LONGITUDINAL DYNAMICS

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based on the course by
Joël Le Duff
Many Thanks!

CAS on Advanced Level Accelerator Physics Course
Trondheim, 18-29 August 2013

Summary of the 2 lectures:

- Acceleration methods
- Accelerating structures
- Phase Stability + Energy-Phase oscillations (Linac)
- Circular accelerators: Cyclotron / Synchrotron
- Dispersion Effects in Synchrotron
- Longitudinal Phase Space Motion
- Stationary Bucket
- Injection Matching
- Adiabatic Damping

Two more related lectures:

- Linear Accelerators I + II – Maurizio Vretanar
- RF Cavity Design - Erk Jensen

Main Characteristics of an Accelerator

Newton-Lorentz Force on a charged particle:

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

2^{nd} term always perpendicular to motion => no acceleration

ACCELERATION is the main job of an accelerator.

- It provides **kinetic energy** to charged particles, hence increasing their **momentum**.
- In order to do so, it is necessary to have an electric field $\vec{E}$ preferably along the direction of the initial momentum (z).

$$\frac{dp}{dt} = eE_z$$

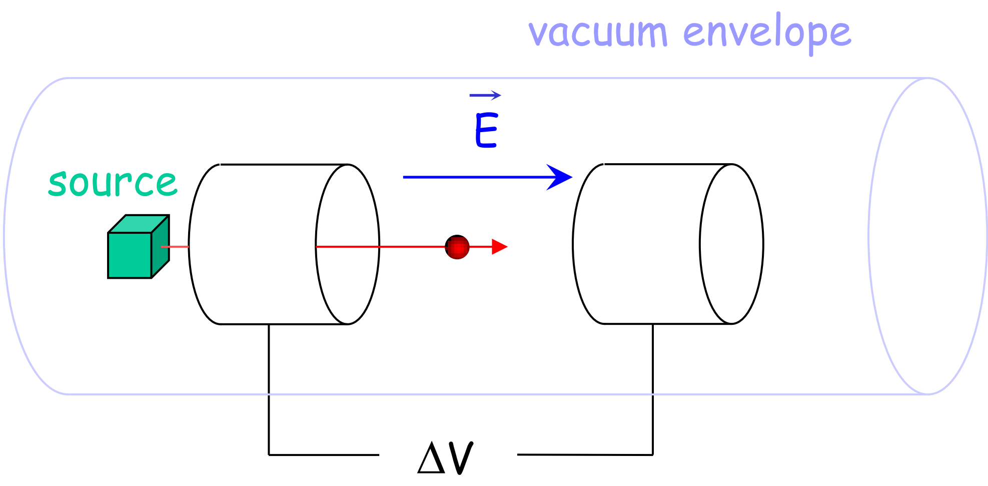
BENDING is generated by a magnetic field perpendicular to the plane of the particle trajectory. The bending radius ρ obeys to the relation :

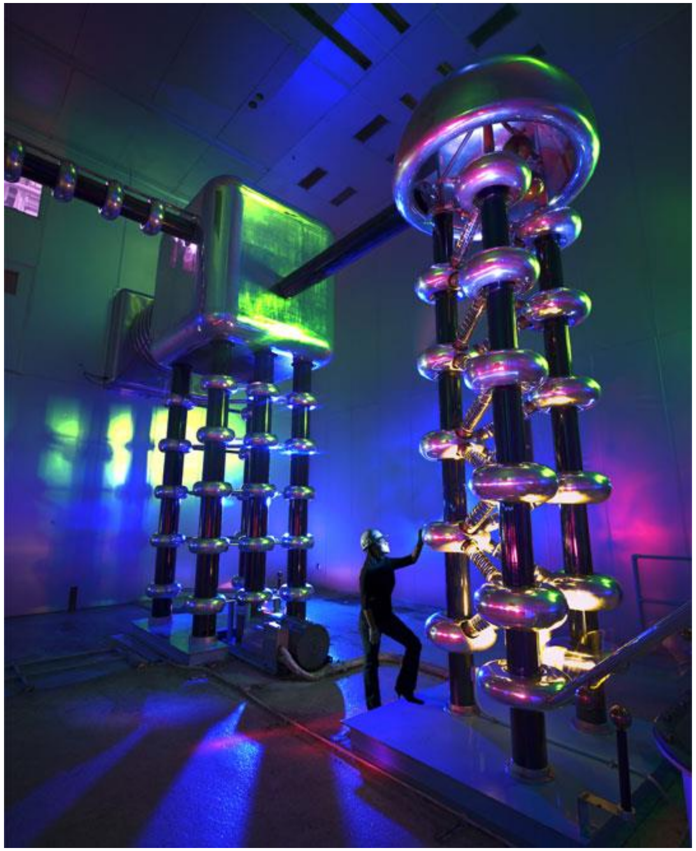
$$\frac{p}{e} = B\rho$$

in practical units: $B\rho\ [\mathrm{Tm}] \approx \frac{p\ [\mathrm{GeV/c}]}{0.3}$

FOCUSING is a second way of using a magnetic field, in which the bending effect is used to bring the particles trajectory closer to the axis, hence to increase the beam density.

Electrostatic Acceleration





750 kV Cockroft-Walton generator at Fermilab (Proton source)

Electrostatic Field:

Energy gain: $W = e\,\Delta V$

Limitation: isolation problems
maximum high voltage (~ 10 MV)

used for first stage of acceleration:
particle sources, electron guns
x-ray tubes

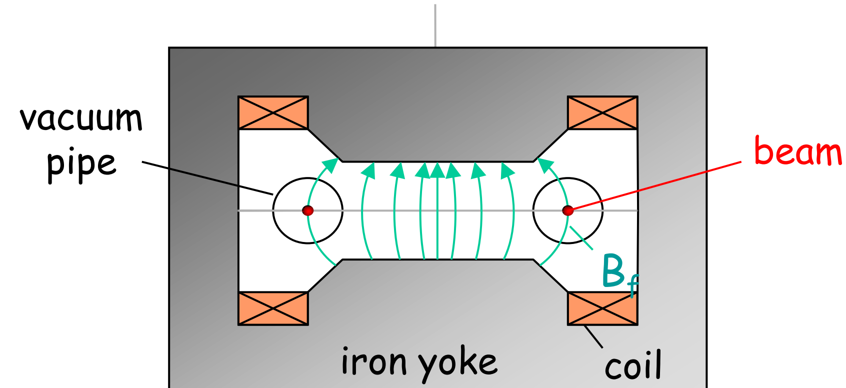
Methods of Acceleration: Induction

From Maxwell's Equations:

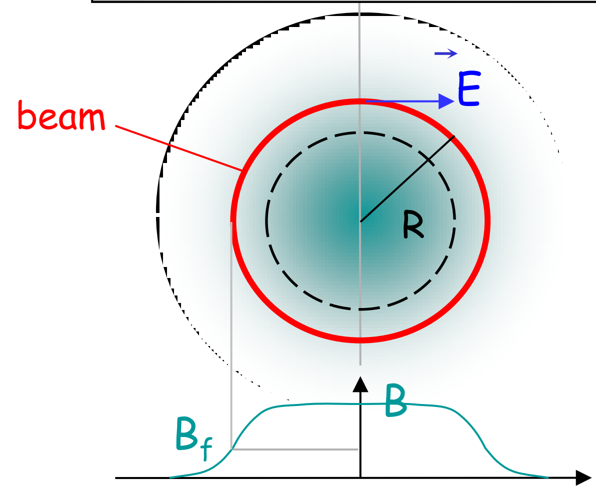
The electric field is derived from a scalar potential φ and a vector potential A
The time variation of the magnetic field H generates an electric field E

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \mu\vec{H} = \vec{\nabla} \times \vec{A}$$

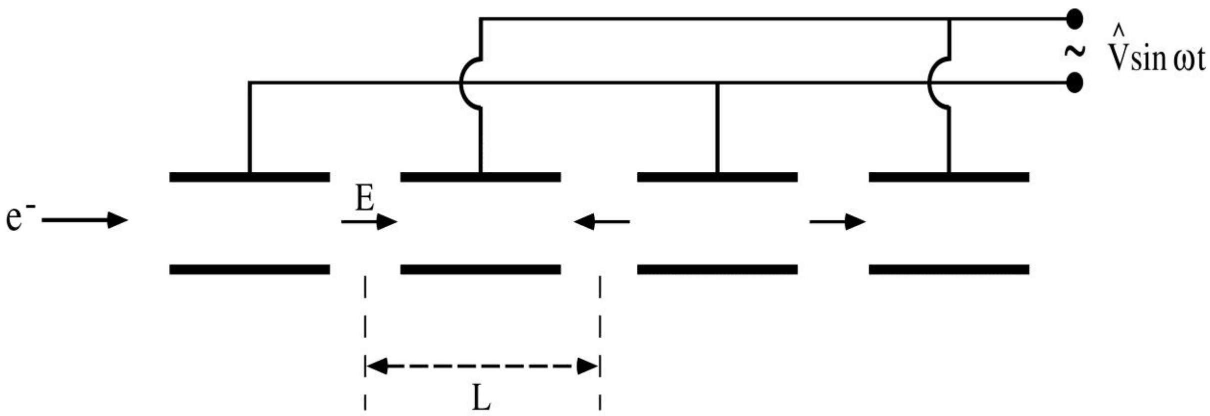


Example: **Betatron**
The varying magnetic field is used to guide particles on a circular trajectory as well as for acceleration.
Limited by saturation in iron



Radio-Frequency (RF) Acceleration

Electrostatic acceleration limited by isolation possibilities => use RF fields

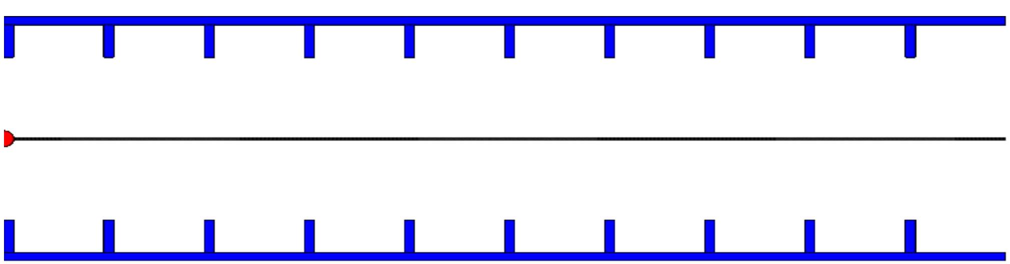


Wideröe-type structure

Cylindrical electrodes (drift tubes) separated by gaps and fed by a RF generator, as shown above, lead to an alternating electric field polarity

Synchronism condition ⟶ $L = v\, T/2$

v = particle velocity
T = RF period



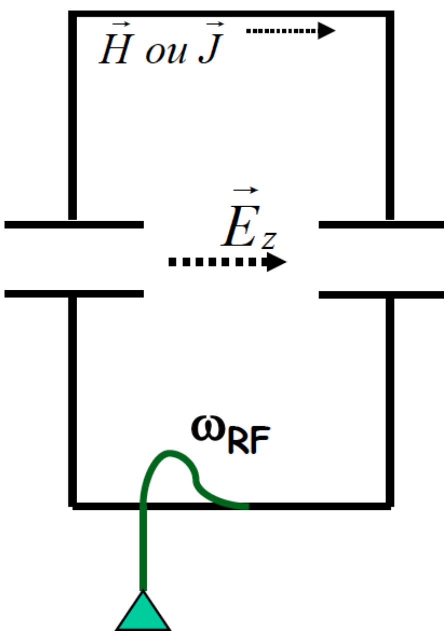
Similar for standing wave cavity as shown (with v≈c)

D.Schulte

Resonant RF Cavities

- Considering RF acceleration, it is obvious that when particles get high velocities the drift spaces get longer and one loses on the efficiency.
  => The solution consists of using a higher operating frequency.

- The power lost by radiation, due to circulating currents on the electrodes, is proportional to the RF frequency.
  => The solution consists of enclosing the system in a cavity which resonant frequency matches the RF generator frequency.



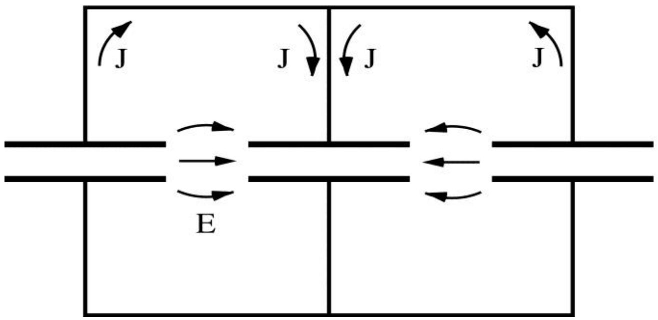
- The electromagnetic power is now constrained in the resonant volume

- Each such cavity can be independently powered from the RF generator

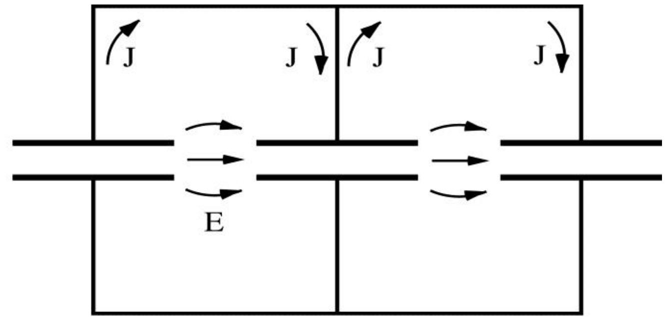
- Note however that joule losses will occur in the cavity walls (unless made of superconducting materials)

Some RF Cavity Examples

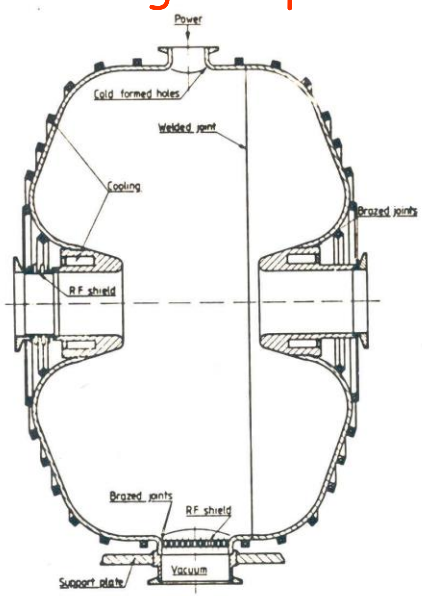
L = vT/2 (π mode)



