Lattice Design in Particle Accelerators
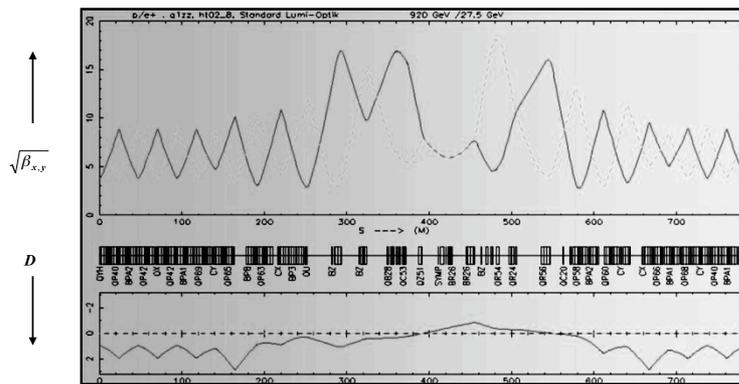## Bernhard Holzer, CERN

1952: Courant, Livingston, Snyder:
Theory of strong focusing in particle beams

Lattice Design: „... *how to build a storage ring*"

High energy accelerators → circular machines

somewhere in the lattice we need a number of dipole magnets,
that are bending the design orbit to a closed ring

Geometry of the ring:

centrifugal force = Lorentz force

$$e * v * B = \frac{mv^2}{\rho}$$

$$\rightarrow e * B = \frac{mv}{\rho} = p / \rho$$

$$\rightarrow B * \rho = p / e$$

p = momentum of the particle,
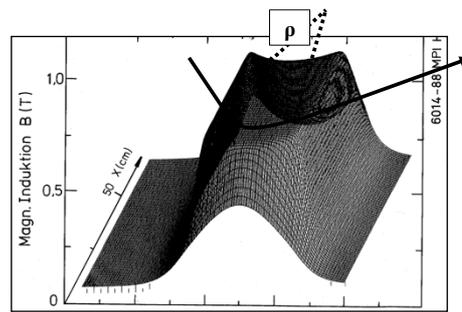*ρ = curvature radius*

Bρ = beam rigidity

Example: heavy ion storage ring TSR
8 dipole magnets of equal bending strength

1.) Circular Orbit:

„… defining the geometry"

field map of a storage ring dipole magnet

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho}$$

$$\alpha = \frac{B * dl}{B * \rho}$$

The angle swept out in one revolution
must be 2π, so

$$\alpha = \frac{\int B\,dl}{B * \rho} = 2\pi \qquad \rightarrow \qquad \int B\,dl = 2\pi * \frac{p}{q} \qquad \text{… for a full circle}$$

Nota bene: $\quad \frac{\Delta B}{B} \approx 10^{-4} \qquad$ *is usually required !!*

Example LHC:

7000 GeV Proton storage ring
dipole magnets $N = 1232$
$l = 15$ m
$q = +1$ e

$$\int B\,dl \approx N\,l\,B = 2\pi\; p\,/\,e$$

$$B \approx \frac{2\pi \;\; 7000 \;\; 10^9 \, eV}{1232 \;\; 15 \, m \;\; 3 \; 10^8 \, \frac{m}{s} \;\; e} = 8.3 \; \textbf{Tesla}$$
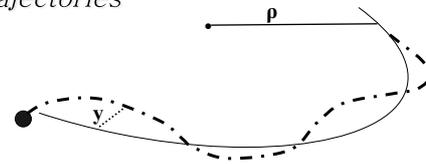
„ Focusing forces ⋯ single particle trajectories"

$$y'' + K * y = 0$$

$$K = -k + 1/\rho^2 \quad \text{hor. plane}$$
$$K = k \quad \quad \text{vert. plane}$$

dipole magnet $\quad \dfrac{1}{\rho} = \dfrac{B}{p/q}$

quadrupole magnet $\quad k = \dfrac{g}{p/q}$

Example: HERA Ring:
Bending radius: $\rho = 580$ m
Quadrupol Gradient: $g = 110$ T/m

$k = 33.64 * 10^{-3} /m^2$
$1/\rho^2 = 2.97 * 10^{-6} /m^2$

For estimates in large accelerators the weak focusing term $1/\rho^2$ can
in general be neglected

Solution for a focusing magnet

$$y(s) = y_0 * \cos(\sqrt{K} * s) + \frac{y'_0}{\sqrt{K}} * \sin(\sqrt{K} * s)$$

$$y'(s) = -y_0 * \sqrt{K} * \sin(\sqrt{K} * s) + y'_0 * \cos(\sqrt{K} * s)$$

Or written more convenient in matrix form:

