Lecture 3

QCD perturbation theory at fixed order: event shapes and jets

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Today

☞ IR safety
☞ event shapes
☞ jet algorithms
☞ A general method for computing QCD jet cross sections
Jet cross sections

Are there less inclusive IR safe observables?
(typical final states have structure)

2 jets

3 jets
Parton-hadron duality

Production probability pattern:

2 jets : 3 jets : 4 jets $\sim O(\alpha_s^0) : O(\alpha_s^1) : O(\alpha_s^2)$

$\Rightarrow$ jets reflect the partonic structure
Jet cross sections: 2 jets

average over event orientation ⇒ $|\mathcal{M}_2|^2$ has no dependence on the parton momenta:

$$\sigma^{\text{LO}} = |\mathcal{M}_2|^2 \int_0^1 \mathrm{d}y_{12} \delta(1 - y_{12}) J_2(p_1, p_2)$$

NLO corrections: jet function

$$d\sigma^R = |\mathcal{M}_2|^2 S_\varepsilon \frac{\mathrm{d}y_{13}}{y_{13}^\varepsilon} \frac{\mathrm{d}y_{23}}{y_{23}^\varepsilon} C_F \frac{\alpha_s}{2\pi} \left[ (1 - \varepsilon) \left( \frac{y_{23}}{y_{13}} + \frac{y_{13}}{y_{23}} \right) + \frac{2y_{12}}{y_{13}y_{23}} - 2\varepsilon \right]$$

$$\times J_3(p_1, p_2, p_3)$$

$$d\sigma^V = |\mathcal{M}_2|^2 S_\varepsilon C_F \frac{\alpha_s}{2\pi} \left( \frac{\mu^2}{s} \right)^\varepsilon \left[ -\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \pi^2 + O(\varepsilon) \right]$$

$$\times \mathrm{d}y_{12} \delta(1 - y_{12}) J_2(p_1, p_2)$$
Subtraction method

Cannot combine the integrands (as for $\sigma_{\text{tot}}$)

$$\sigma^{\text{NLO}} = \sigma_{3}^{\text{NLO}} + \sigma_{2}^{\text{NLO}}$$

regularize both with subtraction:

$$d\sigma_{3}^{\text{NLO}} = d\sigma^{R} J_{3} - d\sigma^{A} J_{2} \quad \text{integrable}$$

$$d\sigma_{2}^{\text{NLO}} = \left(d\sigma^{V} + d\sigma^{A}\right) J_{2} \quad \text{integrable}$$

requires special property of $J_{n}$:

$$\lim_{y_{13}, y_{23} \rightarrow 0} J_{3} \rightarrow J_{2}$$

(IR safety expressed analytically)
Construction of $d\sigma^A$

\[
\frac{y_{23}}{y_{13}} + \frac{y_{13}}{y_{23}} + \frac{2y_{12}}{y_{13}y_{23}} = \frac{y_{23}}{y_{13}} + \frac{1}{y_{13}} \frac{2y_{12}}{y_{13} + y_{23}} + (1 \leftrightarrow 2) \quad \Leftarrow \quad \frac{1}{ab} = \frac{1}{a} \frac{1}{a+b} + \frac{1}{b} \frac{1}{a+b}
\]

\[
= \frac{1}{y_{13}} \left[ y_{23} + \left( 2 \frac{y_{12} + y_{13} + y_{23}}{y_{13} + y_{23}} - 2 \right) \right] + (1 \leftrightarrow 2)
\]

\[
z_1 = \frac{y_{12}}{y_{12} + y_{23}} \quad \Rightarrow \quad y_{13} + y_{23} = 1 - y_{12} = 1 - z_1(1 - y_{13})
\]

\[
\frac{y_{23}}{y_{13}} - \frac{1}{y_{13}} = \frac{y_{23}(1 - y_{13}) - y_{12} - y_{23}}{y_{13}(y_{12} + y_{23})} = \frac{-y_{23}y_{13} - y_{12}}{y_{13}(y_{12} + y_{23})} = -\frac{y_{23}}{y_{12} + y_{23}} - \frac{z_1}{y_{13}}
\]

