho^a = \bar{\Psi} \gamma_\mu \frac{1 - \gamma_5}{2} t^a \Psi \]
However, after quantization one finds the current is not conserved due to triangle loop contributions

\[ p = k + q \]

The sum of diagrams after convoluting with momentum \( p \) is proportional to

\[
\sim \frac{g^2}{8\pi^2} \epsilon^{\mu\nu\alpha\beta} k^\alpha q^\beta \cdot \text{Tr} \left[ t^a \{ t^b t^c \} \right]
\]

In our case there are simultaneously contributions from left and right chiral fermions which contributes to the anomaly with opposite signs

\[
\text{Anom} \sim \text{Tr} \left[ t^a \{ t^b t^c \} \right]_L - \text{Tr} \left[ t^a \{ t^b t^c \} \right]_R
\]

If the anomaly does not vanish the theory loses its gauge invariance, and therefore can not be acceptable.
In QED and QCD left and right fermions obviously couple equally to the gauge bosons. Therefore the left and right chirality contributions compensate exactly each other, the anomaly is equal to zero and the theories perfectly make sense.

In SM left and right states couple with different hypercharges to $U_Y(1)$ and the only left components couple to the $SU_L(2)$ gauge bosons. So, it is not obvious at all apriori that chiral anomalies vanish. In fact, zero anomalies is a requirement for the SM to be a reasonable quantum field theory.

Generators of the $SU_L(2)$ are $t^i = \sigma^i/2$

The only potentially dangerous animalies

$\left(SU_L(2)\right)^2 \cdot U_Y(1)$ and $U_Y(1)^3$
1. \[(SU_L(2))^2 \cdot U_Y(1)\quad \{t^i, t^j\} = \frac{1}{2} \delta^{ij}\]

\[
\text{Anom} \sim \text{Tr} \left[ Y \{t^i, t^j\} \right]_L = \frac{1}{2} \delta^{ij} \text{Tr} Y_L = \frac{1}{2} \delta^{ij} \left[ N_C \cdot 2Y^q_L + 2Y^\ell_L \right]
\]
\[
Y^\ell_L = -3Y^q_L
\]

\[
\text{Anom} \sim \frac{1}{2} \delta^{ij} 2Y^q_L (N_C - 3) \quad \Rightarrow \quad 3 \text{ colors!}
\]

2. \[U_Y(1)^3\]

\[
\text{Anom} \sim \text{Tr} (Y^3_L) - \text{Tr} (Y^3_R) =
\]
\[
= N_C (Y^q_L)^3 \cdot 2 + (Y^\ell_L)^3 \cdot 2 - N_C \left[ (Y^u_R)^3 + (Y^d_R)^3 \right] - (Y^\ell_R)^3
\]

Taking into account:
\[
Y^u_R + Y^d_R = 2Y^q_L \quad Y^\ell_R = 2Y^\ell_L \quad Y^\ell_L = -3Y^q_L
\]

\[
\text{Anom} \sim Y^\ell_L \left[ 2N_C \left( \frac{1}{3}Y^\ell_L + Y^u_R \right)^2 - 6(Y^\ell_L)^2 \right] =
\]
\[
= Y^\ell_L \cdot 6 \left( \frac{1}{3}Y^\ell_L + Y^u_R - Y^\ell_L \right) \left( \frac{1}{3}Y^\ell_L + Y^u_R + Y^\ell_L \right) \quad \Rightarrow \quad Y^u_R = \frac{2}{3}Y^\ell_L \quad \text{or} \quad Y^u_R = -\frac{4}{3}Y^\ell_L
\]
1. Correct (V-A) charge currents
2. Correct electromagnetic interactions
3. No chiral anomalies
4. Predictions of additional neutral currents observed experimentally

But such a theory can not describe the nature:

1. There are no massless EW bosons except the photon
   and no massless fermions except may be the neutrino.
   In experiments massive W-, Z-bosons, leptons and quarks were observed
2. Mass terms for both bosons and fermions violate the basic principle of
   the gauge invariance

\[ M^2_W W^+ W^- \quad \frac{1}{2} M^2_Z Z \cdot Z^\mu \]
\[ V_\mu \rightarrow V_\mu + \partial_\mu \alpha \]
\[ m \bar{\Psi} \Psi = m (\bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L) \]

\text{doublet singlet}
How to make massive particles without violation of basic principle of gauge invariance?