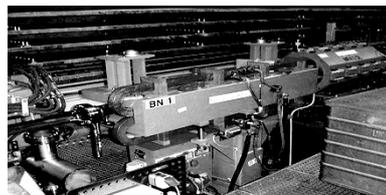$$\begin{pmatrix} y \\ y' \end{pmatrix}_s = M * \begin{pmatrix} y \\ y' \end{pmatrix}_0$$

Hor. focusing Quadrupole Magnet

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l) & \frac{1}{\sqrt{K}}\sin(\sqrt{K} * l) \\ -\sqrt{K}\sin(\sqrt{K} * l) & \cos(\sqrt{K} * l) \end{pmatrix}$$

Hor. defocusing Quadrupole Magnet

$$M_{QD} = \begin{pmatrix} \cosh(\sqrt{K} * l) & \frac{1}{\sqrt{K}}\sinh(\sqrt{K} * l) \\ \sqrt{K}\sinh(\sqrt{K} * l) & \cosh(\sqrt{K} * l) \end{pmatrix}$$

Drift space $\quad M_{Drift} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$

$$\boxed{M_{lattice} = M_{QF1} * M_{D1} * M_{QD} * M_{D1} * M_{QF2} \cdots}$$

2.) Reminder: Beam Dynamics Language

Transfer Matrix M

describes the transformation of amplitude x and angle x' through a number of lattice elements

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

... and can be expressed by the optics parameters

$$M = \begin{pmatrix} \sqrt{\dfrac{\beta_s}{\beta_0}} \left(\cos\psi_s + \alpha_0 \sin\psi_s \right) & \sqrt{\beta_s \beta_0} \sin\psi_s \\[2mm] \dfrac{(\alpha_0 - \alpha_s)\cos\psi_s - (1 + \alpha_0 \alpha_s)\sin\psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\dfrac{\beta_0}{\beta s}} \left(\cos\psi_s - \alpha_s \sin\psi_s \right) \end{pmatrix}$$

* we can calculate the single particle trajectories between two locations in the ring, if we know the α β γ at these positions.
* and nothing but the α β γ at these positions.
*   ... !

Periodic Lattices

In the case of periodic lattices the transfer matrix can be expressed as a function of a set of periodic parameters α, β, γ

$$M(s) = \begin{pmatrix} \cos\psi_{period} + \alpha_s \sin\psi_{period} & \beta_s \sin\psi_{period} \\ -\gamma_s \sin\psi_{period} & \cos\psi_{period} - \alpha_s \sin\psi_{period} \end{pmatrix}$$

$$\psi_{period} = \int\limits_{s}^{s+L} \frac{ds}{\beta(s)}$$

ψ = phase advance per period:

For stability of the motion in periodic lattice structures it is required that

$$\left| trace(M) \right| < 2$$

In terms of these new periodic parameters the solution of the equation of motion is

$$y(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\Phi(s) - \delta)$$

$$y'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta}} * \left\{ \sin(\Phi(s) - \delta) + \alpha \cos(\Phi(s) - \delta) \right\}$$

Transformation of α, β, γ

consider two positions in the storage ring: s_0 , s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$
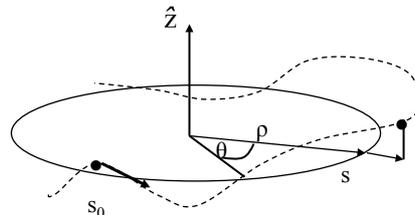
since ε = const:

$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$\varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0^2$$

express x_0, x'_0 as a function of x, x'.

... remember $W = m_{11} m_{22} - m_{12} m_{21} = 1$

$$\left. \begin{aligned} \begin{pmatrix} x \\ x' \end{pmatrix}_0 &= M^{-1} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_s \\ M^{-1} &= \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix} \end{aligned} \right\} \longrightarrow \begin{aligned} x_0 &= m_{22} x - m_{12} x' \\ x_0' &= -m_{21} x + m_{11} x' \end{aligned}$$

inserting into ε $\quad \varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$

$$\varepsilon = \beta_0 (m_{11} x' - m_{21} x)^2 + 2\alpha_0 (m_{22} x - m_{12} x')(m_{11} x' - m_{21} x) + \gamma_0 (m_{22} x - m_{12} x')^2$$

sort via x, x´and compare the coefficients to get ....

