Lecture 4

QCD perturbation theory at fixed order: partons in the initial state

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Today

- Deeply inelastic scattering
  - parton model
  - factorization theorem
  - parton distributions
- Hadron collisions
Roots of pQCD: deeply inelastic scattering

- deeply inelastic: $Q^2 = -q^2 \gg 1 \text{ GeV}^2$
- to study the structure of the proton by measuring the kinematics of the electron
  - inclusive from QCD point of view

$Q^2 = 21475, \quad y = 0.55, \quad M = 198$
DIS: kinematics

DIS kinematic variables:

\[
\begin{align*}
\text{cms energy}^2 &= s = (P + k)^2 \\
\text{mom. transfer} &= q^\mu = k^\mu - k'^\mu \\
|\text{mom. transfer}|^2 &= Q^2 = -q^2 = 2MExy \\
\text{scaling variable} &= x = Q^2 / (2P \cdot q) \\
\text{energy loss} &= \nu = (P \cdot q)/M = E - E' \\
\text{rel. energy loss} &= y = (P \cdot q)/(P \cdot k) = 1 - E'/E \\
\text{recoil mass}^2 &= W^2 = (P + q)^2 = M^2 + \frac{1 - x}{x} Q^2
\end{align*}
\]
xsection for $e(k) + p(P) \rightarrow e(k') + X$: 

$$\sigma = \sum_X \frac{1}{4ME} \int d\phi \frac{1}{4} \sum_{\text{spin}} |\mathcal{M}|^2$$

factorize the phase space and SME into lepton & and hadron part:

$$d\phi = \frac{d^3k'}{(2\pi)^3 2E'} d\phi_X , \quad \frac{1}{4} \sum_{\text{spin}} |\mathcal{M}|^2 = \frac{e^4}{Q^4} L^{\mu\nu} H_{\mu\nu}$$

hadron part: 

(dimensionless)
DIS: parametrizing the target structure

lepton part

PS:
(compute the Jacobian)

\[
E' = (1 - y)E, \quad \cos \vartheta = 1 - \frac{xyM}{(1-y)E}
\]

tensor:

\[
L^{\mu\nu} = \frac{1}{2} Tr [k\gamma^\mu k'^n] = 2(k^\mu k'^\nu + k'^\nu k^\mu - g^{\mu\nu} k \cdot k')
\]

hadron tensor by gauge invariance & Lorentz structure:

\[
W_{\mu\nu}(P, q) = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1(x, Q^2) + \left(P_\mu - q_\mu \frac{P \cdot q}{q^2}\right) \left(P_\nu - q_\nu \frac{P \cdot q}{q^2}\right) \frac{W_2(x, Q^2)}{P \cdot q}
\]
DIS: parametrizing the target structure

\( \frac{d^2 \sigma}{dx \, dy} = \frac{4\pi \alpha^2}{y \, Q^2} \left[ y^2 W_1(x, Q^2) + \left( \frac{1-y}{x} - xy \frac{M^2}{Q^2} \right) W_2(x, Q^2) \right] \)

\( \frac{d^2 \sigma}{dx \, dy} = \frac{4\pi \alpha^2}{y \, Q^2} \left[ (1 + (1-y)^2) F_1 + \frac{1-y}{x} (F_2 - 2xF_1) \right] \)

What are \( F_i \) in this limit?

\( F_1 \) & \( F_2 \) were first measured in the SLAC-MIT experiment
Scaling limit: $F_2(x, Q^2) \rightarrow F_2(x)$
DIS: parton kinematics

\[ e(k) + q(p=\xi P) \rightarrow e(k') + q(p') \]

\[
\frac{1}{2\hat{s}} \int d\phi_2 \frac{1}{4} \sum_{\text{spin}} |\mathcal{M}|^2
\]

\[
\hat{s} = (p + k)^2 = 2p \cdot k
\]

\[
\hat{u} = (p - k')^2 = -2p \cdot k' = (1 - y)\hat{s}
\]