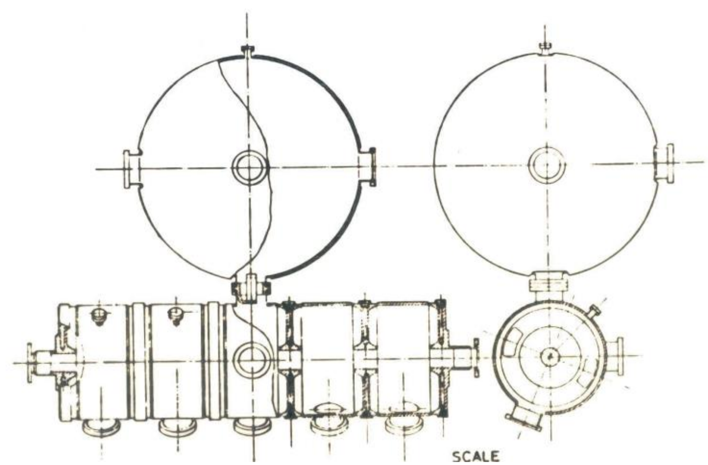
L = vT (2π mode)



Single Gap

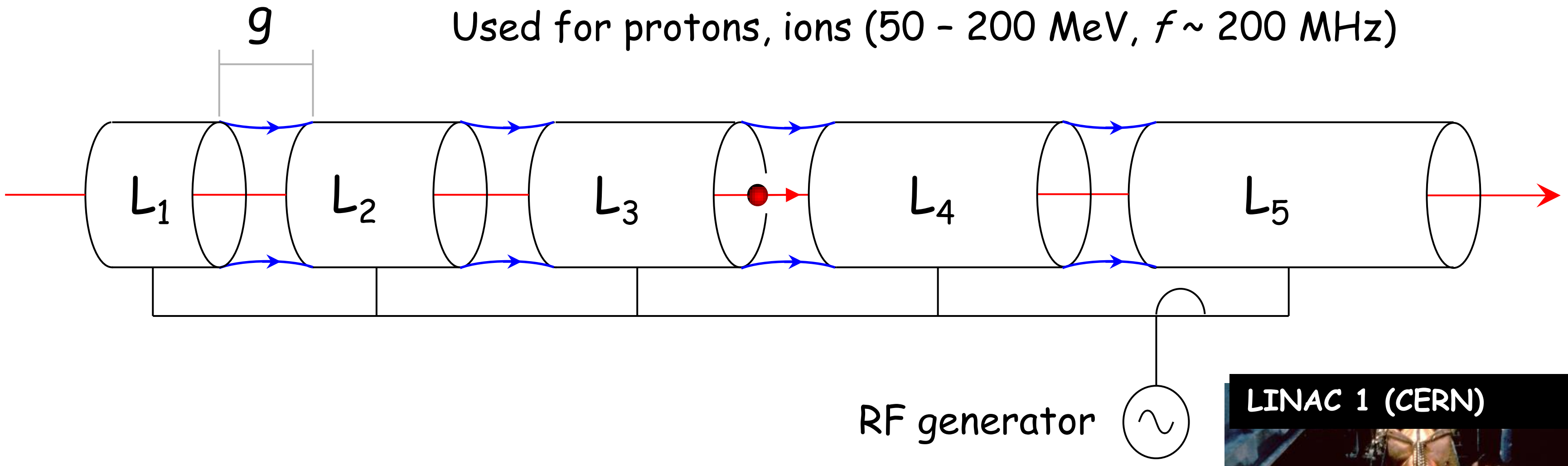


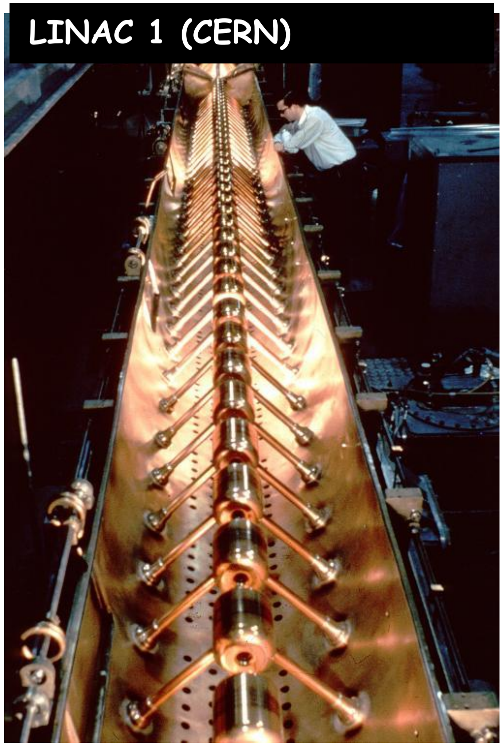
Multi-Gap



RF acceleration: Alvarez Structure

Used for protons, ions (50 – 200 MeV, $f \sim$ 200 MHz)





Synchronism condition $\left(g \ll L\right)$

➡ $$L = v_s \, T_{RF} = \beta_s \, \lambda_{RF}$$

$$\omega_{RF} = 2\pi \frac{v_s}{L}$$

Transit time factor

The accelerating field varies during the passage of the particle
=> particle does not always see maximum field => effective acceleration smaller

Transit time factor defined as:

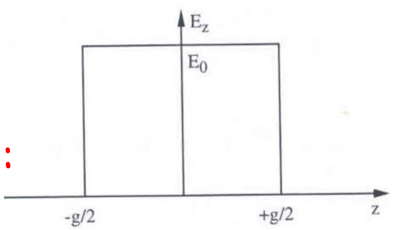
$$T_a = \frac{\text{energy gain of particle with } v = \beta c}{\text{maximum energy gain (particle with } v \to \infty)}$$

In the general case, the transit time factor is:

$$T_a = \frac{\int_{-\infty}^{+\infty} E_1(s,r)\cos\left(\omega_{RF}\frac{s}{v}\right)\mathrm{d}s}{\int_{-\infty}^{+\infty} E_1(s,r)\,\mathrm{d}s}$$

for $E(s,r,t) = E_1(s,r)\cdot E_2(t)$

Simple model uniform field:



$$E_1(s,r) = \frac{V_{RF}}{g} = \text{const.}$$

follows:

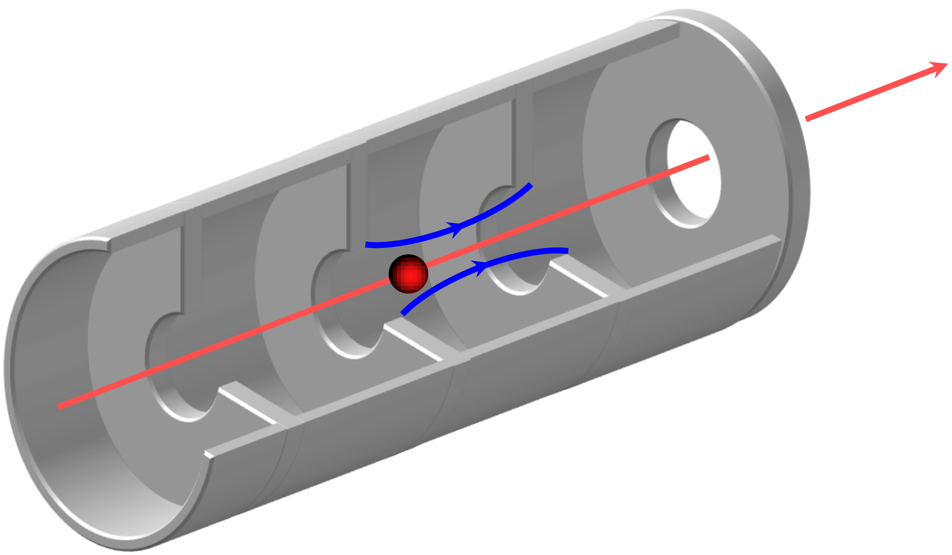
$$T_a = \left|\sin\frac{\omega_{RF}\,g}{2v} \middle/ \frac{\omega_{RF}\,g}{2v}\right|$$

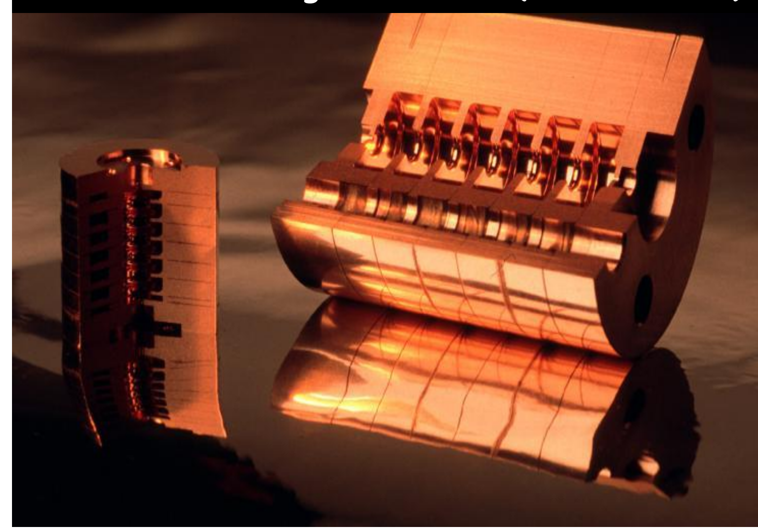
- $0 < T_a < 1$
- $T_a \to 1$ for $g \to 0$, smaller ω_{RF}

Important for low velocities (ions)

Disc loaded traveling wave structures

-When particles gets ultra-relativistic (v~c) the drift tubes become very long unless the operating frequency is increased. Late 40's the development of radar led to high power transmitters (klystrons) at very high frequencies (3 GHz).
-Next came the idea of suppressing the drift tubes using traveling waves. However to get a continuous acceleration the phase velocity of the wave needs to be adjusted to the particle velocity.

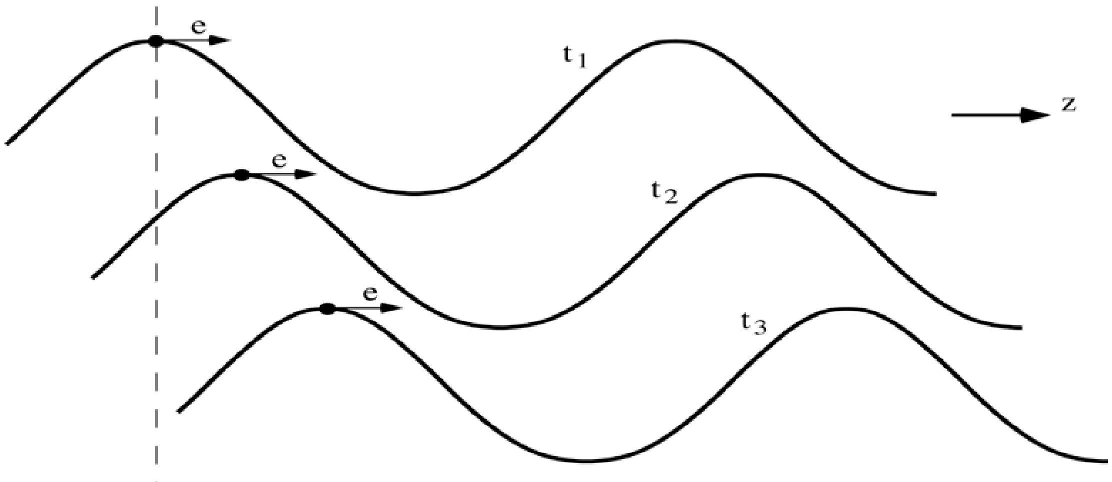




CLIC Accelerating Structures (30 & 11 GHz)

solution: slow wave guide with irises ==> iris loaded structure

The Traveling Wave Case



$$E_z = E_0 \cos\left(\omega_{RF} t - kz\right)$$

$$k = \frac{\omega_{RF}}{v_\varphi} \qquad \text{wave number}$$

$$z = v(t - t_0)$$

v_φ = phase velocity
v = particle velocity

The particle travels along with the wave, and k represents the wave propagation factor.

$$E_z = E_0 \cos\left(\omega_{RF} t - \omega_{RF} \frac{v}{v_\varphi} t - \varphi_0\right)$$

If synchronism satisfied: $v = v_\varphi$ and $E_z = E_0 \cos \varphi_0$

where Φ_0 is the RF phase seen by the particle.

Energy Gain

In relativistic dynamics, total energy E and momentum p are linked by

$$E^2 = E_0^2 + p^2 c^2 \qquad (E = E_0 + W) \qquad W \text{ kinetic energy}$$

Hence: $dE = v\,dp$

The rate of energy gain per unit length of acceleration (along z) is then:

$$\frac{dE}{dz} = v\frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

and the kinetic energy gained from the field along the z path is:

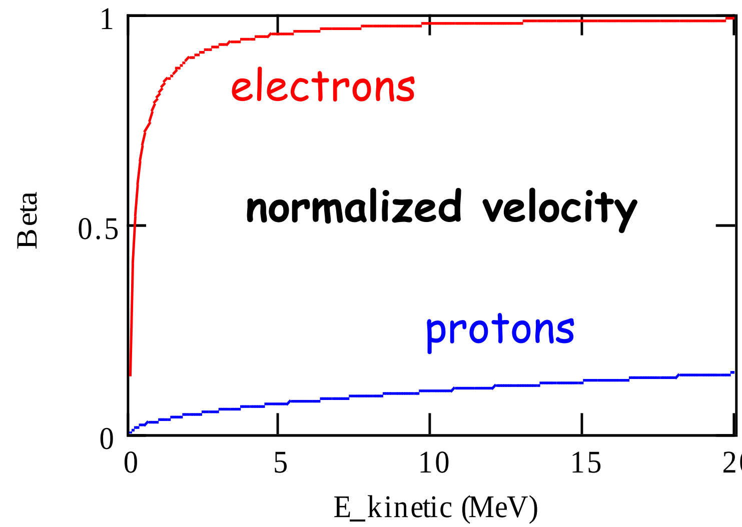
$$dW = dE = eE_z\,dz \quad \rightarrow \quad W = e\int E_z\,dz = eV$$

where V is just a potential.