The new parameters α, β, γ can be transformed through the lattice via the lattice matrix elements defined above.

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{12}m_{21} + m_{22}m_{11} & -m_{12}m_{22} \\ m_{12}^2 & -2m_{22}m_{21} & m_{22}^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s1}$$

the optical parameters depend on the focusing properties of the lattice,
... and can be optimised accordingly !!!

... and here starts the lattice design !!!

Most simple example: drift space

$$M_{drift} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

particle coordinates

$$\begin{pmatrix} x \\ x' \end{pmatrix}_l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \end{pmatrix}_0 \qquad \rightarrow \qquad \boxed{\begin{aligned} x(l) &= x_0 + l * x_0' \\ x'(l) &= x_0' \end{aligned}}$$
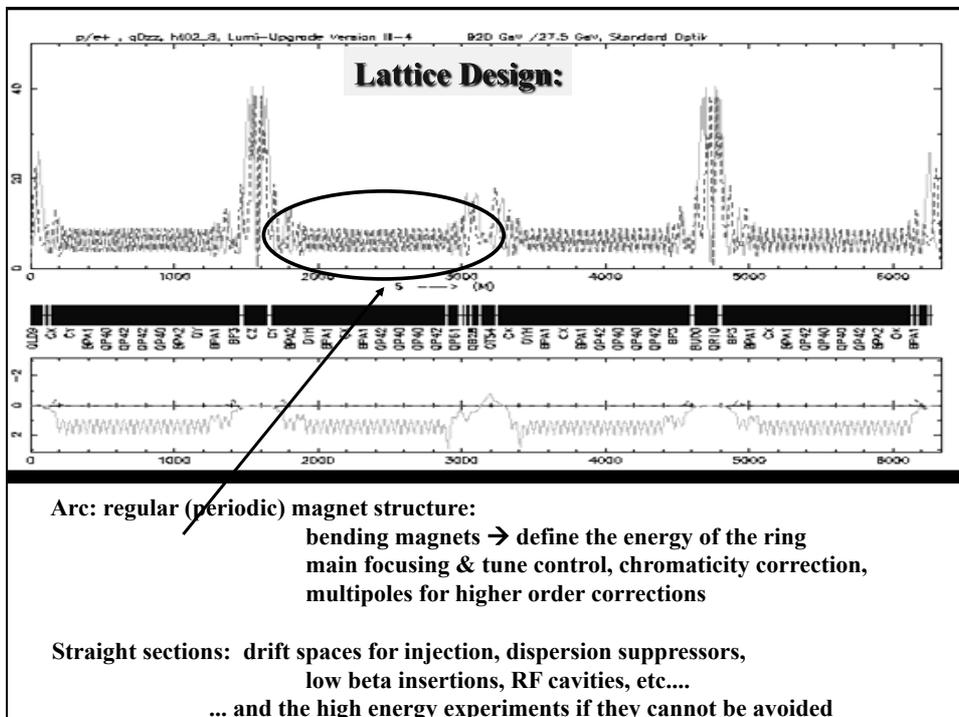
transformation of twiss parameters:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_l = \begin{pmatrix} 1 & -2l & l^2 \\ 0 & 1 & -l \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0 \qquad \boxed{\beta(s) = \beta_0 - 2l * \alpha_0 + l^2 * \gamma_0}$$

Stability ...?

$$trace(M) = 1 + 1 = 2$$

→A periodic solution doesn't exist in a lattice built exclusively out of drift spaces.

Lattice Design:

p/e+ , qDzzz, h602_8, Lumi-Upgrade Version III-4   920 GeV /27.5 GeV, Standard Optik

Arc: regular (periodic) magnet structure:
            bending magnets → define the energy of the ring
            main focusing & tune control, chromaticity correction,
            multipoles for higher order corrections

Straight sections: drift spaces for injection, dispersion suppressors,
            low beta insertions, RF cavities, etc....
   ... and the high energy experiments if they cannot be avoided