\[
y = \frac{P \cdot q}{P \cdot k} = \frac{2p \cdot q}{2p \cdot k} = \frac{\hat{s} + \hat{u}}{\hat{s}} = \frac{Q^2}{\hat{s}}
\]

phase space:

\[
d\phi_2 = \frac{d^3k'}{(2\pi)^3 2E_{k'}} \frac{d^4p'}{(2\pi)^4} \frac{2\pi \delta_+ (p'^2)}{(2\pi)^4} \frac{(2\pi)^4 \delta^4 (k + p - k' - p')}{(2\pi)^4}
\]

\[
(k + p - k')^2 = (p + q)^2 = 2p \cdot q + q^2 = Q^2 \left( \frac{\xi}{x} - 1 \right)
\]
\[ \phi_2 = \frac{d\varphi}{2\pi} \frac{E'}{4\pi} \frac{dE'}{x} \frac{d\cos\vartheta}{Q^2} \frac{x}{\delta(\xi - x)} \]

\[ E' = \frac{\sqrt{s}}{2} (1 - y), \quad \cos\vartheta = 1 - \frac{2yx}{\xi(1 - y)} \]

\[ \text{in } x \text{ and } y: \quad \phi_2 = \frac{d\varphi}{(4\pi)^2} \frac{y^{\hat{s}}}{Q^2} dy \, dx \, \delta(\xi - x) \]

\[ \text{SME:} \quad \frac{1}{4} \sum_{\text{spin}} |\mathcal{M}|^2 = \frac{e^2 e^4}{Q^4} L^\mu\nu Q_{\mu\nu} = 2 \frac{e^2 e^4}{Q^4} (\hat{s}^2 + \hat{u}^2) \]

\[ = 2e^2 e^4 \frac{\hat{s}^2}{Q^4} (1 + (1 - y)^2) \]
**DIS: parton model**

**differential cross section in x and y:**

\[
\frac{d^2 \sigma}{dx \, dy} = \frac{4\pi \alpha^2}{y \, Q^2} \left[ 1 + (1 - y)^2 \right] \frac{1}{2} e_q^2 \delta(\xi - x)
\]

**compare to:**

\[
\frac{d^2 \sigma}{dx \, dy} = \frac{4\pi \alpha^2}{y \, Q^2} \left\{ (1 + (1 - y)^2) F_1 + \frac{1 - y}{x} (F_2 - 2xF_1) \right\}
\]

**parton model prediction:** \( F_1(x) \propto e_q^2 \delta(\xi - x) \)

\( F_2 - 2xF_1 = 0 \) (Callan-Gross relation)

(for spin half quark, for scalar \( F_1 \) would be 0)
Scaling limit: $F_2(x, Q^2) \to F_2(x)$

Fig. 4.2. The $F_2$ structure function from the SLAC-MIT, BCDMS, H1 and ZEUS collaborations.
DIS: parton model

but SLAC-MIT: $F_2(x)$ is not a $\delta$-function, probes the $q$ constituent with $\xi = x$

naïve parton model:

$P$(quark $q$ carries momentum fraction of the proton between $\xi$ and $\xi+d\xi) = f_q(\xi) \, d\xi$

the virtual photon scatters incoherently off the constituents (partons) of the proton

$$F_2(x) = 2xF_1(x) = \sum_q \int_0^1 d\xi \, f_q(\xi) \, x \, e_q^2 \, \delta(x - \xi)$$

$$= x \sum_q e_q^2 \, f_q(xq)$$
DIS: parton model

charged lepton off proton (four flavours):

\[
F_{2}^{\text{em}}(x) = x \left[ \frac{4}{9} \left( u(x) + \bar{u}(x) + c(x) + \bar{c}(x) \right) \right]
\]

\[
f_{q}(x) \equiv q(x) + \frac{1}{9} \left( d(x) + \bar{d}(x) + s(x) + \bar{s}(x) \right)
\]

neutrino off proton (four flavours):

with $W^{-}$:

\[
F_{2}^{\bar{\nu}}(x) = 2x \left[ u(x) + \bar{d}(x) + c(x) + \bar{s}(x) \right]
\]

with $W^{+}$:

\[
F_{2}^{\nu}(x) = 2x \left[ d(x) + \bar{u}(x) + s(x) + \bar{c}(x) \right]
\]