Velocity, Energy and Momentum

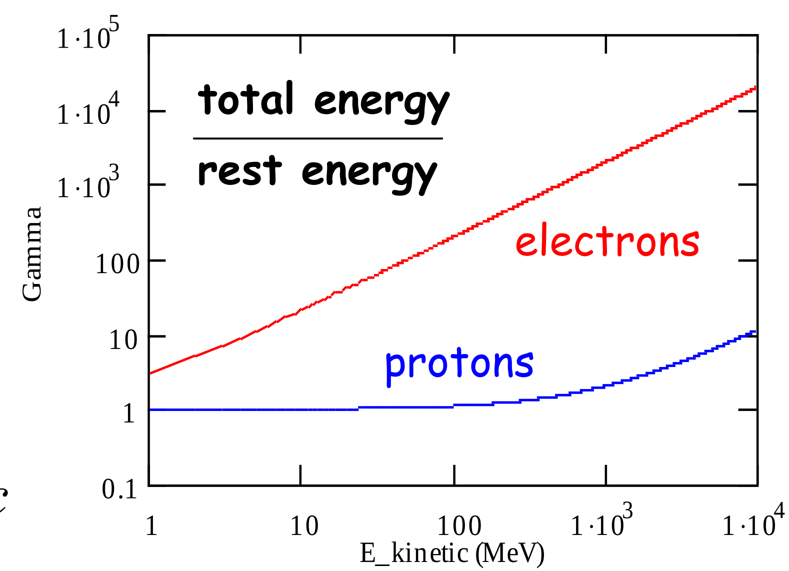
normalized velocity $\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$

=> electrons almost reach the speed of light very quickly (few MeV range)



$\frac{\text{total energy}}{\text{rest energy}}$

$$\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}}$$



Momentum $p = mv = \frac{E}{c^2} bc = b\frac{E}{c} = bgm_0c$

Summary: Relativity + Energy Gain

Newton-Lorentz Force $\vec{F} = \dfrac{\mathrm{d}\vec{p}}{\mathrm{dt}} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$

2[nd] term always perpendicular to motion => no acceleration

Relativistics Dynamics

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \qquad \gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1-\beta^2}}$$

$$p = mv = \frac{E}{c^2}\beta c = \beta\frac{E}{c} = \beta\gamma m_0 c$$

$$E^2 = E_0^2 + p^2c^2 \longrightarrow dE = v\,dp$$

$$\frac{dE}{dz} = v\frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

$$dE = dW = eE_z\,dz \rightarrow W = e\int E_z\,dz$$

RF Acceleration

$$E_z = \hat{E}_z \sin \omega_{RF} t = \hat{E}_z \sin \phi(t)$$

$$\int \hat{E}_z\,dz = \hat{V}$$

$$W = e\hat{V}\sin\phi$$

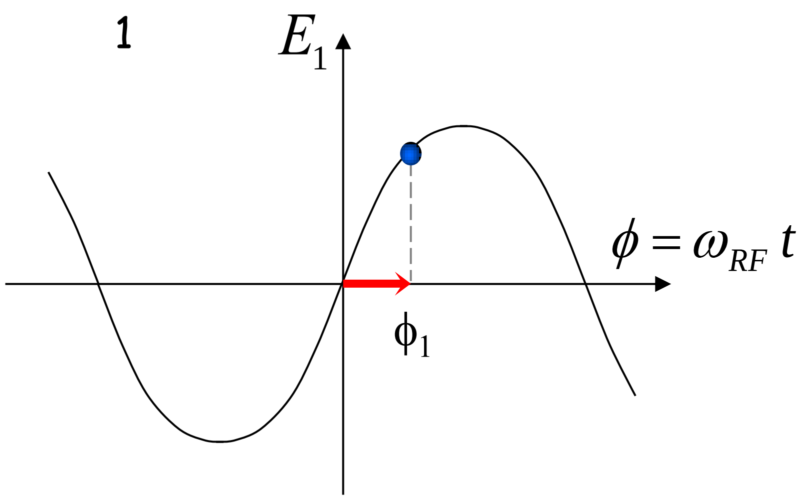
(neglecting transit time factor)

The field will change during the passage of the particle through the cavity
=> effective energy gain is lower

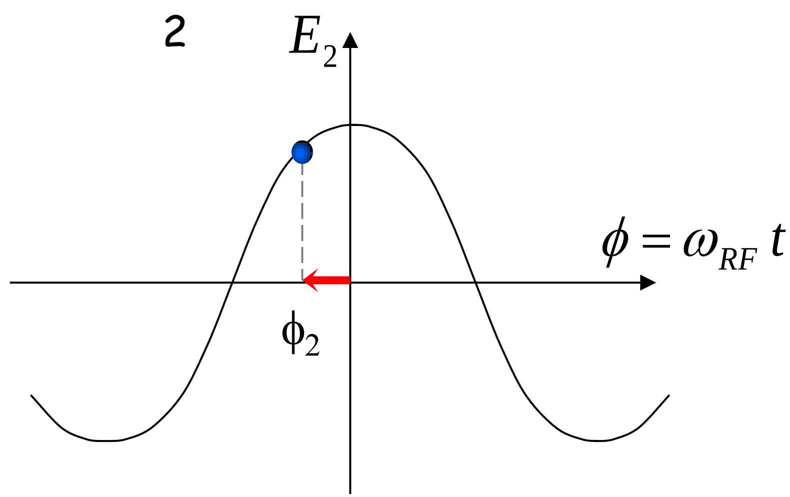
Common Phase Conventions

1. For circular accelerators, the origin of time is taken at the zero crossing of the RF voltage with positive slope
2. For linear accelerators, the origin of time is taken at the positive crest of the RF voltage

Time t= 0 chosen such that:



$$E_1(t) = E_0 \sin\left(w_{RF}\, t\right)$$



$$E_2(t) = E_0 \cos\left(w_{RF}\, t\right)$$

3. I will stick to **convention 1** in the following to avoid confusion

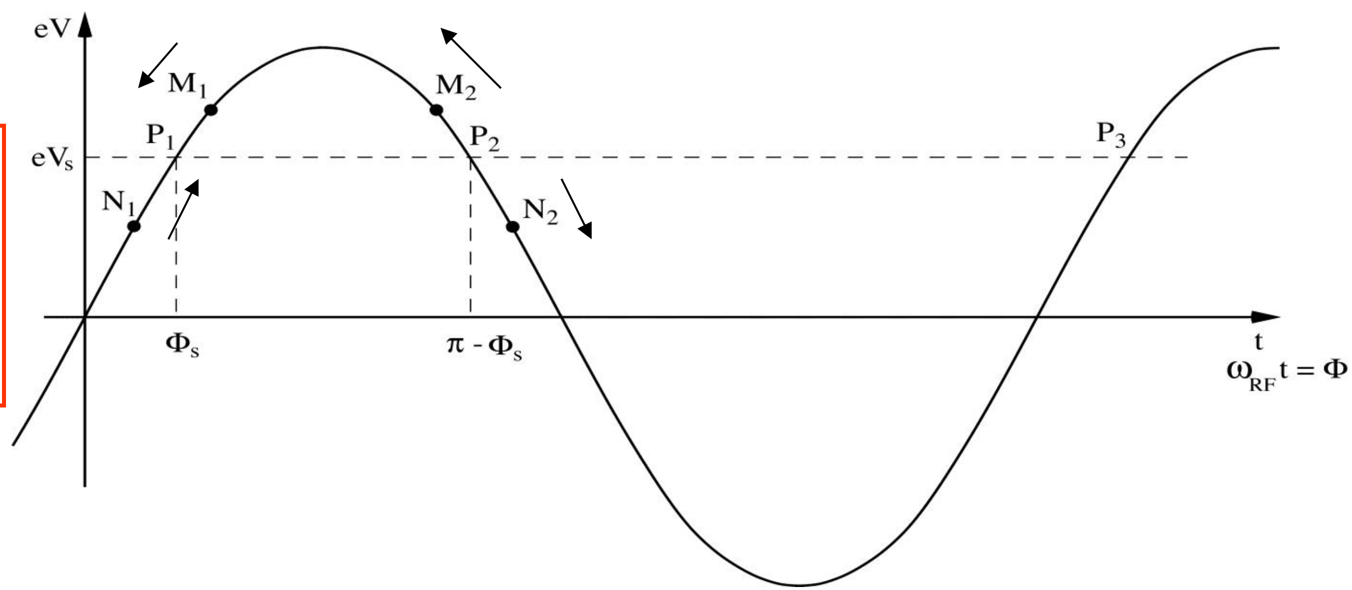
Principle of Phase Stability (Linac)

Let's consider a succession of accelerating gaps, operating in the 2π mode, for which the synchronism condition is fulfilled for a phase Φ_s .

$$eV_s = e\hat{V}\sin\Phi_s$$

is the energy gain in one gap for the particle to reach the next gap with the same RF phase: P_1 ,P_2, …… are fixed points.

For a 2π mode, the electric field is the same in all gaps at any given time.



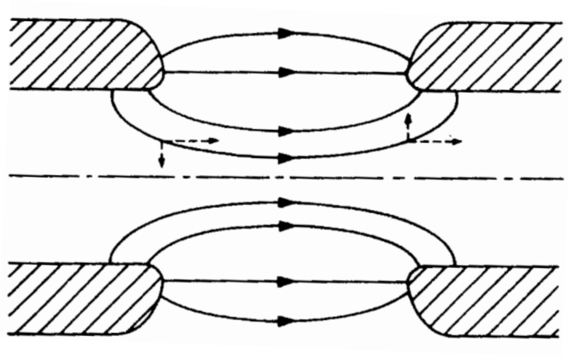
If an energy increase is transferred into a velocity increase =>

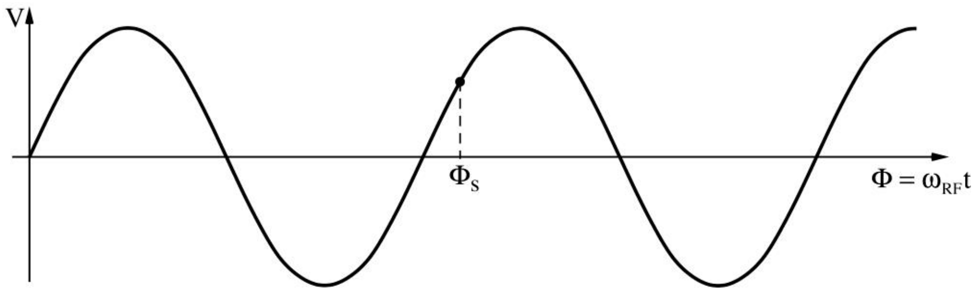
M_1 & N_1 will move towards P_1 => stable

M_2 & N_2 will go away from P_2 => unstable

(Highly relativistic particles have no significant velocity change)

A Consequence of Phase Stability





Transverse focusing fields at the entrance and defocusing at the exit of the cavity.
Electrostatic case: Energy gain inside the cavity leads to focusing
RF case: Field increases during passage => transverse defocusing!

Longitudinal phase stability means : $\frac{\partial V}{\partial t} > 0 \Rightarrow \frac{\partial E_z}{\partial z} < 0$

defocusing RF force

The divergence of the field is zero according to Maxwell : $\nabla . \vec{E} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} > 0$

External focusing (solenoid, quadrupole) is then necessary

Energy-phase Oscillations (1)

- Rate of energy gain for the synchronous particle:

$$\frac{dE_s}{dz} = \frac{dp_s}{dt} = eE_0 \sin\phi_s$$

- Rate of energy gain for a non-synchronous particle, expressed in reduced variables, $w = W - W_s = E - E_s$ and $\varphi = \phi - \phi_s$:

$$\frac{dw}{dz} = eE_0\left[\sin(\phi_s + \varphi) - \sin\phi_s\right] \approx eE_0 \cos\phi_s . \varphi \quad (small\ \varphi)$$

- Rate of change of the phase with respect to the synchronous one:

$$\frac{d\varphi}{dz} = \omega_{RF}\left(\frac{dt}{dz} - \left(\frac{dt}{dz}\right)_s\right) = \omega_{RF}\left(\frac{1}{v} - \frac{1}{v_s}\right) \cong -\frac{\omega_{RF}}{v_s^2}(v - v_s)$$

Since: $$v - v_s = c(\beta - \beta_s) \cong \frac{c}{2\beta_s}\left(\beta^2 - \beta_s^2\right) \cong \frac{w}{m_0 v_s \gamma_s^3}$$

Energy-phase Oscillations (2)

one gets:

$$\frac{d\varphi}{dz} = -\frac{\omega_{RF}}{m_0 v_s^3 \gamma_s^3} w$$

Combining the two 1st order equations into a 2nd order equation gives:

$$\frac{d^2\varphi}{dz^2} + \Omega_s^2 \varphi = 0$$

with

$$\Omega_s^2 = \frac{eE_0 \omega_{RF} \cos\phi_s}{m_0 v_s^3 \gamma_s^3}$$

Stable harmonic oscillations imply: $W_s^2 > 0 \quad \text{and real}$

hence: $\cos\phi_s > 0$

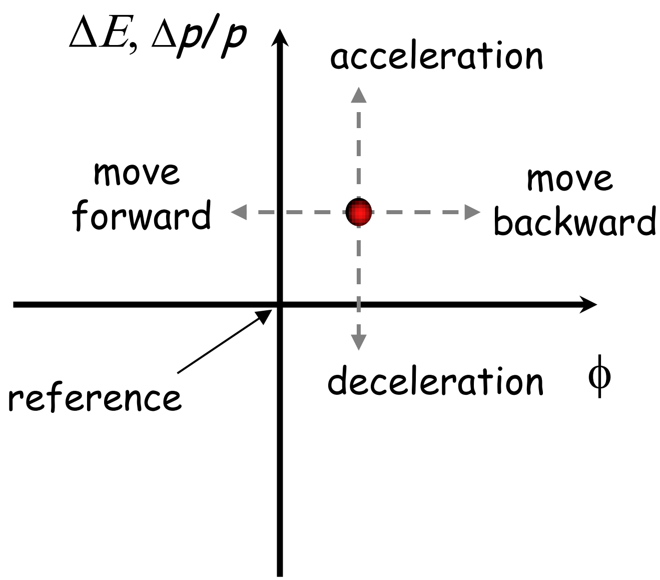
And since acceleration also means: $\sin\phi_s > 0$