3.) The FoDo-Lattice

A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in between.
(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)
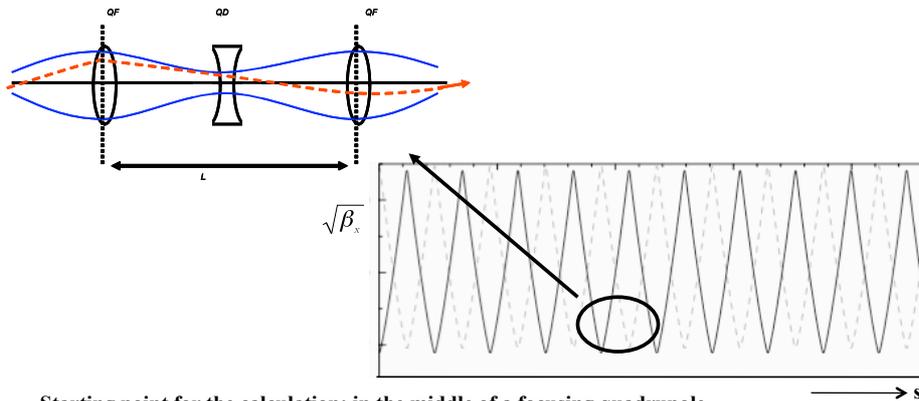
Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell µ = 45°,
→ calculate the twiss parameters for a periodic solution

Periodic Solution of a FoDo Cell
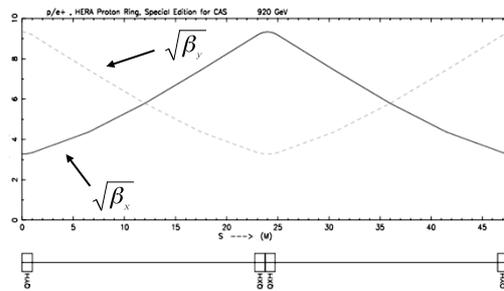
Output of the optics program:

Nr	Type	Length	Strength	β_x	α_x	φ_x	β_z	α_z	φ_z
		m	1/m2	m		1/2π	m		1/2π
0	IP	0,000	0,000	11,611	0,000	0,000	5,295	0,000	0,000
1	QFH	0,250	-0,541	11,228	1,514	0,004	5,488	-0,781	0,007
2	QD	3,251	0,541	5,488	-0,781	0,070	11,228	1,514	0,066
3	QFH	6,002	-0,541	11,611	0,000	0,125	5,295	0,000	0,125
4	IP	6,002	0,000	11,611	0,000	0,125	5,295	0,000	0,125

QX= 0,125   QZ= 0,125

$$0.125 * 2\pi = 45^0$$

7

Can we understand what the optics code is doing ?

matrices

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K}*l_q) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}*l_q) \\ -\sqrt{K}\sin(\sqrt{K}*l_q) & \cos(\sqrt{K}*l_q) \end{pmatrix} \, , \qquad M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1_d \end{pmatrix}$$

strength and length of the FoDo elements

K = +/- 0.54102 m⁻²
lq = 0.5 m
ld = 2.5 m

The matrix for the complete cell is obtained by multiplication of the element matrices

$$M_{FoDo} = M_{qfh} * M_{ld} * M_{qd} * M_{ld} * M_{qf}$$

Putting the numbers in and multiplying out ...

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$

The transfer matrix for 1 period gives us all the information that we need !

1.) is the motion stable? $trace(M_{FoDo}) = 1.415 \rightarrow$ < 2

2.) Phase advance per cell

$$M(s) = \begin{pmatrix} \cos\psi_{cell} + \alpha_x\sin\psi_{cell} & \beta_x\sin\psi_{cell} \\ -\gamma_x\sin\psi_{cell} & \cos\psi_{cell} - \alpha_x\sin\psi_{cell} \end{pmatrix}$$

$$\cos\psi_{cell} = \frac{1}{2}trace(M) = 0.707$$

$$\psi_{cell} = \cos^{-1}\left(\frac{1}{2}trace(M)\right) = 45$$

3.) hor β-function

$$\beta = \frac{m_{12}}{\sin\psi_{cell}} = 11.611\,m$$

4.) hor α-function

$$\alpha = \frac{m_{11} - \cos\psi_{cell}}{\sin\psi_{cell}} = 0$$

Matrix of a focusing quadrupole magnet:

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K}*l) & \dfrac{1}{\sqrt{K}}\sin(\sqrt{K}*l) \\ -\sqrt{K}\sin(\sqrt{K}*l) & \cos(\sqrt{K}*l) \end{pmatrix}$$