\[
= \left[ \frac{1}{y_{13}} \left( \frac{2}{1 - z_1(1 - y_{13})} - 1 - z_1 \right) - \frac{y_{23}}{y_{12} + y_{23}} \right] + (1 \leftrightarrow 2)
\]
Approximate cross section

rewrite the phase space using $y_{13}$ and $z_1$

$$d\sigma^A = |\mathcal{M}_2|^2 S_\varepsilon \left( \frac{\mu^2}{s} \right)^\varepsilon \int_0^1 dy_{13} \int_0^1 dz_1 \left( 1 - y_{13} \right)^{1-2\varepsilon} y_{13}^{-\varepsilon} \left[ z_1 \left( 1 - z_1 \right) \right]^{-\varepsilon} C_F V_{13,2}$$

$$+ (1 \leftrightarrow 2)$$

we shall see: this factorization is universal

$$V_{13,2} (p_1, p_2, p_3) = \frac{\alpha_s}{2\pi} \frac{1}{y_{13}} \left( \frac{2}{1 - z_1 \left( 1 - y_{13} \right)} - 1 - z_1 - \varepsilon \left( 1 - z_1 \right) \right)$$

$$I(\varepsilon) = |\mathcal{M}_2|^2 C_F \frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1-\varepsilon)} \left( \frac{4\pi \mu^2}{s} \right)^\varepsilon \left[ \frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + 10 - \frac{\pi^2}{3} + O(\varepsilon) \right]$$
Integrable subtraction term

\[ \sigma^A = I(\varepsilon)|\mathcal{M}_2|^2 \int dy_{13,2} \delta(1 - y_{13,2}) J_2(\tilde{p}_{13}, \tilde{p}_2) + (1 \leftrightarrow 2) \]

\[ I(\varepsilon)|\mathcal{M}_2|^2 = |\mathcal{M}_2|^2 C_F \frac{\alpha_s}{2\pi} S_\varepsilon \left( \frac{\mu^2}{s} \right)^\varepsilon \left[ \frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + 10 - \frac{\pi^2}{3} + O(\varepsilon) \right] \]

\[ \sigma^V = |\mathcal{M}_2|^2 C_F \frac{\alpha_s}{2\pi} S_\varepsilon \left( \frac{\mu^2}{s} \right)^\varepsilon \left[ -\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \pi^2 + O(\varepsilon) \right] \]

\[ \times \int_0^1 dy_{12} \delta(1 - y_{12}) J_2(p_1, p_2) \]

\[ \sigma_2^{NLO} = |\mathcal{M}_2|^2 C_F \frac{\alpha_s}{\pi} \left[ 1 + \frac{\pi^2}{3} + O(\varepsilon) \right] \int_0^1 dy_{12} \delta(1 - y_{12}) J_2(p_1, p_2) \]
Origin of singular behaviour

Can we construct $d\sigma^A$ universally? (independently of process and observable)

We need study the origin of singularities in $|\mathcal{M}_m|^2$

\[ \theta \to 0: \quad |\mathcal{M}_{m+1}|^2 \approx |\mathcal{M}_m|^2/\theta^2 \]

\[ E_j \to 0: \quad |\mathcal{M}_{m+1}|^2 \approx |\mathcal{M}_m|^2/E_j^2 \]

\[
\frac{1}{(p_i+p_j)^2} = \frac{1}{2 p_i \cdot p_j} = \frac{1}{2 E_i E_j (1-\cos \theta)} \sim \frac{1}{E_i E_j \theta^2}
\]

With phase space factor

\[
\frac{d^3 p_j}{2E_j} = \frac{1}{2} E_j \, dE_j \, d\cos \theta \, d\phi \sim \frac{1}{4} E_j \, dE_j \, d\theta^2 \, d\phi
\]

we find log singularities $dE_j/E_j$ & $d\theta/\theta$
Origin of singular behaviour

arising at kinematically degenerate phase space configurations, i.e. we cannot distinguish (at NLO accuracy)

☞ a single hard parton
☞ single hard parton splitting into two collinear partons
☞ single hard parton emitting a soft gluon

The singularities are artificial, appear if we integrate from 0 (in pQCD), can be tamed into log-enhanced contributions if integrated from carefully chosen lower cut-off → 4th lecture
Are there less inclusive IR safe observables?

