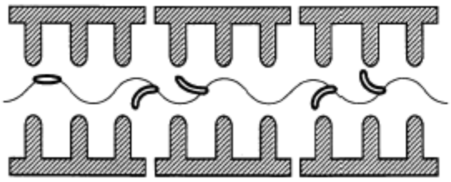
INSTABILITIES IN LINACS

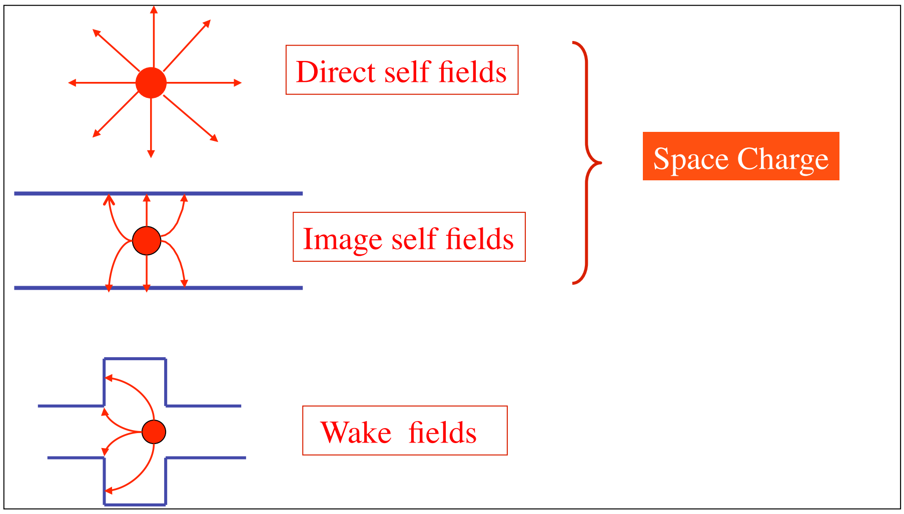
Massimo Ferrario

INFN-LNF



THE CERN ACCELERATOR SCHOOL

SELF FIELDS AND WAKE FIELDS



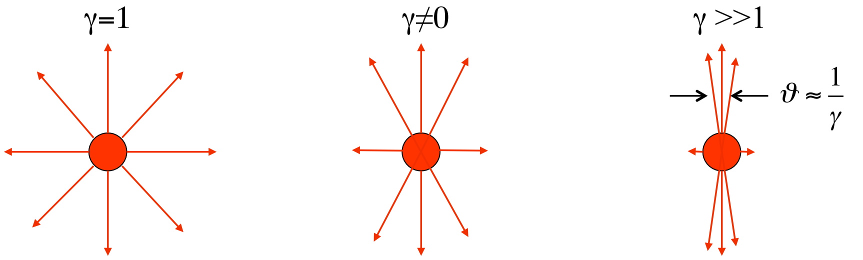
$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_o}\frac{\left(1-\beta^2\right)}{\left(1-\beta^2\sin^2\theta\right)^{3/2}}\frac{\vec{r}}{r^3}$$

$$\beta = 0 \Rightarrow \vec{E} = \frac{q}{4\pi\varepsilon_o}\frac{\vec{r}}{r^3}$$

$$\theta = 0 \Rightarrow E_{//} = \frac{1}{\gamma^2}\frac{q}{4\pi\varepsilon_o}\frac{\vec{r}}{r^3} \xrightarrow{\gamma\to\infty} 0$$

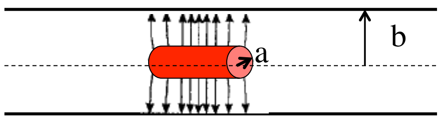
$$\theta = \frac{\pi}{2} \Rightarrow E_{\perp} = \gamma\frac{q}{4\pi\varepsilon_o}\frac{\vec{r}}{r^3} \xrightarrow{\gamma\to\infty} \infty$$



Bunched beam - Circular Perfectly Conducting Pipe

- Beam at Center- Static Approximation $\gamma\to\infty$

$$\rho = \frac{I}{\pi a^2 v}$$



$$\varphi(b) = 0$$

$$E_r = \frac{I}{2\pi\varepsilon_o a^2 v} r \quad \textbf{for} \quad r \le a$$

$$E_r = \frac{I}{2\pi\varepsilon_o v}\frac{1}{r} \quad \textbf{for} \quad r > a$$