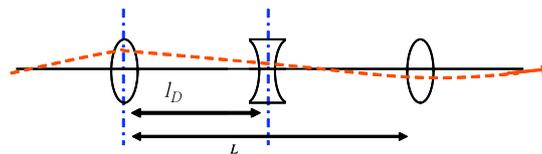**If the focal length f is much larger than the length of the quadrupole magnet,**

$$f = \dfrac{1}{kl_Q} \gg l_Q$$

the transfer matrix can be aproximated using ♪ $\quad kl_q = const,\ l_q \to 0$

$$M = \begin{pmatrix} 1 & 0 \\ \dfrac{1}{f} & 1 \end{pmatrix}$$

4.) FoDo in thin lens approximation

$$l_D = L/2,$$
$$\tilde{f} = 2f$$

Calculate the matrix for a half cell, starting in the middle of a foc. quadrupole:

$$M_{half\ Cell} = M_{QD2} * M_{lD} * M_{QF/2}$$

$$M_{half\ Cell} = \begin{pmatrix} 1 & 0 \\ \dfrac{1}{\tilde{f}} & 1 \end{pmatrix} * \begin{pmatrix} 1 & l_D \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{\tilde{f}} & 1 \end{pmatrix}$$

note: $\tilde{f}$ denotes the focusing strength of half a quadrupole, so $\tilde{f} = 2f$

$$M_{half\ Cell} = \begin{pmatrix} 1 - \dfrac{l_D}{\tilde{f}} & l_D \\ -\dfrac{l_D}{\tilde{f}^2} & 1 + \dfrac{l_D}{\tilde{f}} \end{pmatrix}$$

for the second half cell set $f \to -f$

9

FoDo in thin lens approximation

Matrix for the complete FoDo cell

$$M = \begin{pmatrix} 1 + l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 - l_D/\tilde{f} \end{pmatrix} * \begin{pmatrix} 1 - l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 + l_D/\tilde{f} \end{pmatrix}$$

$$M = \begin{pmatrix} 1 - \dfrac{2l_D^2}{\tilde{f}^2} & 2l_D(1 + \dfrac{l_D}{\tilde{f}}) \\ 2(\dfrac{l_D^2}{\tilde{f}^3} - \dfrac{l_D}{\tilde{f}^2}) & 1 - 2\dfrac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

Now we know, that the phase advance is related to the transfer matrix by

$$\cos\psi_{cell} = \frac{1}{2}trace(M) = \frac{1}{2}*(2 - \frac{4l_d^2}{\tilde{f}^2}) = 1 - \frac{2l_d^2}{f^2}$$

After some beer and with a little bit of trigonometric gymnastics

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2) = 1 - 2\sin^2(x/2)$$

---

we can calculate the phase advance as a function of the FoDo parameter …

$$\cos\psi_{cell} = 1 - 2\sin^2(\psi_{cell}/2) = 1 - \frac{2l_d^2}{f^2}$$

$$\sin(\psi_{cell}/2) = l_d/\tilde{f} = \frac{L_{cell}}{2f}$$

$$\sin(\psi_{cell}/2) = \frac{L_{cell}}{4f}$$

Example:
45-degree Cell

$$L_{Cell} = l_{QF} + l_D + l_{QD} + l_D = 0.5m + 2.5m + 0.5m + 2.5m = 6m$$

$$1/f = k*l_Q = 0.5m*0.541\ m^{-2} = 0.27\ m^{-1}$$

$$\sin(\psi_{cell}/2) = \frac{L_{cell}}{4f} = 0.405$$
$$\rightarrow \quad \psi_{cell} = 47.8°$$
$$\rightarrow \quad \beta = 11.4\ m$$

Remember:
Exact calculation yields:

$$\rightarrow \quad \psi_{cell} = 45°$$
$$\rightarrow \quad \beta = 11.6\ m$$

Stability in a FoDo structure

SPS Lattice

$$M_{FoDo} = \begin{pmatrix} 1 - \dfrac{2l_D^2}{\tilde{f}^2} & 2l_D(1 + \dfrac{l_D}{\tilde{f}}) \\[3mm] 2(\dfrac{l_D^2}{\tilde{f}^3} - \dfrac{l_D}{\tilde{f}^2}) & 1 - 2\dfrac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

Stability requires:

$$\left| Trace(M) \right| \; < \; 2$$

$$\left| Trace(M) \right| \; = \; \left| 2 - \frac{4l_d^2}{\tilde{f}^2} \right| \; < \; 2$$

$$\rightarrow f > \frac{L_{cell}}{4}$$

For stability the focal length has to be larger than a quarter of the cell length
... don't focus to strong !