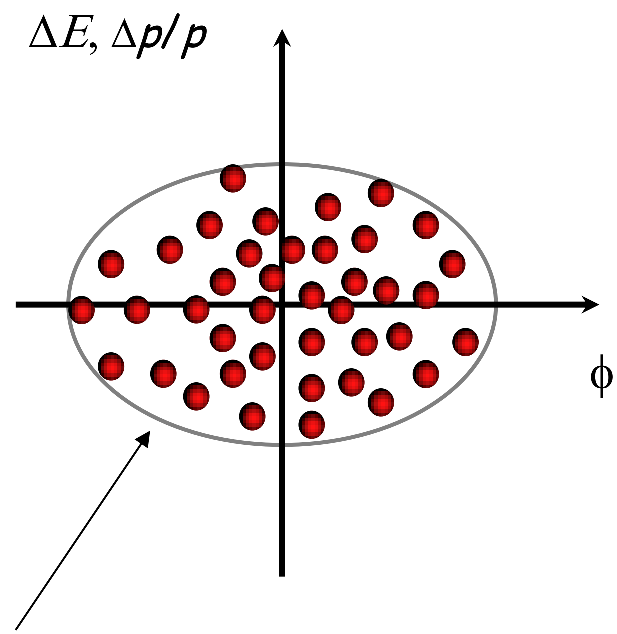
You finally get the result for the stable phase range:

$$0 < \phi_s < \frac{\pi}{2}$$

Longitudinal phase space

The energy – phase oscillations can be drawn in phase space:



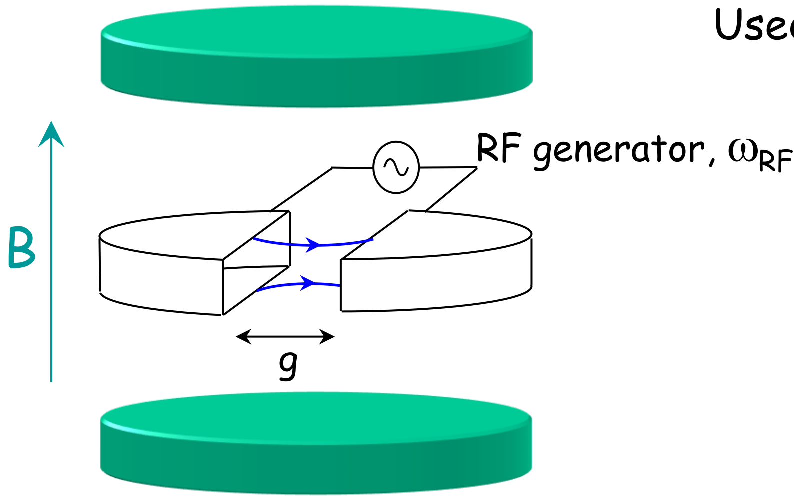


The particle trajectory in the phase space ($\Delta p/p$, ϕ) describes its longitudinal motion.

Emittance: phase space area including all the particles

NB: if the emittance contour correspond to a possible orbit in phase space, its shape does not change with time (matched beam)

Circular accelerators: Cyclotron



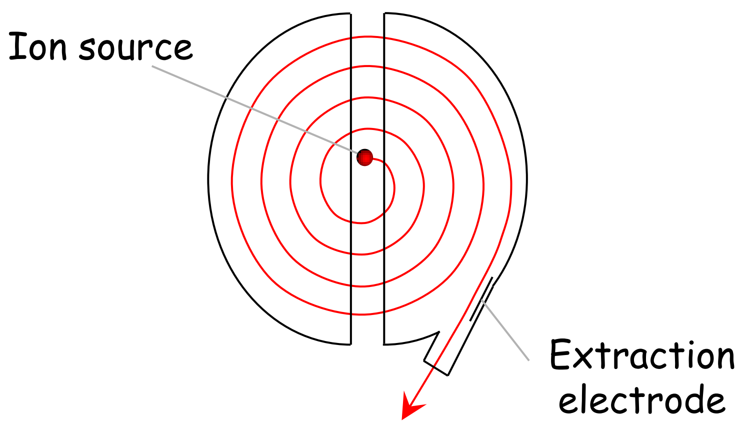
Used for protons, ions

B = constant
ω_{RF} = constant

Synchronism condition

$$\omega_s = \omega_{RF}$$

$$2\pi\,\rho = v_s\,T_{RF}$$



Cyclotron frequency $\omega = \dfrac{q\,B}{m_0\,\gamma}$

1. γ increases with the energy
   $\Rightarrow$ no exact synchronism

2. if **v << c** $\Rightarrow$ $\gamma \cong$ **1**

Cyclotron / Synchrocyclotron

TRIUMF 520 MeV cyclotron Vancouver - Canada

Synchrocyclotron: Same as cyclotron, except a modulation of ω_{RF}

B = constant

$\gamma\, \omega_{RF}$ = constant

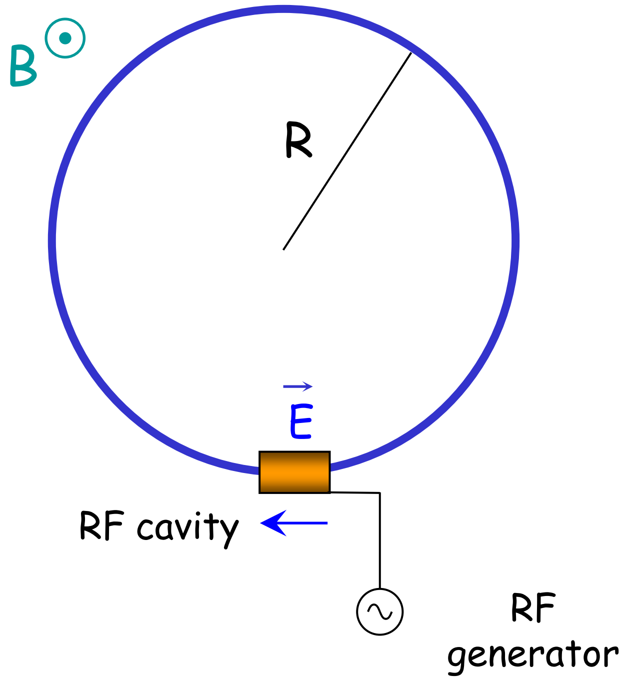
ω_{RF} decreases with time

The condition:

$$\omega_s(t) = \omega_{RF}(t) = \frac{q\,B}{m_0\,\gamma(t)}$$

Allows to go beyond the non-relativistic energies

Circular accelerators: The Synchrotron



1. Constant orbit during acceleration
2. To keep particles on the closed orbit, B should increase with time
3. ω and ω_{RF} increase with energy

Synchronism condition ➡

$$T_s = h\, T_{RF}$$

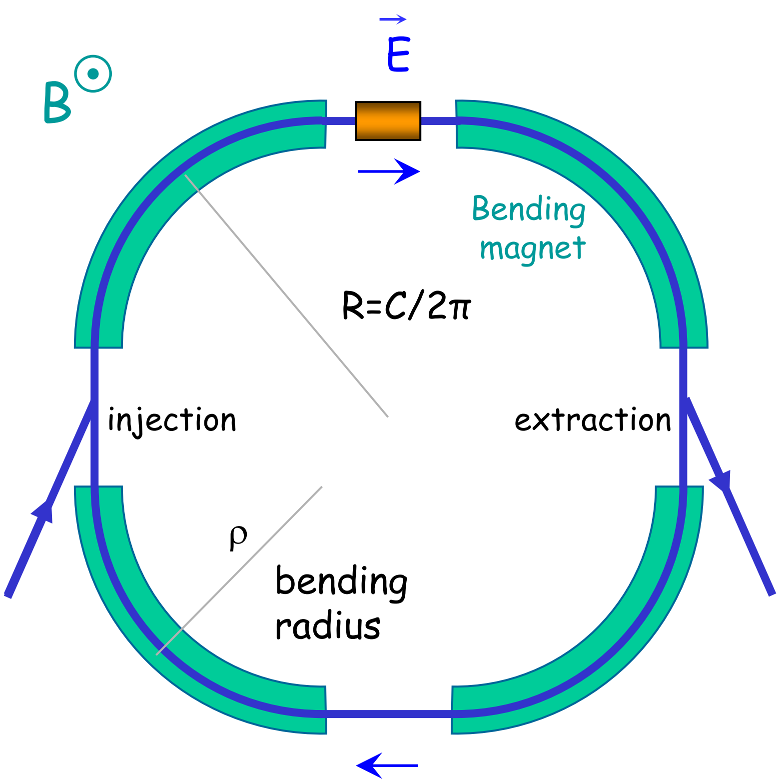
$$\frac{2\pi R}{v_s} = h\, T_{RF}$$

h integer, harmonic number: number of RF cycles per revolution

$$\omega_{RF} = h\, \omega_r$$

The Synchrotron

The synchrotron is a synchronous accelerator since there is a synchronous RF phase for which the energy gain fits the increase of the magnetic field at each turn. That implies the following operating conditions:



$e\hat{V}\sin\Phi \longrightarrow$ **Energy gain per turn**

$\Phi = \Phi_s = cte \longrightarrow$ **Synchronous particle**

$\omega_{RF} = h\omega_r \longrightarrow$ **RF synchronism (h - harmonic number)**

$\rho = cte \quad R = cte \longrightarrow$ **Constant orbit**

$B\rho = P/e \Rightarrow B \longrightarrow$ **Variable magnetic field**

If v≈c, ω_r hence ω_{RF} remain constant (ultra-relativistic e^-)

The Synchrotron

Energy ramping is simply obtained by varying the B field (frequency follows v):

$$p = eB\rho \quad \Rightarrow \quad \frac{dp}{dt} = e\rho\,\dot{B} \quad \Rightarrow (\Delta p)_{turn} = e\rho\,\dot{B}T_r = \frac{2\pi e\rho R\dot{B}}{v}$$

Since:

$$E^2 = E_0^2 + p^2c^2 \quad \Rightarrow \quad \Delta E = v\Delta p$$

$$(\Delta E)_{turn} = (\Delta W)_s = 2\pi e\rho R\dot{B} = e\hat{V}\sin\phi_s$$

Stable phase φ_s changes during energy ramping

$$\sin\phi_s = 2\pi\rho R\frac{\dot{B}}{\hat{V}_{RF}} \quad \Rightarrow \quad \phi_s = \arcsin\left(2\pi\rho R\frac{\dot{B}}{\hat{V}_{RF}}\right)$$

- The number of stable synchronous particles is equal to the harmonic number h. They are equally spaced along the circumference.
- Each synchronous particle satisfies the relation p=eBρ.
  They have the nominal energy and follow the nominal trajectory.

The Synchrotron

During the energy ramping, the RF frequency increases to follow the increase of the revolution frequency :

$$\omega_r = \frac{\omega_{RF}}{h} = \omega(B, R_s)$$

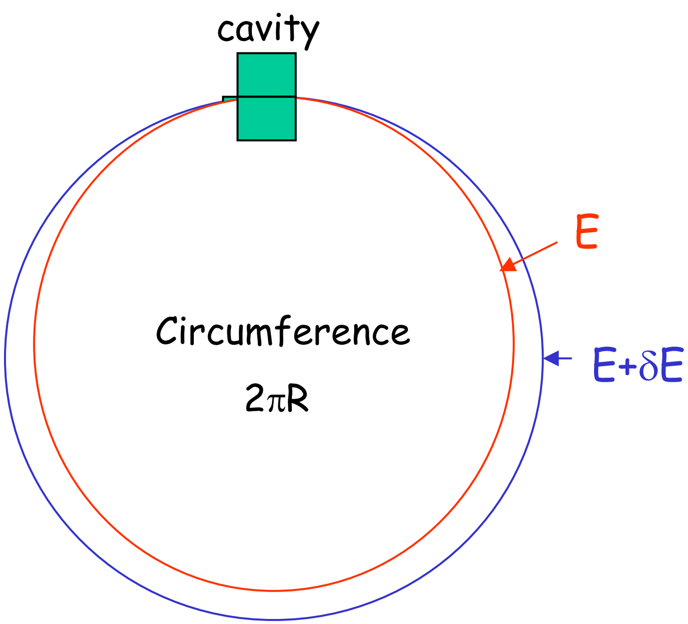
Hence: $\frac{f_{RF}(t)}{h} = \frac{v(t)}{2\pi R_s} = \frac{1}{2\pi} \frac{ec^2}{E_s(t)} \frac{\rho}{R_s} B(t)$ (using $p(t) = eB(t)\rho$, $E = mc^2$)

Since $E^2 = (m_0c^2)^2 + p^2c^2$ the RF frequency must follow the variation of the B field with the law

$$\frac{f_{RF}(t)}{h} = \frac{c}{2\pi R_s} \left\{ \frac{B(t)^2}{(m_0c^2/ec\rho)^2 + B(t)^2} \right\}^{1/2}$$

This asymptotically tends towards $f_r \to \frac{c}{2\pi R_s}$ when B becomes large compared to $m_0c^2/(ec\rho)$ which corresponds to $v \to c$

Dispersion Effects in a Synchrotron



If a particle is slightly shifted in momentum it will have a different orbit and the length is different.