Spontaneous symmetry breaking, Nambu-Goldstone theorem, and Brout-Engelrt-Higgs-Hagen-Guralnik-Kibble mechanism

The situation when the Lagrangian is invariant under some symmetry while the spectrum of the system is not invariant is very common for spontaneous symmetry breaking (for example, Ginzburg-Landau theory)

Simple illustrative example:

$$L = \partial_{\mu} \varphi^{\dagger} \partial^{\mu} \varphi - \mu^2 \varphi^{\dagger} \varphi - \lambda (\varphi^{\dagger} \varphi)^2$$

The Lagrangian is invariant under the phase shift $$\varphi \rightarrow \varphi e^{i\omega}$$, $$w=\text{const}$$

The case $$\mu^2 > 0$$ is trivial and not interesting.

In the case $$\mu^2 = -|\mu^2| < 0$$ the potential has nontrivial minimum

$$V(\varphi) = \mu^2 \varphi^{\dagger} \varphi + \lambda (\varphi^{\dagger} \varphi)^2$$

$$\frac{dV}{d\varphi^{\dagger}}|_{\varphi=0} = -|\mu^2| \varphi + 2\lambda (\varphi^{\dagger} \varphi) \varphi = 0 \Rightarrow |\varphi| = \sqrt{\frac{\mu^2}{2\lambda}} = \frac{v}{\sqrt{2}} > 0$$

$$\varphi_0 = +v/\sqrt{2}$$

A concrete vacuum solution violates the phase shift symmetry
Complex scalar field can be parameterized by two real fields

\[ \varphi = \frac{1}{\sqrt{2}}(v + h(x)) e^{-i\xi(x)/v} \]

\[ L = \partial_\mu \varphi^\dagger \partial^\mu \varphi - \mu^2 \varphi^\dagger \varphi - \lambda (\varphi^\dagger \varphi)^2 \]

\[ L = \quad \frac{1}{2} \partial_\mu h \partial^\mu h - \lambda v^2 h^2 - \lambda v h^3 - \lambda h^4 / 4 + \]
\[ + \frac{1}{2} \partial_\mu \xi \partial^\mu \xi + \frac{2}{v} \partial_\mu \xi \partial^\mu \xi h + \frac{1}{v^2} \partial_\mu \xi \partial^\mu \xi h^2 + \lambda v^4 / 4 \]

The Lagrangian describes the system of massive scalar field \( h \) with mass \( m_h^2 = 2\lambda v^2 \) interacting with massless scalar field \( \xi(x) \). The field \( \xi(x) \) is the Nambu-Goldstone boson field.

This is a particular case of the generic Goldstone theorem.
If the theory is invariant under a global group with \( m \) generators but the vacuum is invariant under transformations generated only by \( \ell \) \((\ell < m)\) generators, then \((m - \ell)\) massless Nambu-Goldstone bosons exist in the theory.

Consider the system

\[
\mathcal{L} = \frac{1}{2} |\partial_{\mu} \phi|^2 - V(\phi)
\]

Let the Lagrangian is invariant under \( i=1...m \) transformations:

\[
\phi \rightarrow \phi' = \phi + \delta \phi, \quad \delta \phi_i = i\delta \theta^A t^A_{ij} \phi_j
\]

The invariance of the potential means:

\[
\delta V = (\partial V/\partial \phi_i) \delta \phi_i = i\delta \theta^A (\partial V/\partial \phi_i) t^A_{ij} \phi_j = 0
\]