Transformation Matrix in Terms of the Twiss Parameters

Transformation of the coordinate vector (x,x´) in a lattice

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M_{s1,s2} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$
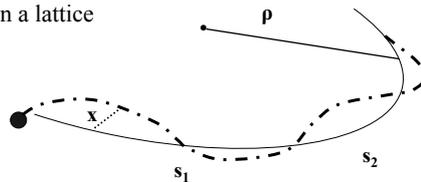
General solution of the equation of motion

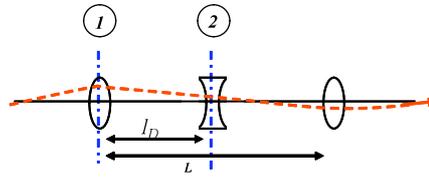$$x(s) = \sqrt{\varepsilon * \beta(s)} * \cos(\psi(s) + \varphi)$$

$$x'(s) = \sqrt{\varepsilon / \beta(s)} * \left\{ \alpha(s)\cos(\psi(s) + \varphi) + \sin(\psi(s) + \varphi) \right\}$$

Transformation of the coordinate vector (x,x´)
expressed as a function of the twiss parameters

$$M_{1 \rightarrow 2} = \begin{pmatrix} \sqrt{\dfrac{\beta_2}{\beta_1}}(\cos\psi_{12} + \alpha_1 \sin\psi_{12}) & \sqrt{\beta_1 \beta_2} \sin\psi_{12} \\[4mm] \dfrac{(\alpha_1 - \alpha_2)\cos\psi_{12} - (1 + \alpha_1\alpha_2)\sin\psi_{12}}{\sqrt{\beta_1\beta_2}} & \sqrt{\dfrac{\beta_1}{\beta_2}}(\cos\psi_{12} - \alpha_2 \sin\psi_{12}) \end{pmatrix}$$

Transfer Matrix for half a FoDo cell:

$$M_{half\,cell} = \begin{pmatrix} 1 - l_D \Big/ \tilde{f} & l_D \\ -l_D \Big/ \tilde{f}^2 & 1 + l_D \Big/ \tilde{f} \end{pmatrix}$$

Compare to the twiss parameter form of M

$$M_{1 \to 2} = \begin{pmatrix} \sqrt{\dfrac{\beta_2}{\beta_1}}(\cos\psi_{12} + \alpha_1 \sin\psi_{12}) & \sqrt{\beta_1 \beta_2}\, \sin\psi_{12} \\ \dfrac{(\alpha_1 - \alpha_2)\cos\psi_{12} - (1 + \alpha_1 \alpha_2)\sin\psi_{12}}{\sqrt{\beta_1 \beta_2}} & \sqrt{\dfrac{\beta_1}{\beta_2}}(\cos\psi_{12} - \alpha_2 \sin\psi_{12}) \end{pmatrix}$$

In the middle of a foc (defoc) quadrupole of the FoDo we allways have α = 0, and the half cell will lead us from β_{max} to β_{min}

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{\dfrac{\check{\beta}}{\hat{\beta}}} \cos\dfrac{\psi_{cell}}{2} & \sqrt{\check{\beta}\hat{\beta}} \sin\dfrac{\psi_{cell}}{2} \\ \dfrac{-1}{\sqrt{\hat{\beta}\check{\beta}}} \sin\dfrac{\psi_{cell}}{2} & \sqrt{\dfrac{\hat{\beta}}{\check{\beta}}} \cos\dfrac{\psi_{cell}}{2} \end{pmatrix}$$
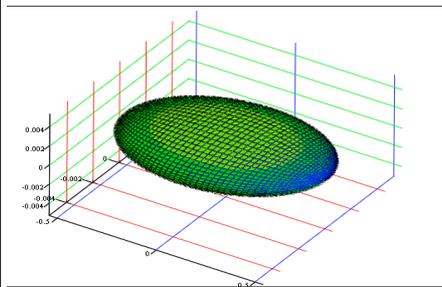

Solving for β_{max} and β_{min} and remembering that …. $\sin\dfrac{\psi_{cell}}{2} = \dfrac{l_d}{f} = \dfrac{L}{4f}$

$$\frac{m_{22}}{m_{11}} = \frac{\hat{\beta}}{\check{\beta}} = \frac{1 + l_d / \tilde{f}}{1 - l_d / \tilde{f}} = \frac{1 + \sin(\psi_{cell}/2)}{1 - \sin(\psi_{cell}/2)}$$

$$\frac{m_{12}}{m_{21}} = \hat{\beta}\check{\beta} = \tilde{f}^2 = \frac{l_d^2}{\sin^2(\psi_{cell}/2)}$$

$$\Rightarrow \qquad \boxed{\hat{\beta} = \frac{(1 + \sin\dfrac{\psi_{cell}}{2})L}{\sin\psi_{cell}}} \, !$$

$$\boxed{\check{\beta} = \frac{(1 - \sin\dfrac{\psi_{cell}}{2})L}{\sin\psi_{cell}}} \, !$$

The maximum and minimum values of the β-function are solely determined by the phase advance and the length of the cell.