The "momentum compaction factor" is defined as:

$$\alpha = \frac{dL/L}{dp/p} \Rightarrow \boxed{\alpha = \frac{p}{L}\frac{dL}{dp}}$$

If the particle is shifted in momentum it will have also a different velocity.
As a result of both effects the revolution frequency changes:

$$\eta = \frac{\mathrm{d} f_r/f_r}{\mathrm{d} p/p} \Rightarrow \boxed{\eta = \frac{p}{f_r}\frac{df_r}{dp}}$$

p=particle momentum

R=synchrotron physical radius

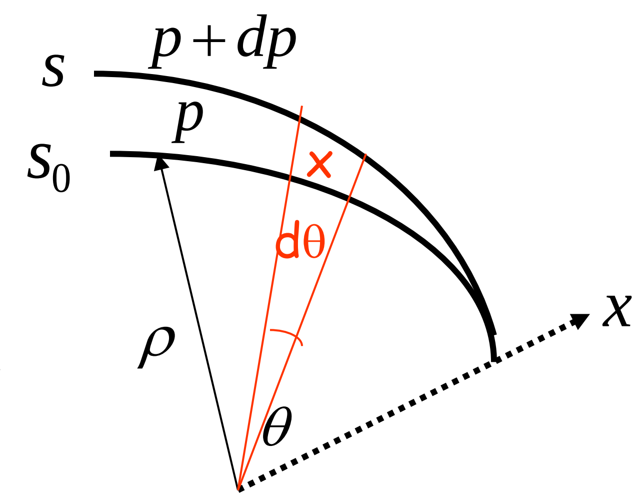
f_r=revolution frequency

Dispersion Effects in a Synchrotron (2)

$$\alpha = \frac{p}{L}\frac{dL}{dp}$$

$$ds_0 = \rho\, d\theta$$

$$ds = (\rho + x)\, d\theta$$



The elementary path difference from the two orbits is:

definition of dispersion D_x

$$\frac{dl}{ds_0} = \frac{ds - ds_0}{ds_0} = \frac{x}{\rho} = \frac{D_x}{\rho}\frac{dp}{p}$$

leading to the total change in the circumference:

$$dL = \oint_C dl = \oint \frac{x}{\rho} ds_0 = \oint \frac{D_x}{\rho}\frac{dp}{p} ds_0$$

$$\alpha = \frac{1}{L}\oint_C \frac{D_x(s)}{\rho(s)} ds_0$$

With ρ=∞ in straight sections we get:

$$\alpha = \frac{\langle D_x \rangle_m}{R}$$

$\langle\ \rangle_m$ means that the average is considered over the bending magnet only

Dispersion Effects in a Synchrotron (3)

$$f_r = \frac{\beta c}{2\pi R} \quad \Rightarrow \quad \frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R} = \frac{d\beta}{\beta} - \alpha \frac{dp}{p}$$

↑ definition of momentum compaction factor

$$p = mv = \beta\gamma \frac{E_0}{c} \quad \Rightarrow \quad \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d\left(1-\beta^2\right)^{-\frac{1}{2}}}{\left(1-\beta^2\right)^{-\frac{1}{2}}} = \underbrace{\left(1-\beta^2\right)^{-1}}_{\gamma^2} \frac{d\beta}{\beta}$$

$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha\right)\frac{dp}{p}$$

$$\frac{df_r}{f_r} = \eta \frac{dp}{p}$$

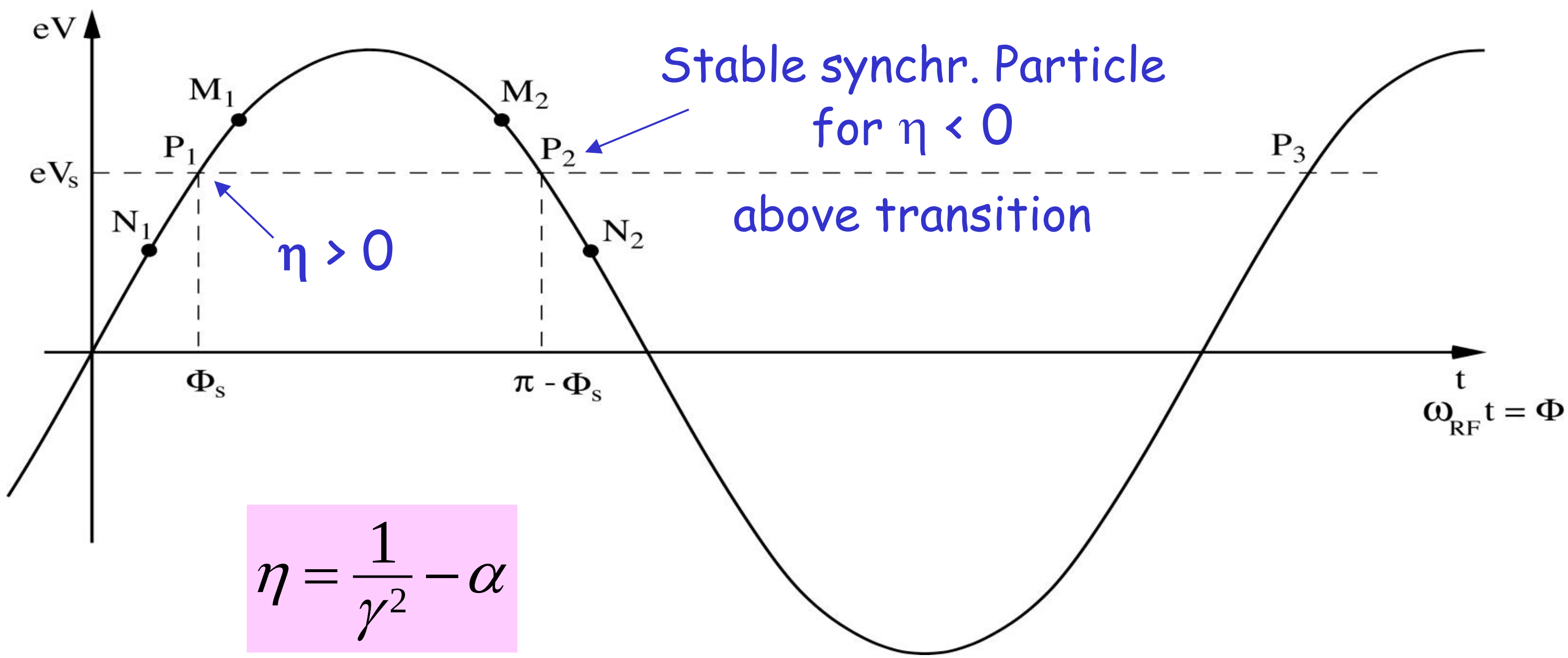
$$\eta = \frac{1}{\gamma^2} - \alpha$$

η=0 at the transition energy

$$\gamma_{tr} = \frac{1}{\sqrt{\alpha}}$$

Phase Stability in a Synchrotron

From the definition of η it is clear that an increase in momentum gives
- below transition ($\eta > 0$) a higher revolution frequency (increase in velocity dominates) while
- above transition ($\eta < 0$) a lower revolution frequency ($v \approx c$ and longer path) where the momentum compaction (generally > 0) dominates.

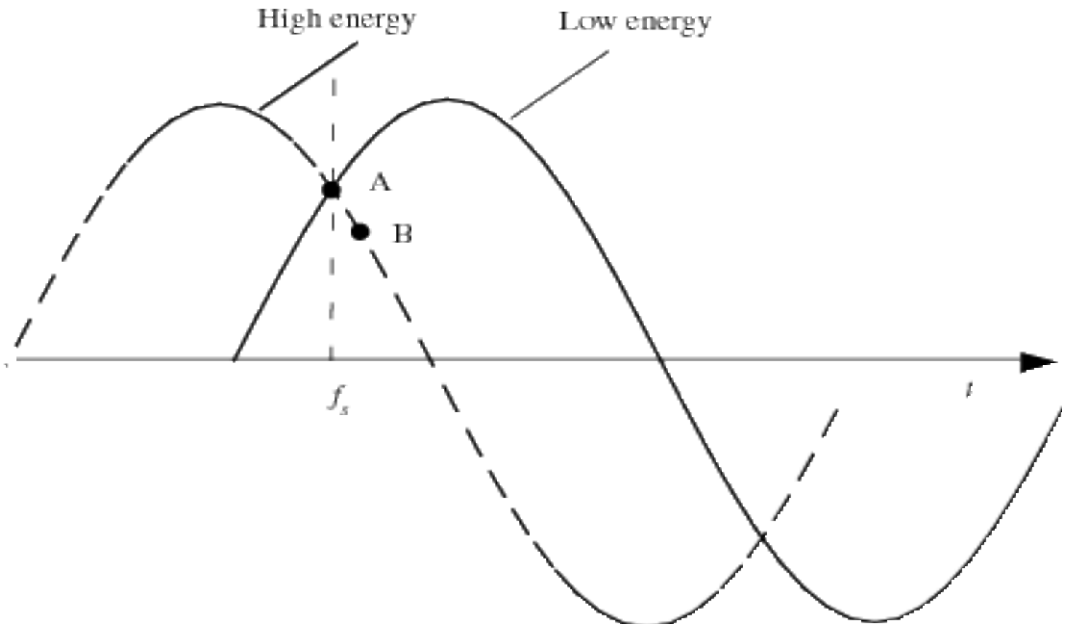


$$\eta = \frac{1}{\gamma^2} - \alpha$$

Crossing Transition

At transition, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

Crossing transition during acceleration makes the previous stable synchronous phase unstable. The RF system needs to make a rapid change of the RF phase, a 'phase jump'.



Synchrotron oscillations

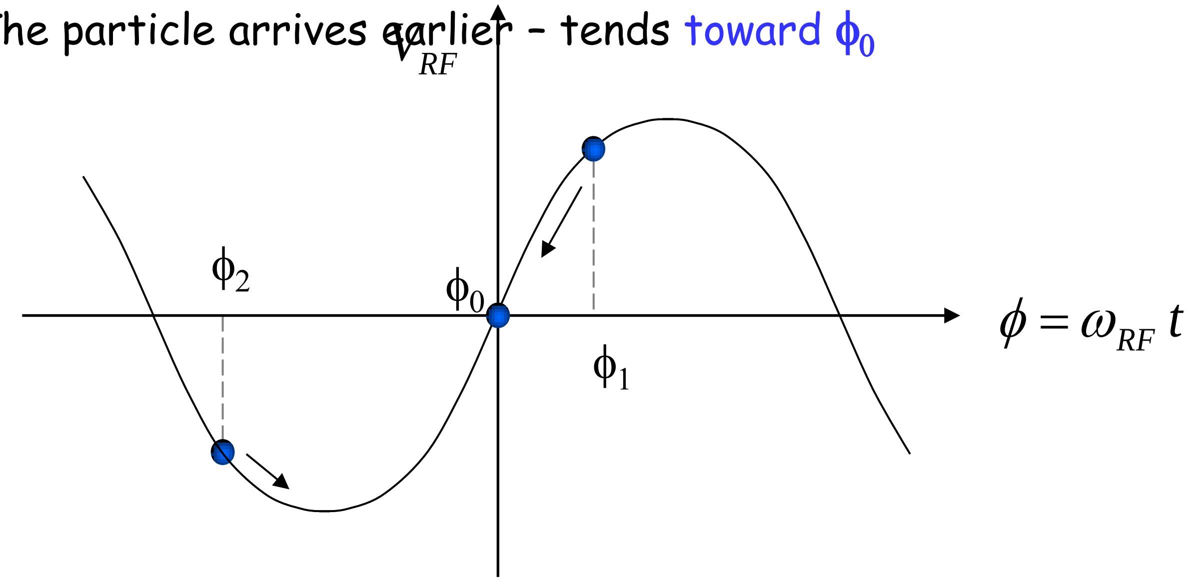
Simple case (no accel.): B = const., below transition $\gamma < \gamma_{tr}$

The phase of the synchronous particle must therefore be $\phi_0 = 0$.

ϕ_1 - The particle is accelerated

- Below transition, an increase in energy means an increase in revolution frequency

- The particle arrives earlier – tends toward ϕ_0

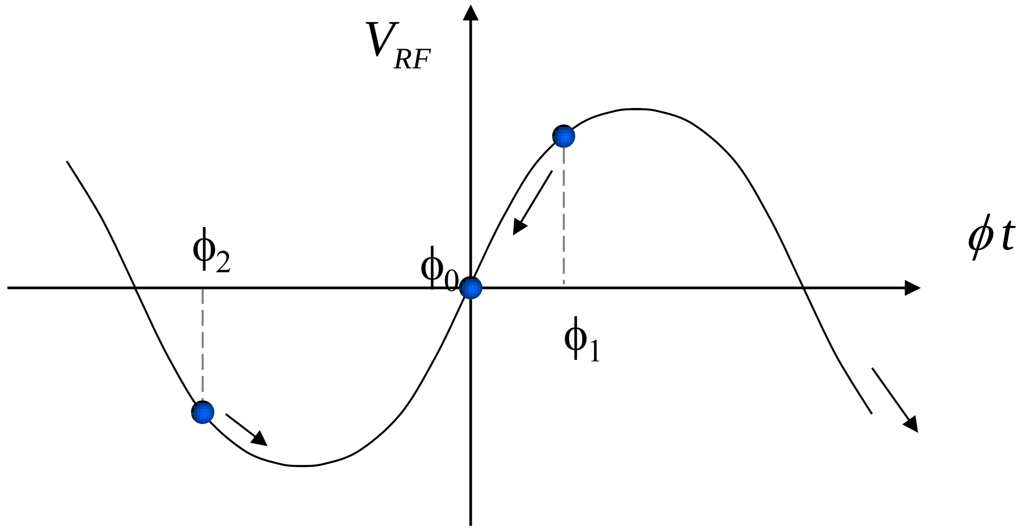


ϕ_2 - The particle is decelerated

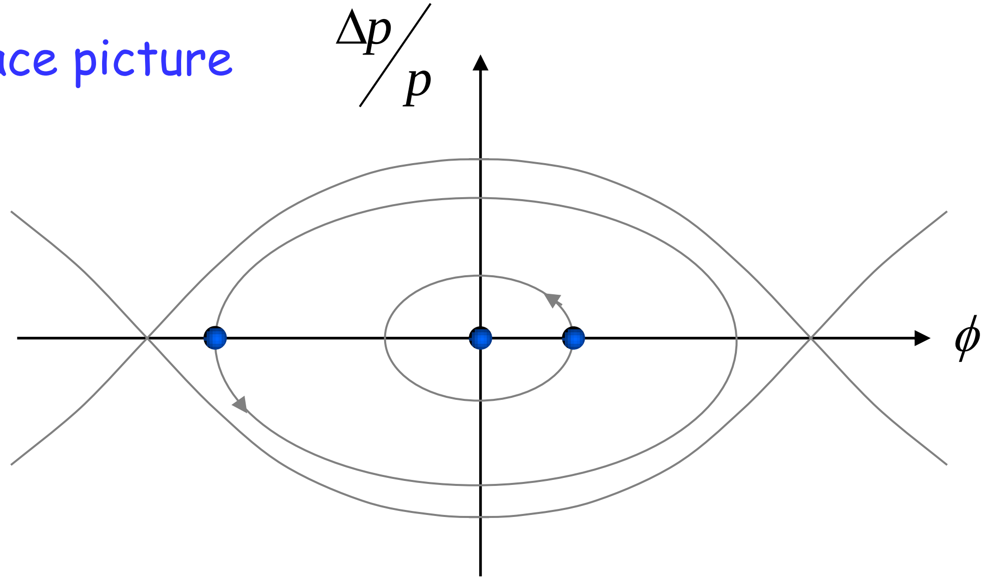
- decrease in energy - decrease in revolution frequency

- The particle arrives later – tends toward ϕ_0

Synchrotron oscillations (2)