Let the potential has a minimum (vacuum) at \( \phi_i = \phi_i^0 : (\partial V/\partial \phi_i)(\phi_i = \phi_i^0) = 0 \)

The second derivative at minimum

\[
\frac{\partial^2 V}{\partial \phi_k \partial \phi_i} (\phi_i = \phi_i^0) t^A_{ij} \phi_j^0 = 0 \quad \Rightarrow \quad \ell
\]

\[
\frac{\partial^2 V}{\partial \phi_k \partial \phi_i} (\phi_i = \phi_i^0) = 0 \quad (m - \ell) \text{ massless \ bosons}
\]
Let us add to the SM Lagrangian

\[ L = -\frac{1}{4} W^{i}_{\mu\nu} (W^{\mu\nu})^{i} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} G_{\mu\nu}^{a} (G^{\mu\nu})^{a} + \]
\[ + \sum_{f=\ell,q} \bar{\Psi}_{L}^{f} (i D_{\mu}^{L} \gamma^{\mu}) \Psi_{L}^{f} + \sum_{f=\ell,q} \bar{\Psi}_{R}^{f} (i D_{\mu}^{R} \gamma^{\mu}) \Psi_{R}^{f} \]

one more complex scalar field, SU\(_{L}(2)\) doublet and U\(_{Y}(1)\) singlet

\[ L_{\Phi} = D_{\mu} \Phi^{\dagger} D^{\mu} \Phi - \mu^{2} \Phi^{\dagger} \Phi - \lambda (\Phi^{\dagger} \Phi)^{4} \]

Covariant derivative
\[ D_{\mu} = \partial_{\mu} - ig_{2} W_{\mu}^{3} t^{i} - ig_{1} \frac{Y_{H}}{2} B_{\mu} \]

\( SU_{L}(2) \) invariance
\[ \Phi(x) \rightarrow \Phi'(x) = \exp \left( ig_{2} \alpha^{i} t^{i} \right) \Phi(x) \]

Parameterization by 4 fields
\[ \Phi(x) = \exp \left( -i \frac{\xi^{i}(x) t^{i}}{v} \right) \left( \begin{array}{c} 0 \\ (v + h)/\sqrt{2} \end{array} \right) \]

In the unitary gauge
\[ g_{2} \alpha^{i}(x) = \xi^{i}(x)/v \]

\[ \Phi = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 0 \\ v + h(x) \end{array} \right) \]
The Lagrangian

\[ L_\Phi = D_\mu \Phi^\dagger D^\mu \Phi - \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^4 \]

in terms of the fields:

\[ W_\mu^\pm = (W_\mu^1 \mp iW_\mu^2) / \sqrt{2} \]

\[ W_\mu^3 = Z_\mu \cos \theta_W + A_\mu \sin \theta_W \]

\[ B_\mu = -Z_\mu \sin \theta_W + A_\mu \cos \theta_W \]

\[ \Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \]

The diagonal mass matrix \( \rightarrow \) physics states with definite masses

\[ L = \frac{1}{2} (\partial_\mu h)^2 - \frac{1}{2} (2\lambda v^2) h^2 - \lambda vh^3 - \frac{\lambda}{4} h^4 + \]

\[ + M_W^2 W_\mu^+ W_\mu^- (1 + h/v)^2 + \frac{1}{2} M_Z^2 Z_\mu Z^\mu (1 + h/v)^2 \]

where

\[ M_H^2 = 2\lambda v^2 \quad M_W = \frac{1}{2} g_2 v \]

\[ M_Z = \frac{1}{2} (g_2 \cos \theta_W + g_1 Y^H \sin \theta_W) v \]

Condition of zero mass for the field \( A \):

\[ -\frac{1}{2} g_2 \sin \theta_W + g_1 \frac{Y^H}{2} \cos \theta_W = 0 \]
Conditions from the lepton sector

\[ g_2 \sin \Theta_W = -g_1 Y_L^\ell \cos \Theta_W \]

tell us: the field A has the correct electromagnetic interactions and has zero mass simultaneously if the charged lepton and Higgs field hypercharges have equal modulus and opposite signs

\[ Y_L^\ell = - Y^H \]

The vacuum \( \Phi_{vac} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \) violates both \( SU_L(2) \) and \( U_Y(1) \) symmetries.