Kinoshita-Lee-Nauenberg (KLN) theorem:
In massless, renormalized QFT in 4 dimensions transition rates are IR safe if summation over kinematically degenerate (initial) and final states is carried out.

IR safe observables must be insensitive to soft and/or collinear parton branching:
- event shape variables (rarely used for hadron collisions)
- jet cross sections (several definitions exist)
Event shapes

thrust, thrust major/minor, C- & D-parameters, oblateness, sphericity, aplanarity, jet masses, jet-broadening, energy-energy correlation, differential jet rates

☞ thrust:

\[ T = \max_{\mathbf{n}} \frac{\sum_{i=1}^{m} |\mathbf{p}_i \cdot \mathbf{n}|}{\sum_{i=1}^{m} |\mathbf{p}_i|} \]

IR safe:

☞ \( p_j \rightarrow 0 \) drops from the sums

☞ replacing \( p_i \) with \( z p_i + (1-z)p_i \) does not change the sums
Thrust at LO

\[
\frac{1}{\sigma} \frac{d\sigma}{dT} = C_F \frac{\alpha_s}{2\pi} \left[ \frac{2(3T^2 - 3T + 2)}{T(1-T)} \ln \left( \frac{2T-1}{1-T} \right) - 3(3T-2) \frac{2-T}{1-T} \right]
\]

- singular at $T=1$
- (2-jet region)
- logs at $n$th order \( \alpha_s^n \log^m (1/(1-T)) \), \[ m \leq n + 1 \]
- may become large at $T \to 1$
- needs resummation

(beyond our scope)
Jet algorithms

☞ to quantify the jet-like structure of the final states (how many jets can you see here?)
☞ has a long history with slow convergence
☞ experimental and theoretical requirements differ
☞ exp: define cones that include almost all hadronic tracks at cheap CPU price
☞ theo: needs to be IR safe and resummable
Experimenters preferred cone jet algorithms ("Snowmass accord")

- start from a cone seed (center) in $\eta-\varphi$ plane: $(\eta_c, \varphi_c)$
- define distance of track from the seed:
  $$d_{ij} = \sqrt{(\eta_i - \eta_c)^2 + (\varphi_i - \varphi_c)^2}$$
  track $i$ belongs to the cone if $d_{ij} < R$
  (a predefined value, usually 0.7)
- IR unsafe
- problem with overlapping cones ➔ abandoned
Theory prefers iterative jet algorithms

Define
- a distance between two momenta (of partons or tracks): $d_{ij}$
- a rule to combine two momenta, $i + j \rightarrow (ij)$

Select a value for resolution $d_{\text{cut}}$ and consider all pairs of momenta:

IF $\min d_{ij} < d_{\text{cut}}$ THEN

\[ i + j \rightarrow (ij); \text{ replace } i \text{ and } j \text{ with } (ij); \text{ start again} \]

ELSE

remaining momenta are the jet momenta

STOP

Becomes very expensive computationally for many particles
Preferred jet algorithm at $e^+e^-$ colliders

- Theory won: Durham (or $k_T$) algorithm

- $d_{ij}: d_{ij} = 2 \frac{\min \left( E_i^2, E_j^2 \right)}{s} \left( 1 - \cos \theta_{ij} \right)$

- $i + j \rightarrow (ij): p_{ij}^\mu = p_i^\mu + p_j^\mu = R_{ij}$

- Resolution parameter $0 < y_{\text{cut}} < 1$

- Log-enhanced terms $(0.118 \ln^2(10^2) = 2.5)$

$$\alpha_s^n \log^m \left( 1/y_{\text{cut}} \right), \quad m = 2n, \ 2n - 1$$

- Can be resummed to all orders
3- and 4-jet rates at LEP

Predictions include hadronization corrections
\( k_T \) algorithms at hadron colliders

- \( d_{ij} = \min(p_{\perp,i}^{2n}, p_{\perp,j}^{2n}) \frac{R_{ij}^2}{R^2} \), \quad n = -1, 0, 1