Combination of measurements gives separate information on $f_{q}(x)$
DIS: measuring \( q(x) \)

use different targets (assume two flavours and isospin symmetry)

proton uud:

\[
F_2^{\text{proton}}(x) = x \left[ \frac{4}{9} (u_p(x) + \bar{u}_p(x)) + \frac{1}{9} (d_p(x) + \bar{d}_p(x)) \right]
\]

neutron udd:

\[
F_2^{\text{neutron}}(x) = x \left[ \frac{4}{9} (u_n(x) + \bar{u}_n(x)) + \frac{1}{9} (d_n(x) + \bar{d}_n(x)) \right]
= x \left[ \frac{1}{9} (u_p(x) + \bar{u}_p(x)) + \frac{4}{9} (d_p(x) + \bar{d}_p(x)) \right]
\]
DIS: measuring $q(x)$

What are these?

NMC F2 data compared with CTEQ parametrization

CTEQ valence and sea quark distributions
Sum rules

_proton: uud (can you write it for s-quarks?)_

\[
\int_0^1 dx \left( u_p(x) - \bar{u}_p(x) \right) = 2, \quad \int_0^1 dx \left( d_p(x) - \bar{d}_p(x) \right) = 1
\]

_proton momentum:

\[
\sum_q \int_0^1 dx \, x f_{q/p}(x) \simeq 0.5
\]

_Oops! – where is the other half?_

Carried by gluons,

but the naïve parton model is insufficient
CT10 parton distribution functions
Improved parton model: QCD

- rewrite the xsection in more usual notation

\[
dy = \frac{dQ^2}{\hat{s}}, \quad \delta(\xi - x) = \frac{1}{\xi} \delta \left( 1 - \frac{x}{\xi} \right)
\]

\[
\frac{d^2\sigma}{dx\,dQ^2} = \int_0^1 \frac{d\xi}{\xi} \sum_i f_i/h(\xi) \frac{d^2\hat{\sigma}}{dx\,dQ^2} \left( \frac{x}{\xi}, Q^2 \right)
\]

- factorization theorem: convolution of PDF (long distance) with hard scattering (short distance) xsection

- derived heuristically, have not proven it

- rigorous proof based on QFT exists
Questions

闰 Summation includes the gluon, but what is the corresponding hard scattering cross-section?
闰 How can we apply perturbation theory?
闰 Scaling was exact in the parton model. Is it so in QCD?

There is a common answer to these questions: DIS in pQCD
Singularities revisited: final state

- hard process with xsection $\sigma_h$, same with an extra gluon (with relative $k_\perp = E\theta$) $\sigma_{h+g} =$

$$\simeq \sigma_h \frac{2C_F}{\pi} \frac{\alpha_s}{E} \frac{dE}{d\theta} = \sigma_h C_F \frac{\alpha_s}{\pi} \frac{dz}{1-z} \frac{dk_\perp^2}{k_\perp^2}$$

- for IR safe observable the corresponding loop correction cancels the divergence,

$$\sigma_{h+V} =$$

$$\simeq -\sigma_h C_F \frac{\alpha_s}{\pi} \frac{dz}{1-z} \frac{dk_\perp^2}{k_\perp^2}$$
Singularities revisited: initial state

\[ \sigma_{h+g}(p) = \sigma_h(zp)C_F \frac{\alpha_s}{\pi} \frac{dz}{1-z} \frac{dk_\perp^2}{k_\perp^2} \]

\[ \sigma_{h+V}(p) = -\sigma_h(p)C_F \frac{\alpha_s}{\pi} \frac{dz}{1-z} \frac{dk_\perp^2}{k_\perp^2} \]