However one can find such a combination of generators which leaves the vacuum unchanged:

\[ e^{iT_i \Theta_i} \Phi_{vac} = \Phi_{vac} \quad \iff \quad T_i \Phi_{vac} = 0 \]

\[ T_3 + \frac{1}{2} Y_H = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2} Y_H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{if} \quad Y_H = +1 \]

The vacuum is neutral

\[ T_3 + \frac{1}{2} Y_H = Q_H = 0 \]

\[ SU_L(2) \times U_Y(1) \rightarrow U_{em}(1) \]
This confirms the Gelmann-Nishijima relations

\[ Y^\ell_H = 1 \quad \Downarrow \quad g_2 \sin \theta_W = g_1 \cos \theta_W \]

\[ Y^\ell_L = -1 \quad \Downarrow \quad M_Z = \frac{1}{2} \left( g_2 \cos \theta_W + g_1 Y^H \sin \theta_W \right) v \]

\[ M_W = \frac{1}{2} g_2 v \]

\[ M_W = M_Z \cos \theta_W \]

\[ Q = (T_3)_L + \frac{Y_L}{2} \]

\[ Q = (T_3)_R + \frac{Y_R}{2} \]

\[ Y^\ell_R = 2Y^\ell_L \]

\[ Y^\ell_L = -3Y^q_L \]

\[ Y^u_R + Y^d_R = 2Y^q_L \]

\[ Y^u_R = -\frac{4}{3} Y^\ell_L \]

It should be so from the relation between SU_L(2) and U_Y(1) generators leading to unbroken U_{em}(1) generator
From the Glodstone theorem we expect $4 - 1 = 3$ massless Nambu-Goldstone bosons. But they are not present in the Lagrangian.

However three bosons ($W^\pm$ and $Z$) become massive – 3 Nambu-Goldstone bosons are “eaten” by the longitudinal modes of $W^\pm$ and $Z$ (Brout-Englert-Higgs-Hagen-Guralnik-Kibble mechanism).

\[
L = \frac{1}{2} (\partial_\mu h)^2 - \frac{1}{2} (2\lambda v^2) h^2 - \lambda v h^3 - \frac{\lambda}{4} h^4 + M_W^2 W^+_\mu W^\mu_- (1 + h/v)^2 + \frac{1}{2} M_Z^2 Z_\mu Z^\mu (1 + h/v)^2
\]
The kinetic terms and self interactions of the gauge fields $W^\pm, A, Z$ comes from the SM Lagrangian after corresponding substitution of $W_i$ and $B$ fields via the physics fields

$$ L_{\text{Gauge}} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{4} F_{\mu \nu} B^{\mu \nu} $$

$$ = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{4} Z_{\mu \nu} Z^{\mu \nu} - \frac{1}{2} W^+ W^- + $$

$$ + e \left[ W^+ W^- A^\nu + h.c. + W^+ W^- F^{\mu \nu} \right] + $$

$$ + e \frac{c_W}{s_W} \left[ W^+ W^- Z^\nu + h.c. + W^+ W^- Z^{\mu \nu} \right] - $$

$$ - e^2 \frac{1}{4 s_W^2} \left[ (W^+ W^+ + W^- W^-) W^\mu W^\nu + h.c. \right] - $$

$$ - e^2 \frac{1}{4 s_W^2} (W^+ A^\nu - W^+ A^\mu) (W^- A^\nu - W^- A^\mu) - $$

$$ - e^2 \frac{c_W^2}{4 s_W^2} (W^+ Z^\nu - W^+ Z^\mu) (W^- Z^\nu - W^- Z^\mu) - $$