- need distance from beam too: \( d_{iB} = p_{\perp,i}^{2n} \)

  IF smallest is \( d_{ij} \) THEN \( i + j \rightarrow (ij) \)

  ELSE \( d_{iB} \) is moved to jets

- unlike the cone algorithms: IR safe

- was very expensive computationally – improved algorithm made it accessible (and standard): FastJet \hspace{1cm} \text{http://fastjet.fr/}
\( k_T \) algorithms at hadron colliders

\( n = 1: \ k_T \)

\( n = -1: \ \text{anti-}k_T \) (\( n = 0: \) Cambridge/Aachen)

\( n = 0 \) jets with weird shapes

more cone-like

NLL resummation does not depend on \( n \)
$k_T$ algorithm: good features

- $R$ plays a similar role as $R$ in cone algorithms:
  - particles within angular separation $R$ tend to combine
  - particles separated by more than $R$ from all other particles become jets
- assigns a clustering sequence to particles within jets \(\Rightarrow\) can look at jet substructure (can also be done with C/A, but not with anti-$k_T$)
$k_T$ algorithm: not so good features

- arbitrary soft particles can form jets
- soft particles tend to cluster first (irregular shapes)
- non-linear dependence on soft particles
- energy calibration difficult
- underlying event correction depends on area
- acceptance corrections are more difficult to compute
anti-$k_T$ algorithm

Particles close in angle cluster first → regular, cone-like shapes

Has become the standard at LHC, but does not provide information on jet substructure

...and now a deep breath
Factorization of $|\mathcal{M}_m|^2$ in the soft limit

- gluon with momentum $p_s^\mu = \lambda q^\mu$, with $\lambda \to 0$, $q^\mu$ fixed

\[ \gamma^\mu \psi_i = -\not{p}_i \gamma^\mu + 2p_i^\mu \]
\[ \bar{u}(p_i) \psi_i = 0 \]

\[ \propto T_i^j g_s \bar{u}(p_i, s_i) \gamma^\mu \frac{\not{p}_i + \not{p}_j}{s_{ij}} \]

\[ p_j \to 0 \approx T_i^j g_s \frac{p_i^\mu}{p_i \cdot p_j} \bar{u}(p_i, s_i) \]

Try to prove:

\[ p_j \to 0 \approx T_i^c g_s \frac{p_i^\mu}{p_i \cdot p_j} \epsilon_\beta(p_i, n) \]
Factorization of $|\mathcal{M}_m|^2$ in the soft limit

- $S_j$ is an operator taking the soft limit, keeping the leading $1/\lambda$ singular term:

$$S_j \langle c | \mathcal{M}_{m+1}(p_j, \ldots) \rangle = g_s \epsilon^\mu (p_j) J_\mu (p_j) | \mathcal{M}_m (\ldots) \rangle$$

with $J_\mu$ soft gluon current:

$$J_\mu (p_j) = \sum_{k=1}^m T^c_k \frac{p_k \mu}{p_k \cdot p_j}$$

- $J_\mu$ conserved:

$$p_\mu J_\mu (s) | \mathcal{M}_m \rangle = \sum_{k=1}^m T^c_k | \mathcal{M}_m \rangle = 0$$

- Soft limit of SME:

$$S_j | \mathcal{M}_{m+1}(p_j, \ldots) \rangle = \sum_{i,k=1}^{m} T_i \cdot T_k | \mathcal{M}_m \rangle$$

gauge terms.
Factorization of $|M_m|^2$ in the soft limit

Insert colour connection with eikonal current squared
Factorization of $|\mathcal{M}_m|^2$ in the collinear limit

 Defined by Sudakov parametrization:

$$p_i^\mu = z_i p^\mu + k_i^\mu - \frac{k_{i\perp}^2}{z_i} \frac{n^\mu}{2 p \cdot n}$$

$$p_j^\mu = z_j p^\mu + k_j^\mu - \frac{k_{j\perp}^2}{z_j} \frac{n^\mu}{2 p \cdot n}$$

With $k_{i\perp}^\mu + k_{j\perp}^\mu = 0$, $z_i + z_j = 1$ and

$$p_i^2 = p_j^2 = p^2 = 0 \quad n^2 = 0 \quad k_{i\perp} \cdot p = k_{j\perp} \cdot n = 0$$