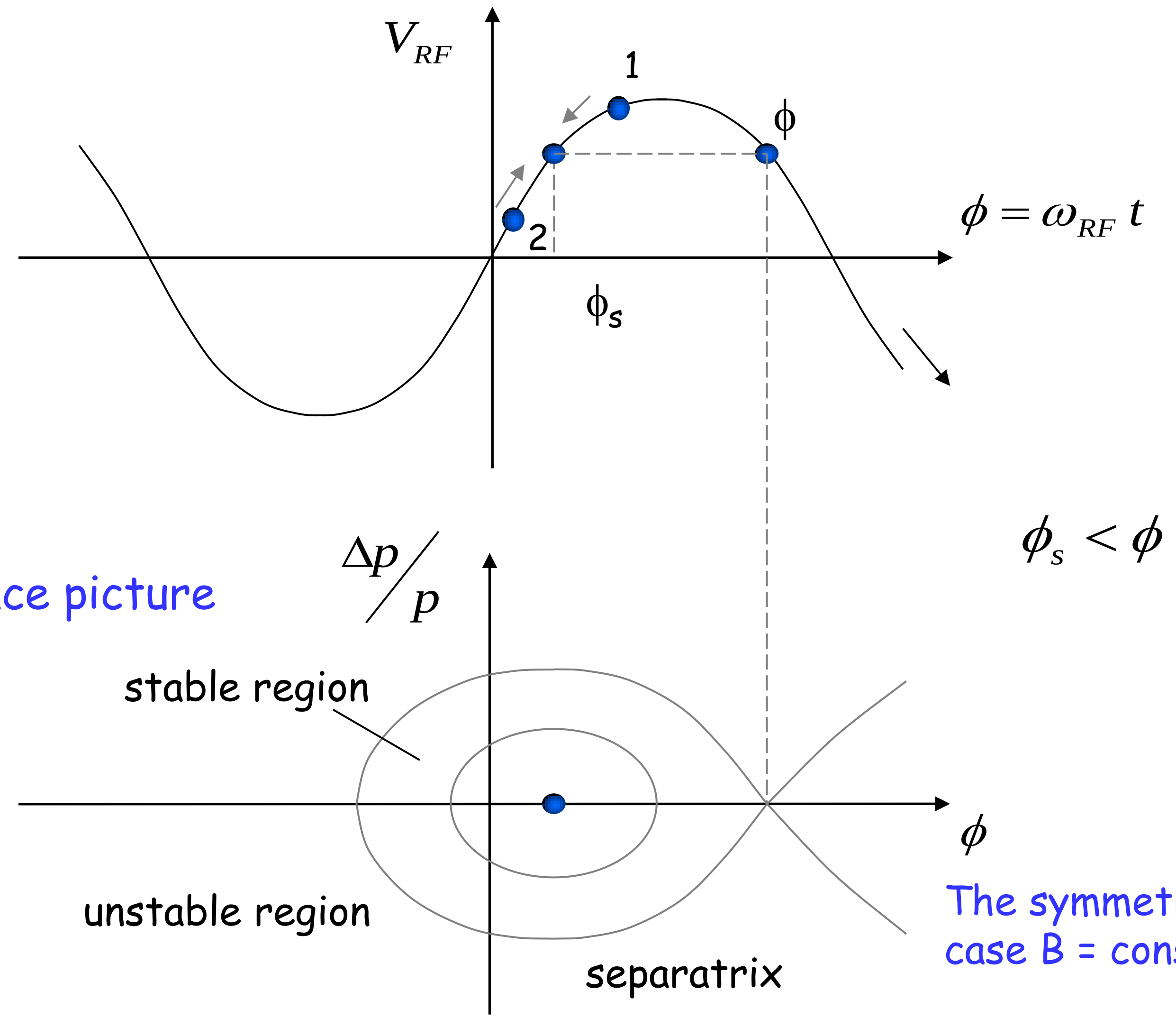


Phase space picture



Synchrotron oscillations (3)

Case with acceleration *B increasing* $\gamma < \gamma_{tr}$



$$\phi_s < \phi < \pi - \phi_s$$

Phase space picture

The symmetry of the case B = const. is lost

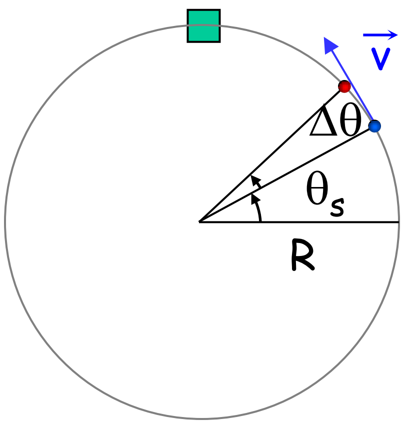
Longitudinal Dynamics in Synchrotrons

It is also often called "synchrotron motion".

The RF acceleration process clearly emphasizes two coupled variables, the energy gained by the particle and the RF phase experienced by the same particle. Since there is a well defined synchronous particle which has always the same phase ϕ_s, and the nominal energy E_s, it is sufficient to follow other particles with respect to that particle.
So let's introduce the following reduced variables:

revolution frequency	:	$\Delta f_r = f_r - f_{rs}$
particle RF phase	:	$\Delta\phi = \phi - \phi_s$
particle momentum	:	$\Delta p = p - p_s$
particle energy	:	$\Delta E = E - E_s$
azimuth angle	:	$\Delta\theta = \theta - \theta_s$

First Energy-Phase Equation



$$f_{RF} = h f_r \quad \Rightarrow \quad \Delta\phi = -h\,\Delta\theta \quad \textit{with} \quad \theta = \int \omega_r\, dt$$

particle ahead arrives earlier
=> smaller RF phase

For a given particle with respect to the reference one:

$$\Delta\omega_r = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h}\frac{d}{dt}(\Delta\phi) = -\frac{1}{h}\frac{d\phi}{dt}$$

Since: $$\eta = \frac{p_s}{\omega_{rs}}\left(\frac{d\omega_r}{dp}\right)_s$$

and

$$E^2 = E_0^2 + p^2c^2$$

$$\Delta E = v_s\,\Delta p = \omega_{rs} R_s\,\Delta p$$

one gets:

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

Second Energy-Phase Equation

The rate of energy gained by a particle is:

$$\frac{dE}{dt} = e\hat{V}\sin\phi\,\frac{\omega_r}{2\pi}$$

The rate of relative energy gain with respect to the reference particle is then:

$$2\pi\Delta\left(\frac{\dot{E}}{\omega_r}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

Expanding the left-hand side to first order:

$$\Delta\left(\dot{E}T_r\right) \cong \dot{E}\,\Delta T_r + T_{rs}\,\Delta\dot{E} = \Delta E\,\dot{T}_r + T_{rs}\,\Delta\dot{E} = \frac{d}{dt}\left(T_{rs}\,\Delta E\right)$$

leads to the second energy-phase equation:

$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}\left(\sin\phi - \sin\phi_s\right)$$

Equations of Longitudinal Motion

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}} \frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}} \dot{\phi}$$

$$2\pi \frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

deriving and combining

$$\frac{d}{dt}\left[\frac{R_s p_s}{h\eta\omega_{rs}} \frac{d\phi}{dt}\right] + \frac{e\hat{V}}{2\pi}(\sin\phi - \sin\phi_s) = 0$$

This second order equation is non linear. Moreover the parameters within the bracket are in general slowly varying with time.

We will study some cases in the following…

Small Amplitude Oscillations

Let's assume constant parameters R_s, p_s, ω_s and η:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

with

$$\Omega_s^2 = \frac{h\eta\omega_{rs} e\hat{V}\cos\phi_s}{2\pi R_s p_s}$$

Consider now small phase deviations from the reference particle:

$$\sin\phi - \sin\phi_s = \sin\left(\phi_s + \Delta\phi\right) - \sin\phi_s \cong \cos\phi_s \Delta\phi$$

(for small $\Delta\phi$)

and the corresponding linearized motion reduces to a harmonic oscillation:

$$\ddot{f} + W_s^2 D f = 0$$

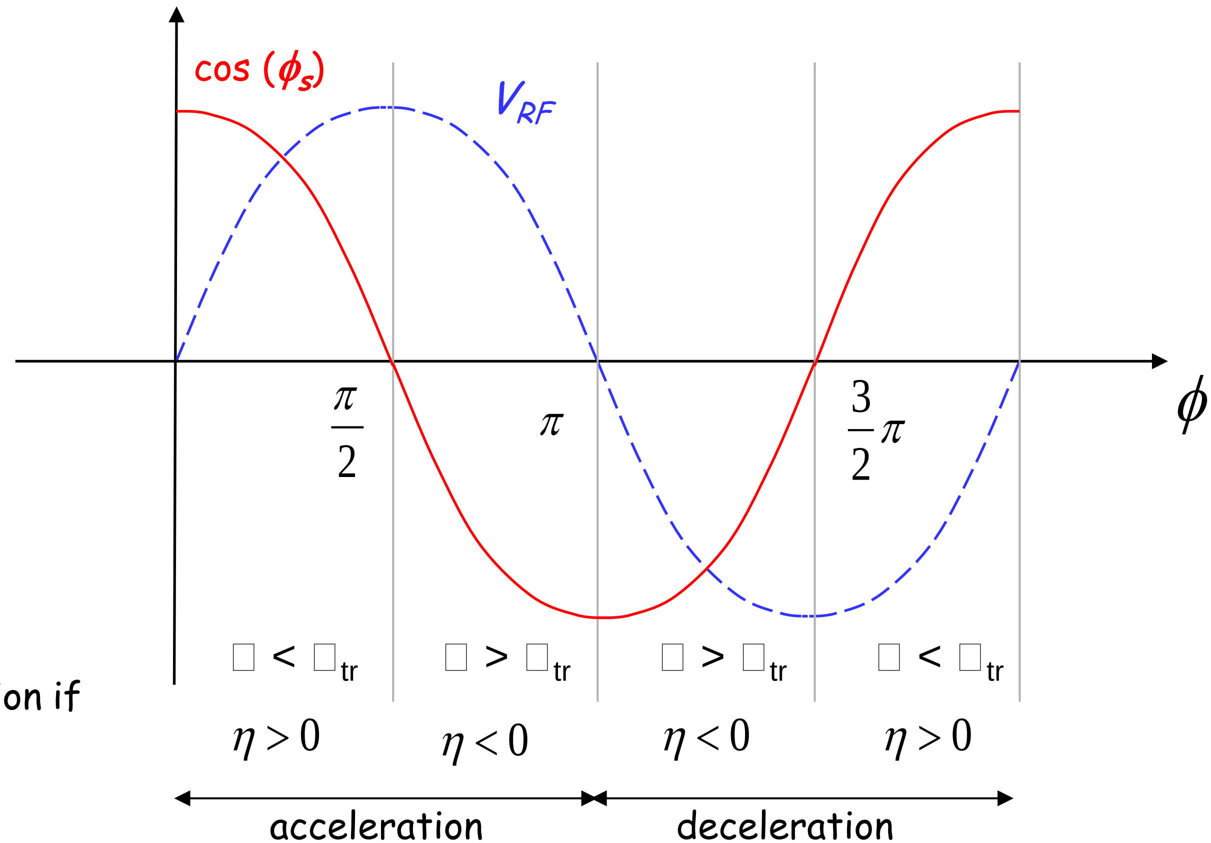
where Ω_s is the synchrotron angular frequency

Stability condition for ϕ_s

Stability is obtained when Ω_s is real and so Ω_s^2 positive:

$$\Omega_s^2 = \frac{e \hat{V}_{RF} \eta h \omega_s}{2\pi R_s p_s} \cos\phi_s \quad \Rightarrow \quad \Omega_s^2 > 0 \quad \Leftrightarrow \quad \eta \cos\phi_s > 0$$

Stable in the region if



Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

(Ω_s as previously defined)

Multiplying by $\dot{\phi}$ and integrating gives an invariant of the motion:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I$$

which for small amplitudes reduces to:

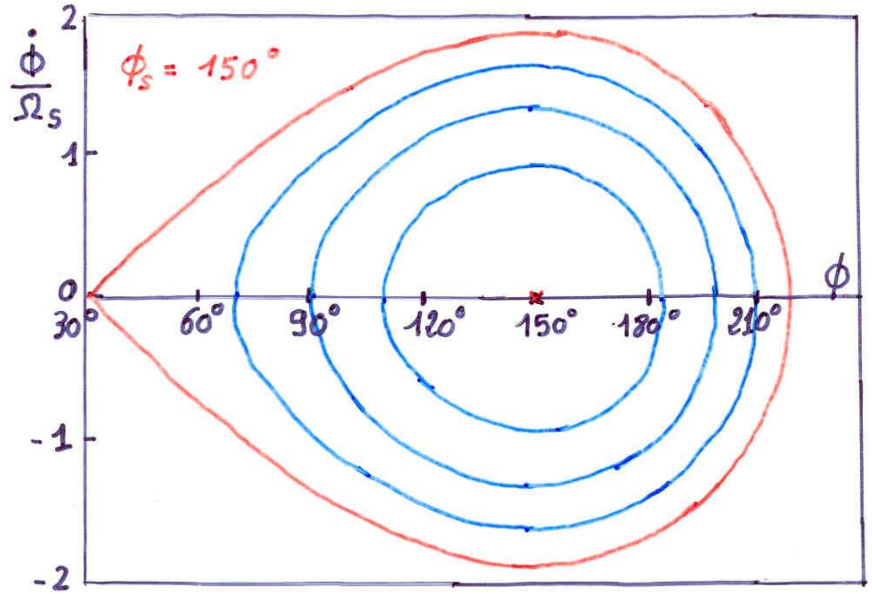
$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\frac{(\Delta\phi)^2}{2} = I'$$

(the variable is $\Delta\phi$, and ϕ_s is constant)

Similar equations exist for the second variable : $\Delta E \propto d\phi/dt$

Large Amplitude Oscillations (2)

When ϕ reaches $\pi-\phi_s$ the force goes to zero and beyond it becomes non restoring.
Hence $\pi-\phi_s$ is an extreme amplitude for a stable motion which in the phase space($\frac{\dot{\phi}}{\Omega_s}, \Delta\phi$) is shown as closed trajectories.



Equation of the separatrix:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}(\cos\phi + \phi\sin\phi_s) = -\frac{\Omega_s^2}{\cos\phi_s}(\cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s)$$

Second value ϕ_m where the separatrix crosses the horizontal axis:

$$\cos\phi_m + \phi_m\sin\phi_s = \cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s$$

Area within this separatrix is called “RF bucket”.