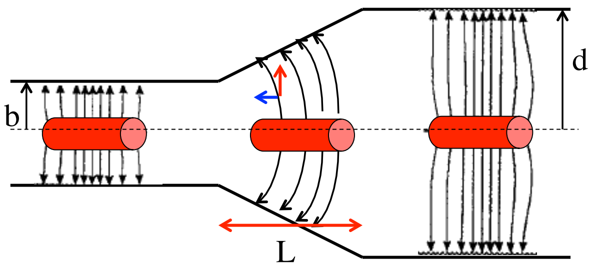
$$B_\vartheta = \frac{\beta}{c}E_r$$

$$\varphi(r) = \int_r^b E_r(r')dr'$$

$$= \frac{I}{4\pi\varepsilon_o v}\left(1 + 2\ln\frac{b}{a} - \frac{r^2}{a^2}\right) \quad \textbf{for} \quad r \le a$$

$$= \frac{I}{2\pi\varepsilon_o v}\ln\frac{b}{r} \quad \textbf{for} \quad a \le r \le b$$

Circular Perfectly Conducting Pipe with Transition



There is a longitudinal $E_z(r,z)$ field in the transition and a test particle experience a voltage given by:

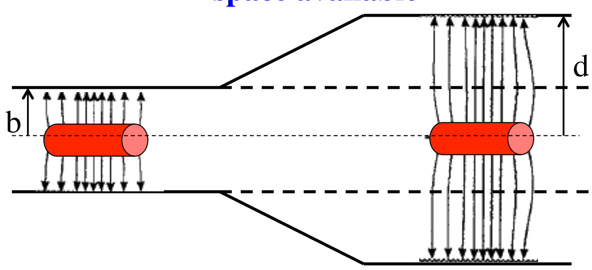
$$V = -\int_0^L E_z(r,z)dz = -(\varphi(r,L) - \varphi(r,0)) = -\frac{I}{2\pi\varepsilon_o v} \ln\frac{d}{b}$$

decelerating if d > b

$$P_b = VI = \frac{I^2}{2\pi\varepsilon_o v} \ln\frac{d}{b}$$

Power lost by the beam

For d > b the power is deposited to the energy of the fields: moving from left to right the beam induces the fields in the additional space available

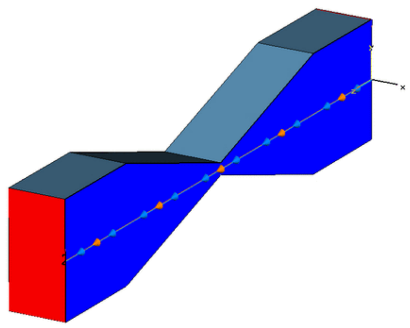


The additional power passing through the right part of the beam pipe is obtained by integrating the Poynting vector throught the sourface $\Delta S = \pi(d^2 - b^2)$

$$P_{em} = \int_{\Delta S} \left(\frac{1}{\mu} \vec{E} \times \vec{B}\right) \cdot d\vec{S} = \int_b^d \frac{E_r B_\vartheta}{\mu} 2\pi r\, dr = \frac{I^2}{2\pi\varepsilon_o v} \ln\frac{d}{b}$$

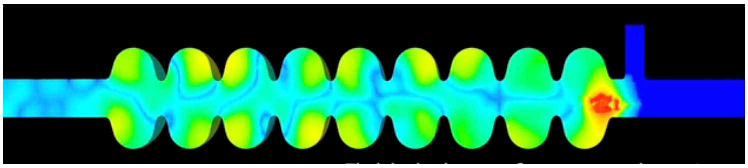
Notice that if d<b the beam gains energy. If d-->∞ the power goes to infinity, such an unphysical result is nevertheless consistent with the original assumption of an infinite energy beam (γ->∞).

Reflected and Diffracted fiels

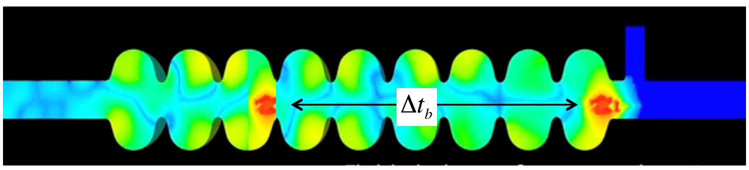


CST MICROWAVE STUDIO®

Short Range Wake Fields Effects ➔ head tail effects

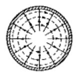
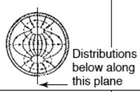
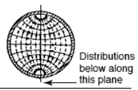
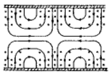
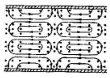
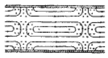
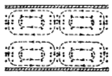
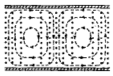