Longer cells lead to larger β

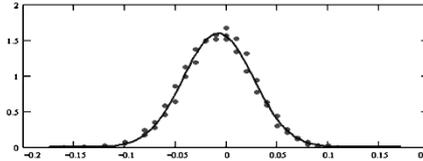*typical shape of a proton bunch in a FoDo Cell*

5.) Beam dimension:

Optimisation of the FoDo Phase advance:

In both planes a gaussian particle distribution is assumed, given by the beam emittance ε and the β-function
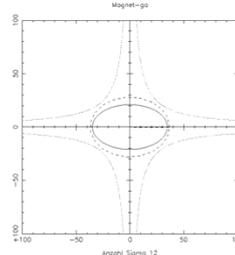
$$\sigma = \sqrt{\varepsilon\beta}$$

HERA beam size

In general proton beams are „*round*" in the sense that

$$\varepsilon_x \approx \varepsilon_y$$

So for highest aperture we have to minimise the β-function in both planes:

$$r^2 = \varepsilon_x\beta_x + \varepsilon_y\beta_y$$

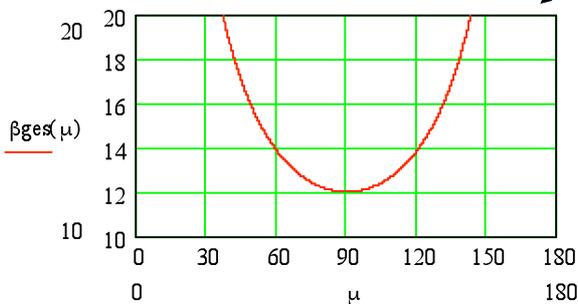*typical beam envelope, vacuum chamber and pole shape in a foc. Quadrupole lens in HERA*

Optimising the FoDo phase advance

$$r^2 = \varepsilon_x\beta_x + \varepsilon_y\beta_y$$

search for the phase advance µ that results in a minimum of the sum of the beta's

$$\hat{\beta} + \bar{\beta} = \frac{(1 + \sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} + \frac{(1 - \sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}}$$

$$\hat{\beta} + \bar{\beta} = \frac{2L}{\sin\psi_{cell}} \qquad \frac{d}{d\psi_{cell}}\left(2L\Big/\sin\psi_{cell}\right) = 0$$
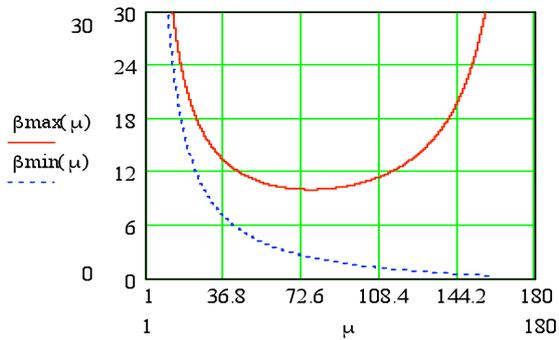

$$\frac{L}{\sin^2\psi_{cell}} * \cos\psi_{cell} = 0 \quad \rightarrow \quad \psi_{cell} = 90°$$

Electrons are different

electron beams are usually flat, $\varepsilon_y \approx 2 - 10\ \%\ \varepsilon_x$
→ *optimise only* β_{hor}

$$\frac{d}{d\psi_{cell}}(\hat\beta) = \frac{d}{d\psi_{cell}}\frac{L(1+\sin\frac{\psi_{cell}}{2})}{\sin\psi_{cell}} = 0 \quad\rightarrow\quad \psi_{cell} = 76°$$
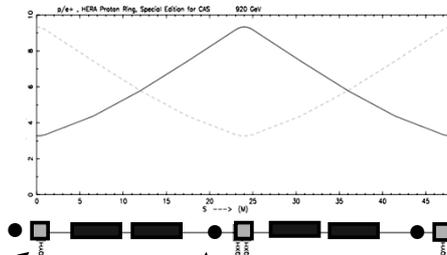
red curve: β_{max}
blue curve: β_{min}
as a function of the phase advance ψ

βmax(μ)
βmin(μ)