Energy Acceptance

From the equation of motion it is seen that $\dot{\phi}$ reaches an extreme when $\ddot{\phi} = 0$, hence corresponding to $\phi = \phi_s$.
Introducing this value into the equation of the separatrix gives:

$$\dot{\phi}^2_{max} = 2\Omega_s^2 \left\{ 2 + \left(2\phi_s - \pi\right) \tan\phi_s \right\}$$

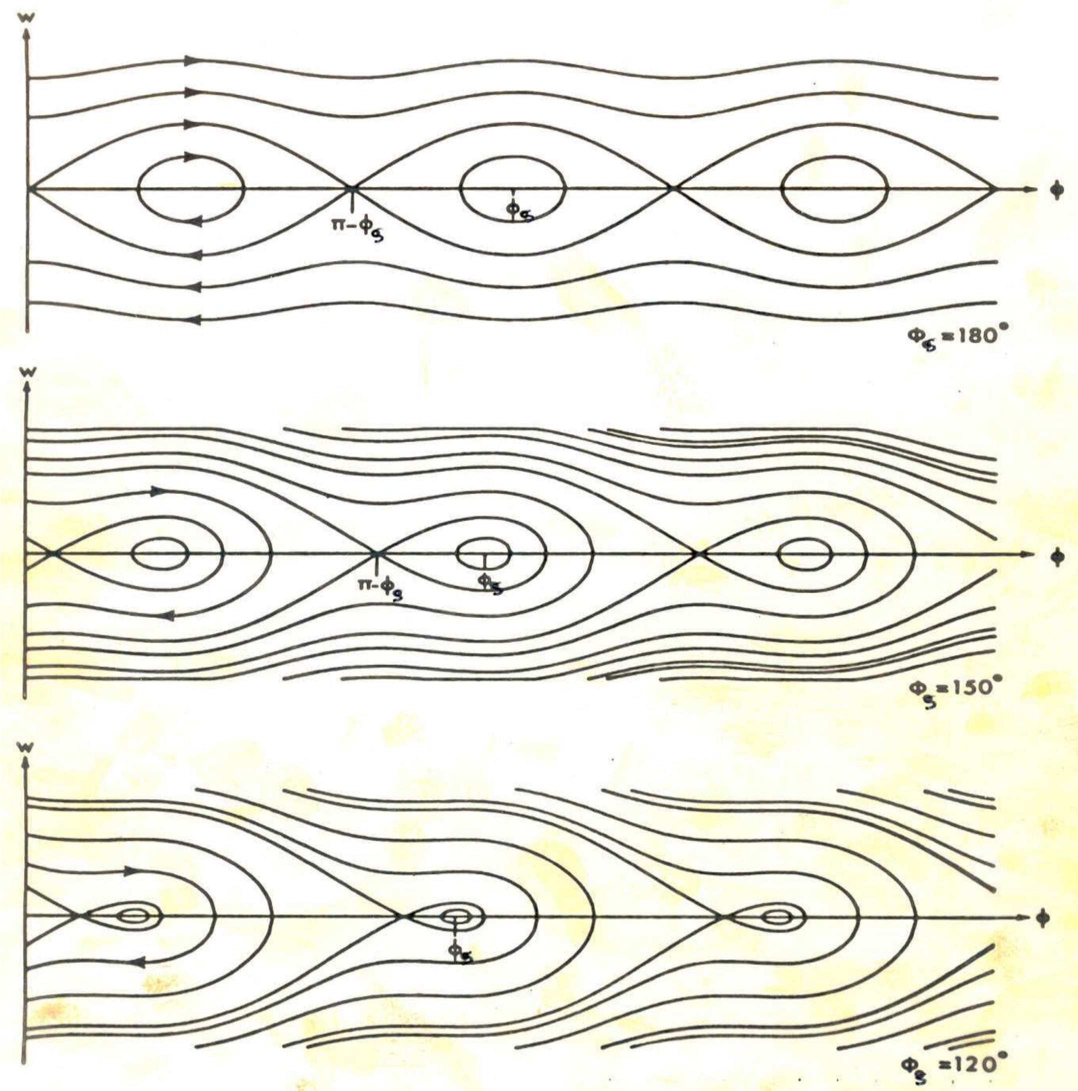
That translates into an acceptance in energy:

$$\left(\frac{\Delta E}{E_s}\right)_{max} = \mp\beta \sqrt{-\frac{e\hat{V}}{\pi h \eta E_s} G\left(\phi_s\right)}$$

$$G\left(\phi_s\right) = \left[2\cos\phi_s + \left(2\phi_s - \pi\right)\sin\phi_s\right]$$

This "RF acceptance" depends strongly on ϕ_s and plays an important role for the capture at injection, and the stored beam lifetime.

RF Acceptance versus Synchronous Phase



The areas of stable motion (closed trajectories) are called "BUCKET".

As the synchronous phase gets closer to 90° the buckets gets smaller.

The number of circulating buckets is equal to "h".

The phase extension of the bucket is maximum for ϕ_s =180° (or 0°) which correspond to no acceleration . The RF acceptance increases with the RF voltage.

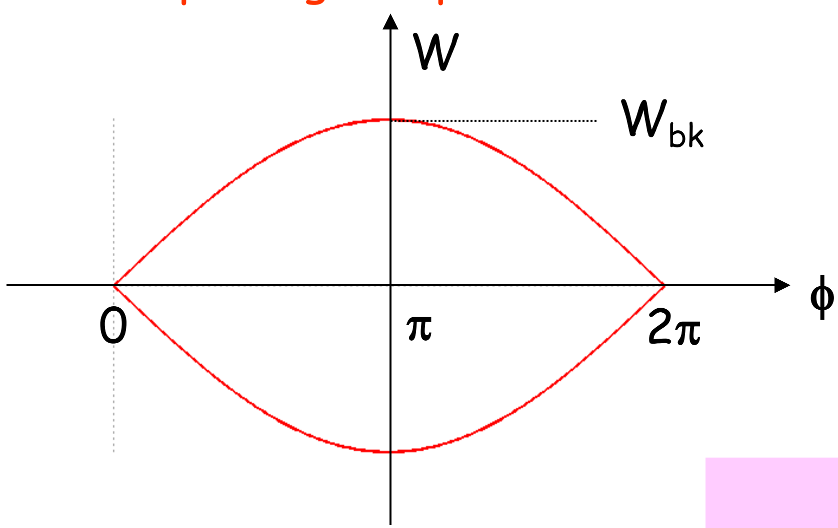
Stationnary Bucket - Separatrix

This is the case $\sin\phi_s=0$ (no acceleration) which means $\phi_s=0$ or π . The equation of the separatrix for $\phi_s= \pi$ (above transition) becomes:

$$\frac{\dot{\phi}^2}{2}+\Omega_s^2\cos\phi=\Omega_s^2$$

$$\frac{\dot{\phi}^2}{2}=2\Omega_s^2\sin^2\frac{\phi}{2}$$

Replacing the phase derivative by the (canonical) variable W:



$$W=2\pi\frac{\Delta E}{\omega_{rs}}=-2\pi\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

and introducing the expression for Ω_s leads to the following equation for the separatrix:

$$W=\pm 2\frac{C}{c}\sqrt{\frac{-e\hat{V}E_s}{2phh}}\sin\frac{f}{2}=\pm W_{bk}\sin\frac{f}{2}$$

with $C=2\pi R_s$

Stationnary Bucket (2)

Setting ϕ=π in the previous equation gives the height of the bucket:

$$W_{bk} = 2\frac{C}{c}\sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}}$$

This results in the maximum energy acceptance:

$$DE_{\max} = \frac{w_{rs}}{2p}W_{bk} = b_s\sqrt{2\frac{-e\hat{V}_{RF}E_s}{phh}}$$

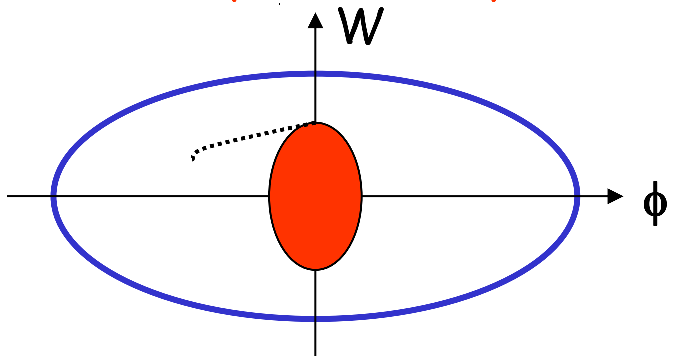
The area of the bucket is: $A_{bk} = 2\int_0^{2\pi} W\,d\phi$

Since: $\int_0^{2\pi}\sin\frac{\phi}{2}\,d\phi = 4$

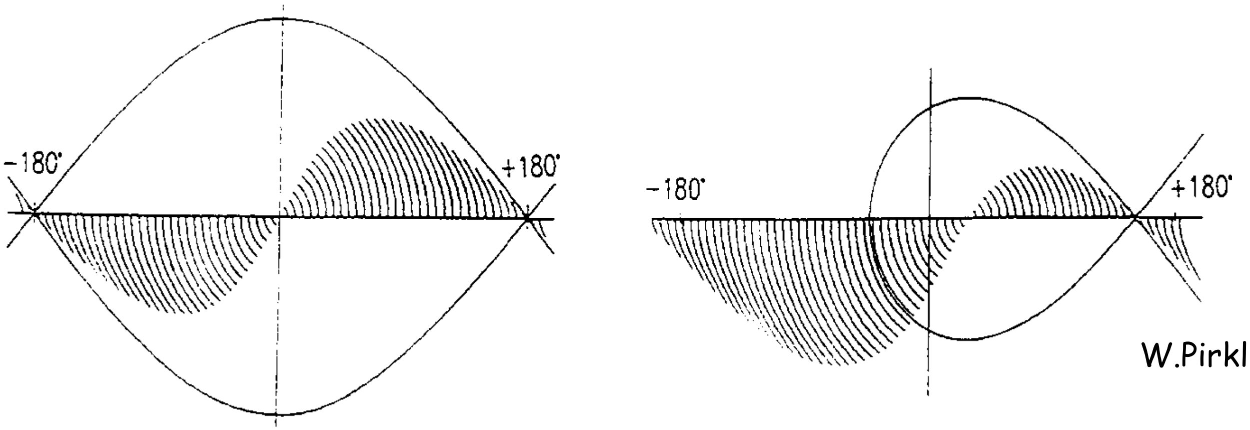
one gets: $$A_{bk} = 8W_{bk} = 16\frac{C}{c}\sqrt{\frac{-e\hat{V}E_s}{2phh}}$$ → $$W_{bk} = \frac{A_{bk}}{8}$$

Effect of a Mismatch

Injected bunch: short length and large energy spread
after 1/4 synchrotron period: longer bunch with a smaller energy spread.



For larger amplitudes, the angular phase space motion is slower
(1/8 period shown below) => can lead to filamentation and emittance growth

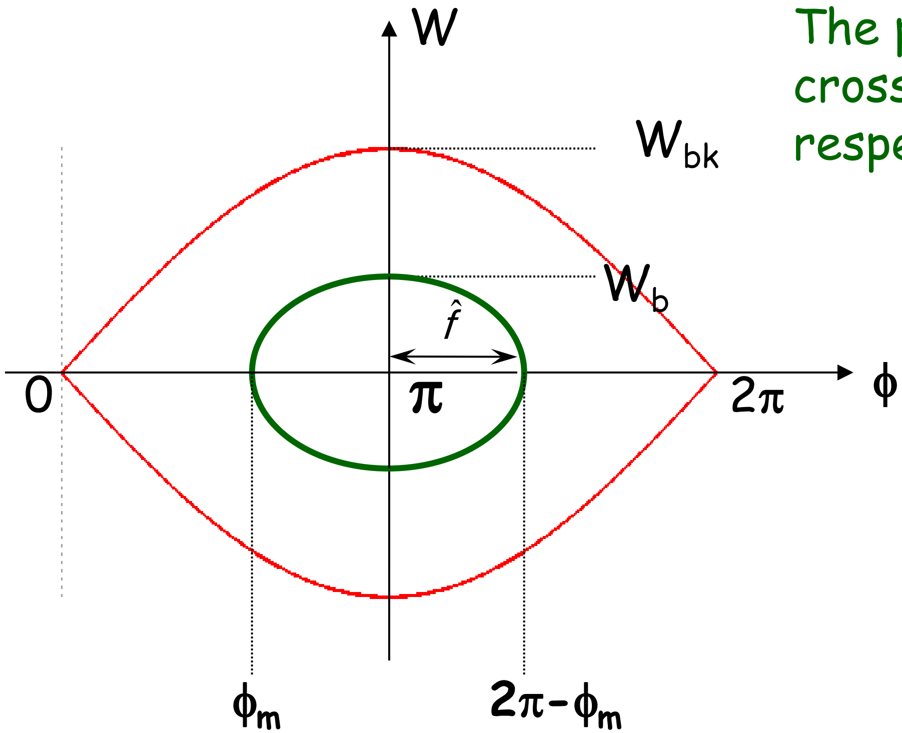


stationary bucket accelerating bucket

Bunch Matching into a Stationnary Bucket

A particle trajectory inside the separatrix is described by the equation:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I \quad \xrightarrow{\phi_s = \pi} \quad \frac{\dot{\phi}^2}{2} + \Omega_s^2\cos\phi = I$$



The points where the trajectory crosses the axis are symmetric with respect to $\phi_s = \pi$

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\cos\phi = \Omega_s^2\cos\phi_m$$

$$\dot{\phi} = \pm\Omega_s\sqrt{2\left(\cos\phi_m - \cos\phi\right)}$$

$$W = \pm W_{bk}\sqrt{\cos^2\frac{j_m}{2} - \cos^2\frac{j}{2}}$$

$$\cos(f) = 2\cos^2\frac{f}{2} - 1$$

Bunch Matching into a Stationnary Bucket (2)

Setting $\phi = \pi$ in the previous formula allows to calculate the bunch height:

$$W_b = W_{bk}\cos\frac{\phi_m}{2} = W_{bk}\sin\frac{\hat{\phi}}{2}$$

or:

$$W_b = \frac{A_{bk}}{8}\cos\frac{\phi_m}{2}$$

$$\longrightarrow \quad \left(\frac{\Delta E}{E_s}\right)_b = \left(\frac{\Delta E}{E_s}\right)_{RF}\cos\frac{\phi_m}{2} = \left(\frac{\Delta E}{E_s}\right)_{RF}\sin\frac{\hat{\phi}}{2}$$

This formula shows that for a given bunch energy spread the proper matching of a shorter bunch (ϕ_m close to π, $\hat{\phi}$ small) will require a bigger RF acceptance, hence a higher voltage