even for IR safe observable the corresponding loop correction does not cancel the divergence,

\[ \sigma_{h+g} + \sigma_{h+V} \simeq C_F \frac{\alpha_s}{\pi} \int \frac{Q^2}{m_g^2} \frac{dk_\perp^2}{k_\perp^2} \int_0^1 \frac{dz}{1-z} \left[ \sigma_h(zp) - \sigma_h(p) \right] \]

infinite if \( m_g = 0 \)

finite
DIS in pQCD

hard scattering xsection in PT at LO

\[
\hat{F}_{2,q}(x) = \frac{d^2\hat{\sigma}}{dx\,dQ^2} \bigg|_{F_2} = e_q^2 x \, \delta(1 - x)
\]

\[
\hat{F}_{2,g}(x) = \frac{d^2\hat{\sigma}}{dx\,dQ^2} \bigg|_{F_2} = \sum_q e_q^2 x \, 0
\]
DIS in pQCD

- hard scattering xsection in PT at NLO
- finite in the UV and final state IR, but un-cancelled singularity remains in the initial state IR, regularized with a small mass here

\[ \hat{F}_{2,q}(x) = \left. \frac{d^2 \hat{\sigma}}{dx \, dQ^2} \right|_{F_2} = e_q^2 x \left[ \delta(1-x) + \frac{\alpha_s}{4\pi} \left( P_{qq}(x) \log \frac{Q^2}{m^2_q} + C^q_2(x) \right) \right] \]

\[ \hat{F}_{2,g}(x) = \left. \frac{d^2 \hat{\sigma}}{dx \, dQ^2} \right|_{F_2} = \sum_q e_q^2 x \left[ 0 + \frac{\alpha_s}{4\pi} \left( P_{qg}(x) \log \frac{Q^2}{m^2_q} + C^g_2(x) \right) \right] \]

this was not yet physical
only the structure function physical (measurable)

\[ F_{2,q}(x, Q^2) = x \sum_i e_{qi}^2 \left[ f_i^{(0)}(x) \right. \]

\[ + \frac{\alpha_s}{2\pi} \int_0^1 \frac{d\xi}{\xi} f_i^{(0)}(\xi) \left( P_{qqg} \left( \frac{x}{\xi} \right) \log \frac{Q^2}{m_g^2} + C_2^q \left( \frac{x}{\xi} \right) \right) \]

renormalization of PDF’s: if the divergences are universal (they are: do not depend on the hard scattering), can absorb those into the PDF’s, e.g.:

\[ f_q(x, \mu_F) = f_q^{(0)}(x) + \frac{\alpha_s}{2\pi} \int_0^1 \frac{d\xi}{\xi} \left[ f_i^{(0)}(\xi) P_{qqg} \left( \frac{x}{\xi} \right) \log \frac{\mu_F^2}{m_g^2} + z_{qq} \left( \frac{x}{\xi} \right) \right] \]
the structure function is “factorized”

\[
F_{2,q}(x, Q^2) = x \sum_i e_{q_i}^2 \left[ f_i(x, \mu_F) \\
+ \frac{\alpha_s(\mu_R)}{2\pi} \int_0^1 \frac{d\xi}{\xi} f_i(\xi, \mu_F) \left( P_{qq} \left( \frac{x}{\xi} \right) \log \frac{Q^2}{\mu_F^2} + (C_2^q - z_{qq}) \left( \frac{x}{\xi} \right) \right) \right]
\]

actually, a convolution in x-space

\[
\equiv x \sum_i e_{q_i}^2 f_i(\mu_F) \otimes_x \hat{F}_{2,i}(\mu_R, t), \quad t = \log \frac{Q^2}{\mu_F^2}
\]

defined by

\[
f \otimes_x g \equiv \int_0^1 \frac{d\xi}{\xi} f(\xi) g \left( \frac{x}{\xi} \right)
\]
Factorization in DIS