$$ - e^2 \frac{c_W}{2 s_W} (W^+ A^\nu - W^+ A^\mu) (W^+ Z^\nu - W^- Z^\mu) + h.c. $$

$$ + (M_W^2 W^+ W^- + \frac{1}{2} M_Z^2 Z_{\mu \nu} Z^{\mu \nu}) $$

The mass term comes from the scalar part

The propagators are given by inverting quadratic form. In the unitary gauge:

$$ V^\nu \left( \Box g_{\mu \nu} - \partial^\mu \partial^\nu + g_{\mu \nu} M_V^2 \right) V^\mu $$

$$ D_{\mu \nu}(p) = \frac{-i}{p^2 - M_V^2} \left[ g_{\mu \nu} - \frac{p_\mu p_\nu}{M_V^2} \right] $$

$M_V$ is $M_W$ or $M_Z$
The propagator \( (p_\mu p_\nu / M_V^2) \) has a bad ultraviolet behavior. This lead to the problem of proving renormalizibility of SM. However, one can use another gauge in which the bad ultraviolet behavior is absent.

It is convenient to express the Higgs field as follows:

\[
\Phi(x) = \begin{pmatrix}
- i w_g^+ \\
(\nu + h + iz_g) / \sqrt{2}
\end{pmatrix}
\]

\( w_g^\pm \) and \( z_g \) Goldstone bosons

Covariant derivative in term of \( W^\pm, A, Z \) fields:

\[
D_\mu \Phi = \begin{pmatrix}
\partial_\mu - i e (1 - 2 s_W^2) / 2 s_W c_W Z_\mu - i e A_\mu & -i e / \sqrt{2} s_W W_\mu^+ \\
-i e / \sqrt{2} s_W W_-^\mu & \partial_\mu + i e / 2 s_W c_W Z_\mu
\end{pmatrix} \Phi
\]
The Higgs-gauge part of the Lagrangian

\[ L = (D_\mu \Phi)^\dagger(D^\mu \Phi) - \lambda (\Phi \Phi^\dagger - v^2/2)^2 = \]

\[ = \frac{1}{2} (\partial^\mu h)(\partial_\mu h) + M_W^2 W_\mu^+ W_\mu^- (1 + h/v)^2 + \frac{1}{2} M_Z^2 Z_\mu Z^\mu (1 + h/v)^2 - \]

\[ - M_h^2 h^2 - \lambda v h^3 - \frac{\lambda}{4} h^4 - \]

\[ - M_W \partial_\mu w_g^+ W_\mu^- - M_W \partial_\mu w_g^- W_\mu^+ - M_Z \partial_\mu z_g Z_\mu + \]

\[ + \partial_\mu w_g^+ \partial^\mu w_g^- + \frac{1}{2} \partial_\mu z_g \partial^\mu z_g - \]

\[ - \lambda h(h + 2v) (w_g^- w_g^+ + z_g/2) - \lambda (w_g^- w_g^+ + z_g/2)^2 + \]

+ more cubic and quartic terms involving \( w_g^\pm \) and \( z_g \)

Massless Goldstone bosons

Mixing terms
The gauge fixing term allows to cancel kinetic mixing parts

\[ L_{GF} = -\frac{1}{\xi} \left( \partial_\mu W_\mu^+ - \xi M_W w_g^+ \right) \left( \partial_\nu W_\nu^- - \xi M_W w_g^- \right) - \frac{1}{2\xi} \left( \partial_\mu Z^\mu - \xi M_Z z_g \right)^2 \]