Theorem: in a physical gauge, leading collinear singularities $(k_{(i \& j)\perp} \to 0)$ are due to the collinear splitting of an external parton
Factorization of $|\mathcal{M}_m|^2$ in the collinear limit

need to compute:

for three cases:

<table>
<thead>
<tr>
<th>$f_{(ij)}$</th>
<th>$f_i + f_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q$</td>
<td>$q + q$</td>
</tr>
<tr>
<td>$g$</td>
<td>$q + \bar{q}$</td>
</tr>
<tr>
<td>$g$</td>
<td>$g + g$</td>
</tr>
</tbody>
</table>
The computation is lengthy but doable on paper (try it). In the collinear limit

\[ C_F g_S^2 \mu^{2\epsilon} \frac{p_i + p_j}{s_{ij}} \gamma^\mu \gamma^\nu d_{\mu\nu}(p_j, n) \frac{p_i + p_j}{s_{ij}} \]

and the collinear limit of SME:

\[ C_{ij} \left| \mathcal{M}^{(0)}_{m+1} \right|^2 = 8\pi \alpha_s \mu^{2\epsilon} \frac{1}{s_{ij}} \left\langle \mathcal{M}^{(0)}_m(p, \ldots) \right| \hat{P}^{(0)}_{qq}(z_i, z_j, k_\perp; \epsilon) \left| \mathcal{M}^{(0)}_m(p, \ldots) \right\rangle \]
Splitting kernels

$q \rightarrow qg$: \[ \langle s|\hat{P}_{qg}^{(0)}|s'\rangle = C_F \left[ 2\frac{z_i}{z_j} + (1 - \epsilon)z_j \right] \delta_{ss'} \]
\[ = C_F \left[ \frac{2}{1 - z_i} - 1 - z_i - \epsilon(1 - z_i) \right] \delta_{ss'} \]

$g \rightarrow q\bar{q}$:
\[ \langle \mu|\hat{P}_{q\bar{q}}^{(0)}(z_i, z_j, k_\perp; \epsilon)|\nu\rangle = T_R \left[ -g^{\mu\nu} + 4z_i z_j \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} \right] \]

$g \rightarrow gg$:
\[ \langle \mu|\hat{P}_{gg}^{(0)}(z_i, z_j, k_\perp; \epsilon)|\nu\rangle = 2C_A \left[ -g^{\mu\nu} \left( \frac{z_i}{z_j} + \frac{z_j}{z_i} \right) - 2(1 - \epsilon)z_i z_j \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} \right] \]
Note my different notation from the literature:

usually $P_{ij}(z)$ denotes the splitting

$$f_i(p) \rightarrow f_j(z \, p) + f_k((1-z) \, p)$$

for me $P_{ij}(z)$ denotes the splitting

$$f_k(p) \rightarrow f_i(z_i \, p) + f_j(z_j \, p)$$

where the parton type $f_k$ is determined uniquely by the flavour summation rules:

$$q + g = q$$
$$q + \bar{q} = g$$
$$g + g = g$$

The reason is visible in $1 \rightarrow 3$ splittings only
Towards general subtractions
follow 4 sketchy slides, understand structure

\[ \langle s | \hat{P}_{qq}^{(0)} | s' \rangle = C_F \left[ \frac{2}{1 - z_i} - 1 - z_i - \epsilon (1-z_i) \right] \delta_{ss'} \]

to

\[ V_{ij,k} (p_i, p_j, p_k) = \frac{\alpha_S}{2\pi} \frac{1}{y_{ij}} \left( \frac{2}{1 - z_i (1 - y_{ij})} - 1 - z_i - \epsilon (1-z_i) \right) \]

\( \Rightarrow \) in the collinear limit

\[ C_F V_{ij,k} \rightarrow \frac{\alpha_S}{2\pi} \frac{1}{y_{ij}} \hat{P}_{ij}^{(0)} (z_i, z_j, k_\perp ; \epsilon) \]

in the soft limit (recall \( z_i = \frac{s_{ik}}{s_{ik} + s_{kj}} \))

\[ V_{ij,k} \rightarrow \frac{\alpha_S}{2\pi} \frac{2s_{ik}}{s_{ij} + s_{kj}} \]

compare
Subtraction terms (not unique)