物理截面是一个 convoluation

$$F_{2,q}(x, Q^2) = x \sum_i e_{q_i}^2 f_i(\mu_F) \otimes x \hat{F}_{2,i}(\mu_R, t), \quad t = \log \frac{Q^2}{\mu_F^2}$$

- 长距离物理被因子化为 PDF’s，依赖于 factorization scale $\mu_F$
- 短距离物理被因子化为硬散射截面，依赖于 both factorization scale $\mu_F$ and renormalization scale $\mu_R$
- $\mu_F$ 是一个 arbitrary, unphysical scale
- $z_{qq}$ 定义因子化方案 — 也 not unique，must be the same in short & long distance ($\overline{\text{MS}}$ is standard) in all computations
Factorization in DIS

\[ F_{2,q}(x, Q^2) = x \sum_i e_{q_i}^2 f_i(\mu_F) \otimes x \hat{F}_{2,i}(\mu_R, t), \quad t = \log \frac{Q^2}{\mu^2_F} \]

- \( F_2 \) does not depend on scales \( \mu_R \) & \( \mu_F \)
  - to be understood perturbatively
  - expressed by RG equations
- short distance component can be computed in PT (see previous lectures)
  - recall UV RG equation on \( \mu_R \)
- long distance component cannot be computed in PT

What do we know about it?
Mellin transforms

- defined by \( f(N) \equiv \int_0^1 dx \, x^{N-1} f(x) \)

- turns a convolution into a real product

\[
\int_0^1 dx \, x^{N-1} \int_0^1 \frac{d\xi}{\xi} f(\xi) g\left(\frac{\xi}{x}\right) = \int_0^1 dx \, x^{N-1} \int_0^1 d\xi \int_0^1 dy \delta(x - y\xi) f(\xi) g(y) \\
= \int_0^1 d\xi \int_0^1 dy \, (\xi y)^{N-1} f(\xi) g(y) = f(N)g(N)
\]

- so \( F_{2,q}(N, Q^2) = x \sum_i e_{q_i}^2 f_i(N, \mu_F) \hat{F}_{2,i}(N, \mu_R, t) \)

is independent of \( \mu_F \)

\[
\mu_F \frac{dF_2}{d\mu_F} = 0 \left( \equiv \mathcal{O}(\alpha_s^{n+1}) \right)
\]
Evolution of PDF’s

- for simplicity assume one parton type

\[ F_{2,q}(N, Q^2) = x e^2 q_i q(N, \mu_F) \hat{F}_{2,i}(N, \mu_R, t) \]

- RGE:

\[ \hat{F}_{2,q}(N, t) \frac{dq}{dt} (N, \mu_F) + q(N, \mu_F) \frac{d\hat{F}_{2,q}}{dt} (N, t) = 0 \]

- divide by \( F_{2,q} \)

\[ \mu_F \frac{d \log q}{d \mu_F} (N, \mu_F) = -\mu_F \frac{d \log \hat{F}_{2,q}}{d \mu_F} (N, t) = -\gamma_{qg}(N) \]

\[ = -\frac{\alpha_s(\mu_R)}{2\pi} P_{qq}(N) \]

\( \propto \) Mellin transform of \( P_{qq} \) (computable)
DGLAP equation

How to choose scales? $\mu_R^2 = \mu_F^2 = Q^2$
if we want to avoid large logs that spoil convergence of perturbative series

RGE: $Q^2 \frac{d \log q}{dQ^2}(N, Q^2) = -\frac{1}{2} \gamma_{qg} \left( N, \alpha_s(Q^2) \right)$
which is the Mellin transform of

$$Q^2 \frac{dq}{dQ^2}(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} P_{qg} \otimes x q(Q^2)$$
called DGLAP equation – discussion was highly simplified by neglecting mixing of partons, if not:

$$Q^2 \frac{df_{ij}}{dQ^2}(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \sum_i P_{ij} \otimes x f_i(Q^2)$$
Solution of the RGE

☞ RGE:
\[
Q^2 \frac{d \log q}{dQ^2} (N, Q^2) = -\frac{1}{2} \gamma_{qqg} \left( N, \alpha_s(Q^2) \right)
\]