Long Range Wake Fields Effects ➔ multibunch instabilities

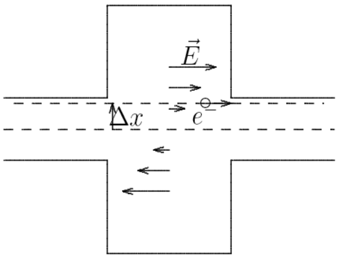


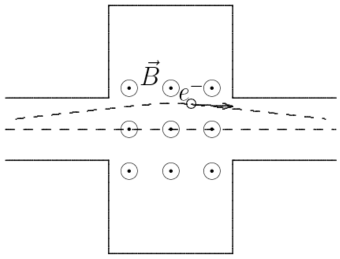
$$\Delta t_b \approx \tau = \frac{2Q}{\omega} \begin{cases} \approx \mu\text{s} \Rightarrow \text{Normal Conducting Cavities} \\ \approx \text{ms} \Rightarrow \text{Superconducting Cavities} \end{cases}$$

Table 3 Mode Patterns in Circular Waveguide.

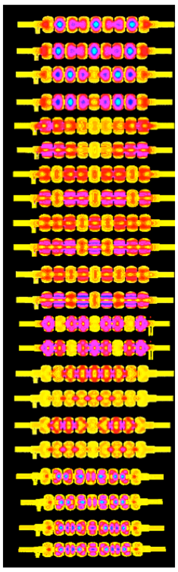
Wave Type	TM_{01}	TM_{02}	TM_{11}	TE_{01}	TE_{11}
Field distributions in cross-sectional plane, at plane of maximum transverse fields					
Field distributions along guide					
Field components present	E_z, E_r, H_ϕ	E_z, E_r, H_ϕ	E_z, E_r, E_ϕ, H_r, H_ϕ	H_z, H_r, E_ϕ	H_z, H_r, H_ϕ, E_r, E_ϕ

Energy exchange if: $\boxed{\frac{d\gamma}{dt} = \frac{e}{mc}\vec{E}\cdot\vec{\beta} = \frac{e}{mc}\left(E_{//}\beta_{//} + E_{\perp}\beta_{\perp}\right) \neq 0}$

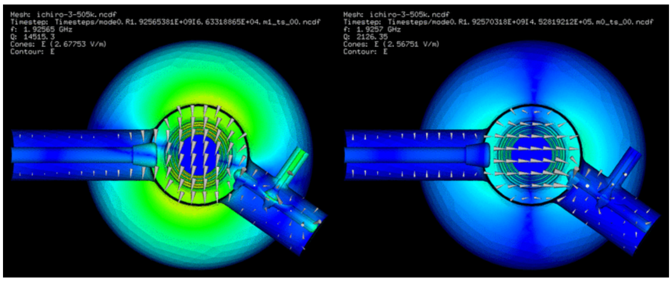




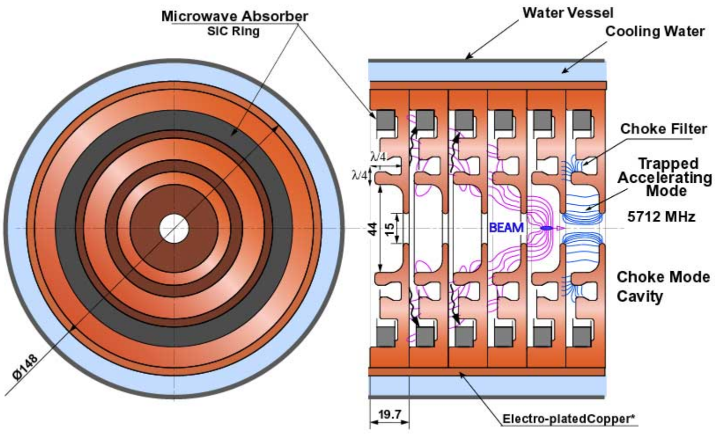
Mode	f [MHz]	$(R/Q)^*$ [Ω/cm^n]	Q_{ext}
M: TM010-9	1300.00	1161	$8\cdot10^5$
D: TE111-7a	1717.15	5.0	$4\cdot10^4$
D: TE111-7b	1717.21	5.0	$5\cdot10^4$
D: TE111-8a	1738.12	3.0	$6\cdot10^4$
D: TE111-8b	1738.15	3.0	$8\cdot10^4$
D: TM110-2a	1882.15	3.4	$6\cdot10^3$
D: TM110-2b	1882.47	3.4	$6\cdot10^3$
D: TM110-4a	1912.04	4.6	$9\cdot10^3$
D: TM110-4b	1912.21	4.6	$1\cdot10^4$
D: TM110-5a	1927.10	15.6	$1.5\cdot10^4$
D: TM110-5b	1927.16	15.6	$1.5\cdot10^4$
D: TM110-6a	1940.25	12.1	$2\cdot10^4$
D: TM110-6b	1940.27	12.1	$2\cdot10^4$
M: TM011-6	2177.48	192	10^4
M: TM011-7	2182.81	199	10^4
D: 3-rd-1a	2451.07	31.6	$1\cdot10^5$
D: 3-rd -1b	2451.15	31.6	$2\cdot10^5$
D: 3-rd 1-2a	2457.04	22.2	$5\cdot10^4$
D: 3-rd 1-2b	2457.09	22.2	$5\cdot10^4$
D: 5-th – 7a	3057.43	0.5	$3\cdot10^5$
D: 5-th – 7b	3057.45	0.5	$3\cdot10^5$
D: 5-th – 8a	3060.83	0.4	$8\cdot10^5$
D: 5-th – 8b	3060.88	0.4	$9\cdot10^5$