\( W \) and \( Z \) boson propagators come from:

\[
-\frac{1}{4} Z_\mu Z^{\mu \nu} + \frac{1}{2} M_Z^2 Z_\nu Z^{\nu} - \frac{1}{2\xi} \left( \partial_\mu Z^\mu \right)^2 - \\
-\frac{1}{2} W_\mu^+ W_-^{\mu \nu} + M_W^2 W_\nu^+ W_-^{\nu} - \frac{1}{\xi} \left( \partial_\mu W^{+\mu} \right) \left( \partial_\nu W^{-\nu} \right)
\]

\[ Z_{\mu \nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu, \quad W_\mu^{\pm} = \partial_\mu W_\nu^{\pm} - \partial_\nu W_\mu^{\pm} \]

inverting the quadratic form

\[
D_\mu^{\xi} = \frac{-i}{k^2 - M_V^2} \left[ g_{\mu \nu} - \left( 1 - \xi \right) \frac{k_\mu k_\nu}{k^2 - \xi M_V^2} \right]
\]

\[ \xi = 0 \quad - \text{Landau gauge} \]

\[ \xi = 1 \quad - \text{t’Hooft-Feynman gauge} \]

**Goldstone boson and Faddeev-Popov ghost propagators:**

(Complete set of Feynman rules in codes like CompHEP, Madgraph...)

**Good ultraviolet behavior - the SM is renormalizable theory**
How the Higgs mechanism of spontaneous symmetry breaking works in case of fermion fields?

There are only two gauge invariant dimension 4 operators preserving the SM gauge invariance – the Yukawa type operators:

\[ \bar{Q}_L \Phi d_R \quad \text{and} \quad \bar{Q}_L \Phi^C u_R \quad \text{with} \quad Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \]

\[ \Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h \end{pmatrix} \quad \text{and} \quad \Phi^C = i\sigma^2 \Phi^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} v + h \\ 0 \end{pmatrix} \]

Charge conjugated operators:

\[ (\bar{Q}_L \Phi d_R)^\dagger = d_R^\dagger \Phi^\dagger (\bar{Q}_L)^\dagger = d_R^\dagger \gamma^0 \gamma^0 \Phi^\dagger \gamma^0 Q_L = \bar{d}_R \Phi^\dagger Q_L \]

\[ (\bar{Q}_L \Phi^C u_R)^\dagger = \bar{u}_R (\Phi^C)^\dagger Q_L \]

After spontaneous symmetry breaking such operators generate needed fermion masses of Dirac type:

\[ (\bar{u}_L \bar{d}_L) \begin{pmatrix} 0 \\ v \end{pmatrix} d_R + \bar{d}_R (0 \quad v) \begin{pmatrix} u_L \\ d_L \end{pmatrix} = \bar{d}_L d_R + v\bar{d}_R d_L = v (\bar{d}_L d_R + \bar{d}_R d_L) = v\bar{d}d \]

and similar for the up-type quarks with the field \( \Phi^C \).
Most general gauge invariant Lagrangian with possible mixing of Yukawa type operators:

\[ L_{Yukawa} = -\Gamma d^i j \bar{Q}^i L^j \Phi d_R^j + h.c. - \Gamma u^i j \bar{Q}^i L^j \Phi u_R^j + h.c. - \Gamma e^i j \bar{L}^i L^j \Phi e_R^j + h.c. \]

In the unitary gauge one can rewrite the Lagrangian as follows

\[ L_{Yukawa} = - \left[ M_d^i j \bar{d}^i_L d_R^j + M_u^i j \bar{u}^i_L u_R^j + M_e^i j \bar{e}^i_L e_R^j + h.c. \right] \cdot \left( 1 + \frac{h}{\nu} \right) \]

\[ M^i j = \Gamma^i j \nu / \sqrt{2} \]

The physics states are the states with definite mass. One should diagonalize matrices in order to get the physical states for quark and leptons

\[ d_L^i = (U_L^d)_{ij} d_{Lj}; \quad d_R^i = (U_R^d)_{ij} d_{Rj}; \quad u_L^i = (U_L^u)_{ij} u_{Lj}; \quad u_R^i = (U_R^u)_{ij} u_{Rj} \]

\[ \ell_L^i = (U_L^\ell)_{ij} \ell_{Lj}; \quad \ell_R^i = (U_R^\ell)_{ij} \ell_{Rj} \]

\[ U_L U_L^\dagger = 1, \quad U_R U_R^\dagger = 1, \quad U_L^\dagger U_L = 1. \]