☞ solution:
\[
q(N, Q^2) = q(N, Q_0^2) \exp \left[ -\int_{t_0}^{t} dt \gamma_{qqg} \left( N, \alpha_s(\Lambda^2 e^t) \right) \right]
\]

☞ recall:
\[
\alpha_s(Q^2) = \frac{1}{b_0 t}, \quad t = \ln \frac{Q^2}{\Lambda^2}
\]

☞ let \( \gamma_{qqg}(N) \leq 0 \), then
\[
q(N, Q^2) = q(N, Q_0^2) \exp \left[ \gamma_{qqg}(N) \int_{t_0}^{t} \frac{dt}{t} \right]
\]

or
\[
q(N, Q^2) = q(N, Q_0^2) \left( \frac{t}{t_0} \right)^{\gamma_{qqg}(N)}
\]

scaling violation
Scaling is violated, especially at low $x$

- $d_{qg}(1) = 0 \Rightarrow \int_0^1 dx \, q(x, Q^2) = \text{valence } q$
- Independent of $Q^2$
- Higher moments vanish more rapidly $\Rightarrow$ average $x$ decreases as $Q^2$ increases
- $q(x, Q^2)$
  - Increases at small $x$
  - Decreases at large $x$
PDF’s evolve

Beautiful art
Factorization theorem in hadron collisions

including fragmentation into identified hadron

\[
\frac{d\sigma_{pp\to Z+\pi+X}(s, x, \alpha_s, \mu_R, \mu_F)}{d^2x} = \\
\sum_{i,j,k} \int_0^1 dx_1 f_{i/p}(x_1, \alpha_s, \mu_F) \int_0^1 dx_2 f_{j/p}(x_2, \alpha_s, \mu_F) \\
\times \int_x^1 \frac{dz}{z} d\hat{\sigma}_{pp\to Z+\pi+X}(\hat{s}, z, \alpha_s, \mu_R, \mu_F) D_{\pi/j} \left( \frac{x}{z}, \hat{s} \right) + O \left( \frac{\Lambda}{Q} \right)^p
\]
Is factorization theorem predictive?

YES!

☞ you can compute the hard scattering cross section in PT
  ➡ renormalize UV divergences
  ➡ use subtraction method to cancel IR ones
  ➡ absorb initial state collinear ones into renormalization of PDF’s

☞ you cannot compute PDF’s & FF’s but extract those from cross section measurements at a reference scale
  ➡ PDF’s (FF’s) are universal, so can be used to predict other cross sections
  ➡ their evolution can be predicted in PT (DGLAP)
Summary: we are in a good shape

_found the theoretical framework: the factorization theorem_

perturbatively computable:
- hard scattering xsection
- evolution of the PDF’s

universal:
- PDF’s
- subtraction method for computing NLO corrections
Are we happy?

✝ Theorists: yes 😊
we can make precision predictions for distributions of IR safe observables

✝ Experimentalists: no 😞
A tool that can simulate real events at correct rates would be much more handy

So, let us try to model events
1. High-$Q^2$ Scattering

2. Parton Shower

3. Hadronization

4. Underlying Event
1. High-$Q^2$ Scattering

- where new physics lies
- process dependent
- first principles description
- it can be systematically improved

3. Hadronization

4. Underlying Event
1. High-$Q^2$ Scattering

QCD - "known physics"
- universal/ process independent
- first principles description

2. Parton Shower

3. Hadronization

4. Underlying Event
1. High-$Q^2$ Scattering
2. Parton Shower

[Diagram showing particle interactions with text]

3. Hadronization

- low $Q^2$ physics
- universal/ process independent
- model dependent

4. Underlying Event
1. High-$Q^2$ Scattering

2. Parton Shower

3. Hadronization

- low $Q^2$ physics
- energy and process dependent
- model dependent

4. Underlying Event
We have (almost) put the gear together

...it’s time to climb the mountain

Enjoy!