Define

\[ D_{ij,k}(p_1, \ldots, p_{m+1}) = -\frac{1}{s_{ij}} \left\langle \tilde{\mathcal{M}}_m^{(0)} \right| \frac{T_k \cdot T_{ij}}{T_{ij}^2} V_{ij,k} \right| \tilde{\mathcal{M}}_m^{(0)} \right\rangle \]

Then prove:

\[ \left| \mathcal{M}_{m+1}^{(0)} \right|^2 J_{m}^{(m+1)} s_{ij} \to 0 \sum_{k \neq i,j} D_{ij,k}(p_1, \ldots, p_{m+1}) J_{m}^{(m)}(\tilde{p}_1, \ldots, \tilde{p}_{m+1}) \]

tilde means phase space mapping that preserves momentum conservation with on-shell new momenta (needed for gauge inv.)

\[ \tilde{p}_{i\nu} = p_{i\nu} + p_{j\nu} - \frac{y}{1 - y} p_{k\nu} \quad \tilde{p}_{k\nu} = \frac{1}{1 - y} p_{k\nu} \]
Integrated subtraction terms

Phase space factorizes exactly:

\[ \frac{d\phi_3(p_i, p_j, p_k; Q)}{d\phi_2(\tilde{p}_{ij}, \tilde{p}_k; Q)} = \frac{d\phi_2(\tilde{p}_{ij}, \tilde{p}_k; Q)}{d\phi_j(\tilde{p}_{ij}, \tilde{p}_k)} \]

where (this is tricky to derive):

\[
\int [dp_j] = \frac{(2\tilde{p}_{ij} \cdot \tilde{p}_k)^{1-\epsilon}}{16\pi^2} \int \frac{d^{d-3}\Omega}{(2\pi)^{1-2\epsilon}} \\
\times \int_0^1 dz_j [z_j (1 - z_j)]^{-\epsilon} \int_0^1 dy (1 - y)^{1-2\epsilon} y^{-\epsilon}
\]

so we can integrate the splitting matrices (without reference to the jet function!)

\[
\mathcal{V}_{ij,k} = \int [dr] \frac{1}{s_{ij}} \langle V_{ij,k} \rangle
\]
the rest is (cumbersome) algebra, giving

\[
\int_{m+1} d\sigma^A = \int_m d\sigma^B \otimes I_m(\epsilon)
\]

\[
|\mathcal{M}_m|^2 \otimes I_m(\epsilon) = \langle \mathcal{M}_m | I_m(\epsilon) | \mathcal{M}_m \rangle
\]

where \( I_m \) is a matrix in colour space

\[
I_m = -\frac{\alpha_s}{2\pi} S_\epsilon \sum_{i=1}^m \frac{1}{T_i^2} \nu_i(\epsilon) \sum_{k \neq i} T_i \cdot T_k \left( \frac{\mu^2}{s_{ik}} \right)^\epsilon
\]

\[
\nu_i(\epsilon) = T_i^2 \left( \frac{1}{\epsilon^2} - \frac{\pi^2}{3} \right) + \frac{1}{\epsilon} \gamma_i + \gamma_i + K_i
\]
Other uses of

- factorization in the soft/collinear limits
  - in checking numerical implementations of SME
  - can also be derived for loop SME
  - in all order resummation of log-enhanced terms
  - in devising a parton shower (PS) algorithm splitting kernels:
    - evolution equations for PDF’s (see next lecture)
Thrust at NLO and NNLO

Effect of hadronization is significant
Three-jet rate in PT

\[ \alpha_s(m_Z) = 0.118 \]
\[ \sqrt{s} = 35 \text{ GeV} \]
preliminary
Jade data for three-jet rate vs PT
Summary

- Identified sources of IR singularities
- understood how to cancel those in the final state
- Showed that pQCD can provide approximations to distributions of IR safe observables (event shapes, jet xsections)

Let us try to collide with protons