The matrices \( U \) are chosen such:

\[ (U_L^u)^\dagger M_u U_R^u = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}; \quad (U_L^d)^\dagger M_d U_R^d = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}; \quad (U_L^\ell)^\dagger M_\ell U_R^\ell = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \]
The Yukawa Lagrangian after diagonalization

\[ L_{Yukawa} = - \left[ m_d d_i \bar{d}^i + m_u u_i \bar{u}^i u^i + m_\ell \bar{\ell}^i \ell^i \right] \cdot \left( 1 + \frac{h}{v} \right) \]

contains masses of particles and the interaction of the fermions with the Higgs boson.

Neutral currents have the same structure with respect to flavors as the mass terms. And they become diagonal simultaneously with the mass terms:

\[ \Psi' \rightarrow U \Psi \]  
\[ \bar{\Psi}' \hat{O}_N \Psi' \rightarrow \bar{\Psi} \hat{O} \Psi \]

But charge currents contain fermions rotated by different matrices:

\[ J_C \sim \bar{u}_L \hat{O}_{ch} d_L \quad u' \rightarrow (U_L^u) u, \quad d' \rightarrow (U_L^d) d. \]

\[ J_C \sim (U_L^u)^\dagger U_L^d \bar{u}_L \hat{Q} d_L \]

The unitary matrix is called the Cabbibo-Kobayashi-Moskawa mixing matrix:

\[ V_{CKM} = (U_L^u)^\dagger U_L^d \]
Concrete values for the elements of the CKM matrix are not predicted in SM. One can show that arbitrary unitary matrix with $N \times N$ complex elements may be parametrized by $N(N-1)/2$ real angles and $(N-1)(N-2)/2$ complex phases.

So the CKM matrix contains 3 real parameters and 1 complex phase. Presence of this phase lead to CP violation, which in this sence is a prediction of the Standard Model with three generations.

\[
V = \begin{pmatrix}
V_{ud} & V_{us} & V_{ub} \\
V_{cd} & V_{cs} & V_{cb} \\
V_{td} & V_{ts} & V_{tb}
\end{pmatrix} \approx \begin{pmatrix}
1 - \lambda^2/2 & \lambda & A\lambda^3 (\rho - i\eta) \\
-\lambda & 1 - \lambda^2/2 & A\lambda^2 \\
A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1
\end{pmatrix} + \mathcal{O}(\lambda^4)
\]

\[
\lambda = 0.2257^{+0.0009}_{-0.0010}, \quad A = 0.814^{+0.021}_{-0.022}
\]

\[
\bar{\rho} = 0.135^{+0.031}_{-0.016}, \quad \bar{\eta} = 0.349^{+0.015}_{-0.017}
\]
\[ L_{SM} = L_{Gauge} + L_{FG} + L_{H} \]

Kinetic terms for the gauge fields;
Interaction terms of the gauge fields

Kinetic terms for fermions;
Interactions of fermions with the gauge fields (NC and CC currents)

Kinetic and self-interaction terms for the higgs boson fields;
Higgs - gauge boson interaction terms;
Higgs-fermion interaction terms;
Mass terms for the gauge bosons and fermions;
+ Goldstone bosons and ghosts interactions

\[
L_H = \frac{1}{2} \left( \partial^\mu h \right) \left( \partial_\mu h \right) + \frac{M_h^2}{2} h^2 - \frac{M_h^2}{2v} h^3 - \frac{M_h^2}{8v^2} h^4 + \\
+ \left( M_W^2 W_\mu^+ W^{-\mu} + \frac{1}{2} M_Z^2 Z_\mu Z^{\mu} \right) \left( 1 + \frac{h}{v} \right)^2 - \sum_f m_f \bar{f} f \left( 1 + \frac{h}{v} \right)
\]