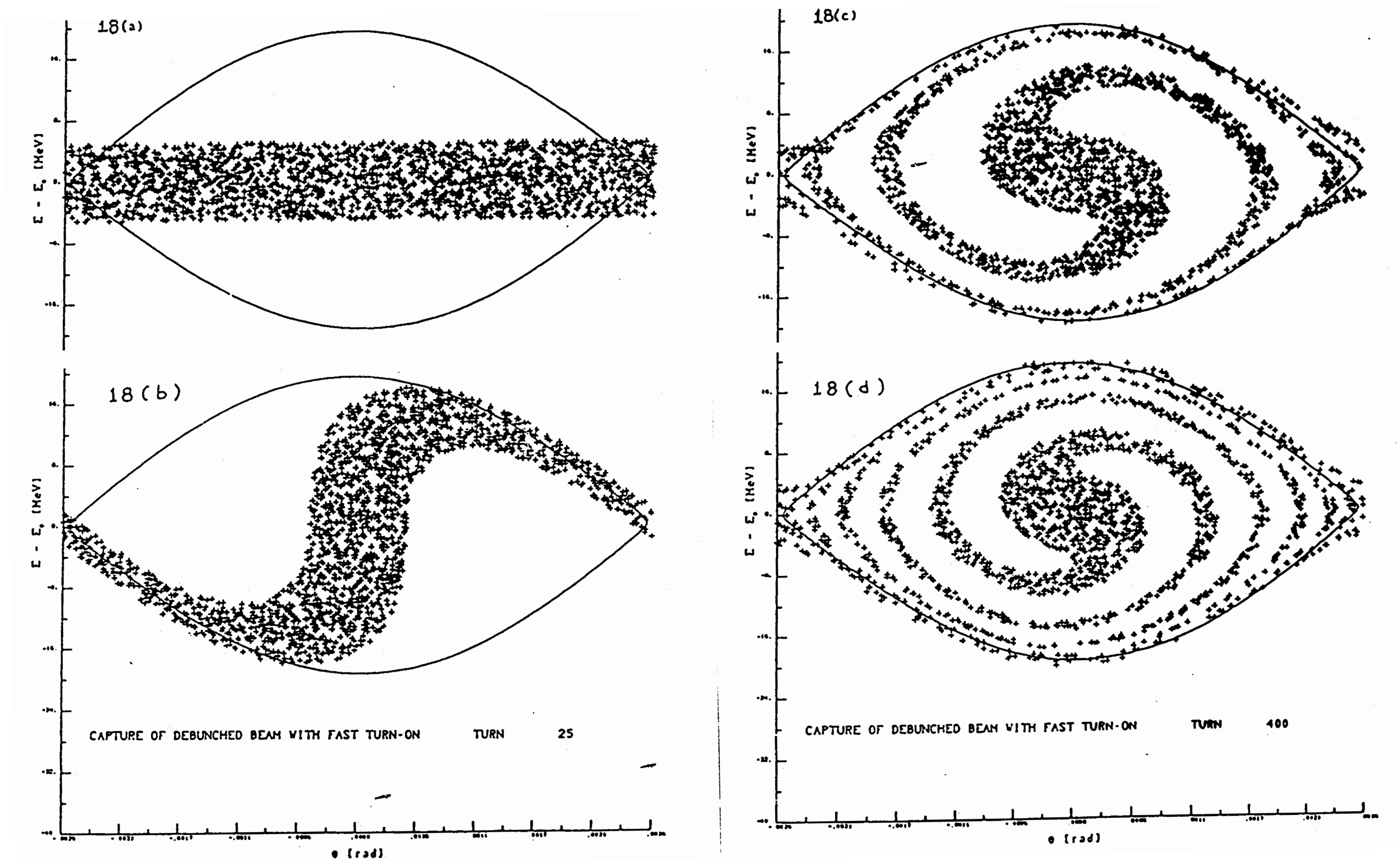
For small oscillation amplitudes the equation of the ellipse reduces to:

$$W = \frac{A_{bk}}{16}\sqrt{\hat{\phi}^2 - (\Delta\phi)^2} \quad \longrightarrow \quad \left(\frac{16\,W}{A_{bk}\hat{\phi}}\right)^2 + \left(\frac{\Delta\phi}{\hat{\phi}}\right)^2 = 1$$

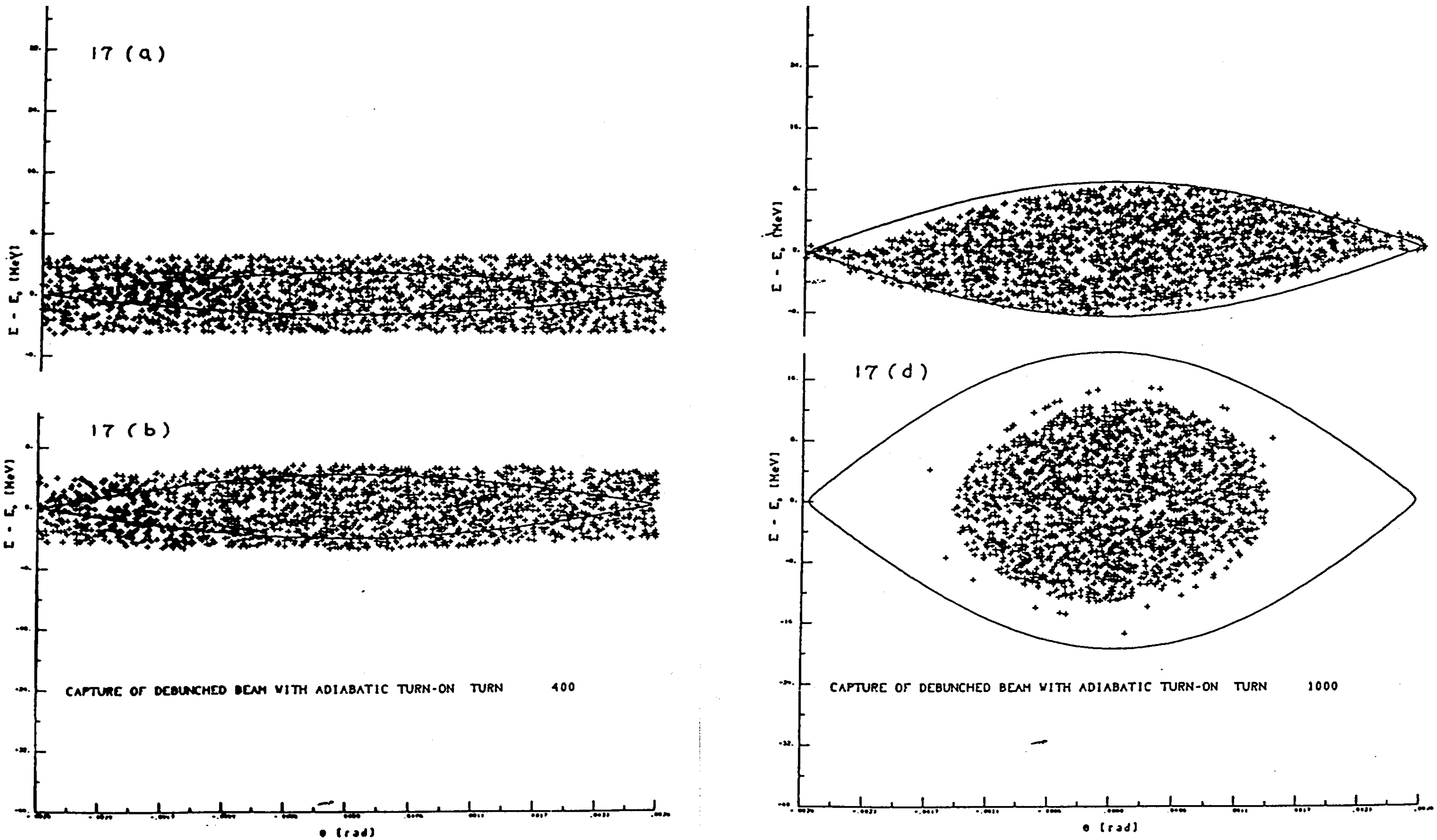
Ellipse area is called longitudinal emittance

$$A_b = \frac{\pi}{16}A_{bk}\hat{\phi}^2$$

Capture of a Debunched Beam with Fast Turn-On



Capture of a Debunched Beam with Adiabatic Turn-On

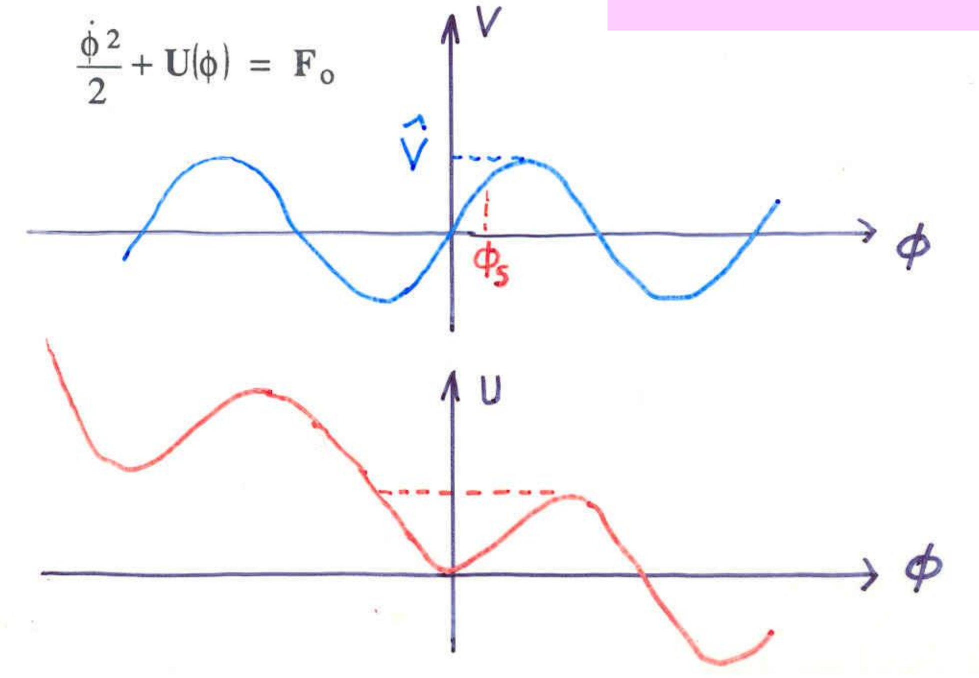


Potential Energy Function

The longitudinal motion is produced by a force that can be derived from a scalar potential:

$$\frac{d^2\phi}{dt^2} = F(\phi) \qquad F(\phi) = -\frac{\partial U}{\partial \phi}$$

$$U = -\int_0^\phi F(\phi)\, d\phi = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) - F_0$$



The sum of the potential energy and kinetic energy is constant and by analogy represents the total energy of a non-dissipative system.

Hamiltonian of Longitudinal Motion

Introducing a new convenient variable, W, leads to the 1st order equations:

$$W = 2\pi\left(\frac{\Delta E}{\omega_{rs}}\right) = 2\pi R_s \Delta p \quad \longrightarrow \quad \begin{aligned} \frac{d\phi}{dt} &= -\frac{1}{2\pi}\frac{h\eta\omega_{rs}}{p_s R_s} W \\ \frac{dW}{dt} &= e\hat{V}(\sin\phi - \sin\phi_s) \end{aligned}$$

The two variables ϕ,W are canonical since these equations of motion can be derived from a Hamiltonian H(ϕ,W,t):

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial W} \qquad \frac{dW}{dt} = -\frac{\partial H}{\partial \phi}$$

$$H(\phi, W, t) = e\hat{V}[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s] - \frac{1}{4\pi}\frac{h\eta\omega_{rs}}{R_s p_s} W^2$$

Adiabatic Damping

Though there are many physical processes that can damp the longitudinal oscillation amplitudes, one is directly generated by the acceleration process itself. It will happen in the synchrotron, even ultra-relativistic, when ramping the energy but not in the ultra-relativistic electron linac which does not show any oscillation.

As a matter of fact, when E_s varies with time, one needs to be more careful in combining the two first order energy-phase equations in one second order equation:

$$\frac{d}{dt}\left(E_s\dot{\phi}\right) = -\Omega_s^2 E_s \Delta\phi$$

$$E_s\ddot{\phi} + \dot{E}_s\dot{\phi} + \Omega_s^2 E_s \Delta\phi = 0$$

$$\ddot{\phi} + \frac{\dot{E}_s}{E_s}\dot{\phi} + \Omega_s^2\left(E_s\right)\Delta\phi = 0$$

The damping coefficient is proportional to the rate of energy variation and from the definition of Ω_s one has:

$$\frac{\dot{E}_s}{E_s} = -2\frac{\dot{\Omega}_s}{\Omega_s}$$

Adiabatic Damping (2)

So far it was assumed that parameters related to the acceleration process were constant. Let's consider now that they vary slowly with respect to the period of longitudinal oscillation (adiabaticity).

For small amplitude oscillations the hamiltonian reduces to:

$$H(\phi,W,t) \cong -\frac{e\hat{V}}{2}\cos\phi_s (\Delta\phi)^2 - \frac{1}{4\pi}\frac{h\eta\omega_{rs}}{R_s p_s}W^2$$

with

$$W = \hat{W}\cos\Omega_s t$$

$$\Delta\phi = (\Delta\hat{\phi})\sin\Omega_s t$$

Under adiabatic conditions the Boltzman-Ehrenfest theorem states that the action integral remains constant:

$$I = \oint W\, d\phi = const.$$

(W, ϕ are canonical variables)

Since:

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial W} = -\frac{1}{2\pi}\frac{h\eta\omega_{rs}}{R_s p_s}W$$

the action integral becomes:

$$I = \oint W\frac{d\phi}{dt}dt = -\frac{1}{2\pi}\frac{h\eta\omega_{rs}}{R_s p_s}\oint W^2 dt$$

Adiabatic Damping (3)

Previous integral over one period:

$$\oint W^2 dt = \pi \frac{\hat{W}^2}{\Omega_s}$$

leads to:

$$I = -\frac{h\eta\omega_{rs}}{2R_s p_s}\frac{\hat{W}^2}{\Omega_s} = const.$$

From the quadratic form of the hamiltonian one gets the relation:

$$\hat{W} = \frac{2\pi p_s R_s \Omega_s}{h\eta\omega_{rs}}\Delta\hat{\phi}$$

Finally under adiabatic conditions the long term evolution of the oscillation amplitudes is shown to be:

$$\Delta\hat{\phi} \propto \left[\frac{\eta}{E_s R_s^2 \hat{V}\cos\phi_s}\right]^{1/4} \propto E_s^{-1/4}$$

$$\hat{W} \; or \; \Delta\hat{E} \propto E_s^{1/4}$$

$$\hat{W} \cdot \Delta\hat{\phi} = invariant$$

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