Orbit distortions in a periodic lattice

field error of a dipole/distorted quadrupole

$$\rightarrow \delta(mrad) = \frac{ds}{\rho} = \frac{\int B ds}{p/e}$$

the particle will follow a new closed trajectory, the distorted orbit:

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi Q)}\oint\sqrt{\beta(\tilde s)}\frac{1}{\rho(\tilde s)}\cos(|\psi(\tilde s) - \psi(s)| - \pi Q)d\tilde s$$

* the orbit amplitude will be large if the β function at the location of the kick is large
   $\beta(\tilde s)$ indicates the sensitivity of the beam → here orbit correctors should be
      placed in the lattice

* the orbit amplitude will be large at places where in the lattice β(s)
   is large → here beam position monitors should be installed

14

Orbit Correction and Beam Instrumentation in a storage ring
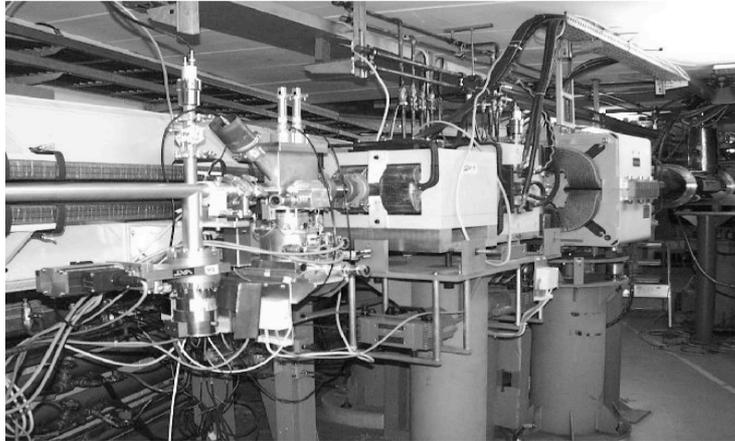
Elsa ring, Bonn

*

Resumé:

1.) Dipole strength

$$\int B ds = N * B_0 * l_{eff} = 2\pi \frac{p}{q}$$

l_{eff} effective magnet length, N number of magnets

2.) Stability condition

$$Trace(M) < 2$$

for periodic structures within the lattice / at least for the transfer matrix of the complete circular machine

3.) Transfer matrix for periodic cell

$$M(s) = \begin{pmatrix} \cos\psi_{cell} + \alpha_s \sin\psi_{cell} & \beta_s \sin\psi_{cell} \\ -\gamma_s \sin\psi_{cell} & \cos\psi_{cell} - \alpha_s \sin\psi_{cell} \end{pmatrix}$$

α,β,γ depend on the position s in the ring, μ (phase advance) is independent of s

4.) Thin lens approximation

$$M_{QF} = \begin{pmatrix} 1 & 0 \\ \dfrac{1}{f_Q} & 1 \end{pmatrix} \qquad f_Q = \frac{1}{k_Q l_Q}$$

focal length of the quadrupole magnet $f_Q = 1/(k_Q l_Q) \gg l_Q$

5.) Tune (rough estimate)

$$\psi_{period} = \int\limits_{s}^{s+L} \frac{ds}{\beta(s)}$$

Tune = phase advance in units of 2π

$$Q = N * \frac{\psi_{period}}{2\pi} = \frac{1}{2\pi} * \oint \frac{ds}{\beta(s)} \approx \frac{1}{2\pi} * \frac{2\pi\overline{R}}{\overline{\beta}} = \frac{\overline{R}}{\overline{\beta}}$$

$\overline{R}, \overline{\beta}$ average radius and β-function

$$Q \approx \frac{\overline{R}}{\overline{\beta}}$$

6.) Phase advance per FoDo cell
(thin lens appro

$$\sin \frac{\psi_{cell}}{2} = \frac{l_q}{f} = \frac{L_{cell}}{4 f_Q}$$

L_{Cell} length of the complete FoDo cell, f_Q focal length of the quadrupole, μ phase advance per cell

7.) Stability in a FoDo cell
(thin lens approx)

$$f_Q > \frac{L_{Cell}}{4}$$

8.) Beta functions in a FoDo cell
(thin lens approx)

$$\hat{\beta} = \frac{(1 + \sin \frac{\psi_{cell}}{2}) L_{cell}}{\sin \psi_{cell}} \qquad \overline{\beta} = \frac{(1 - \sin \frac{\psi_{cell}}{2}) L_{cell}}{\sin \psi_{cell}}$$

L_{Cell} length of the complete FoDo cell, μ phase advance per cell