Example of two dipoles overlapping modeling in the TESLA cavity with Omega3P



"Choke Mode Cavity"



Causality and the Catch-Up distance

The image charges travel with the same particle velocity v.
Since both the particles and the image charges move on parallel paths, in the limit **v = c** they do not interact with each other, no matter how close to the wall the particles are.

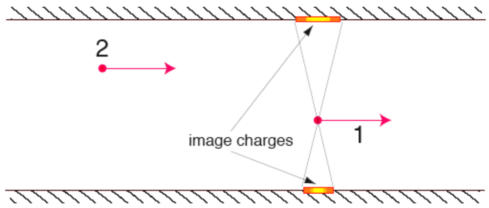


FIGURE 2. Particles traveling inside a perfectly conducting pipe of arbitrary cross section. Shown are the image charges on the wall generated by the leading charge.

If a particle moves along a straight line with the speed of light, the electromagnetic field of this particle scattered off the boundary discontinuities will not overtake it and, furthermore, will not affect the charges that travel ahead of it.

The field can interact only with the trailing charges in the beam that move behind it.
This constitutes the principle of causality in the theory of wake fields

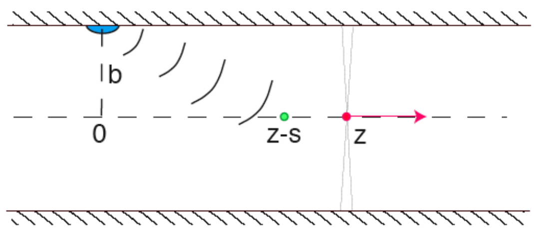
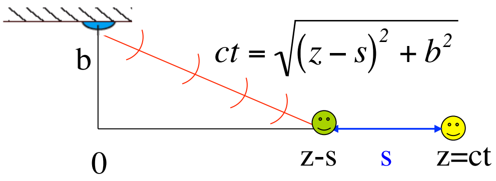


FIGURE 3. A wall discontinuity located at $z = 0$ scatters the electromagnetic field of an ultrarelativistic particle. When the particle moves to location z, the scattered field arrives to point $z - s$.

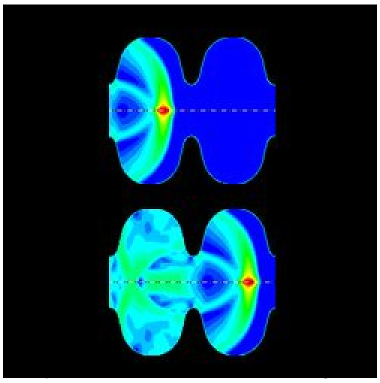
We can estimate the distance at which the electromagnetic field produced by a leading charge reaches a trailing particles traveling at a distance s behind.



$$z^2 = (z-s)^2 + b^2 \quad ==> \quad z_{catch-up} \approx \frac{b^2}{2s} \quad \textit{\textbf{for s << b}}$$

Only after the leading charge has traveled $\mathbf{z_{catch\text{-}up}}$ away from the discontinuity, can a particle at point **s behind it** feel the field generated by the discontinuity.

Numerical Analysis

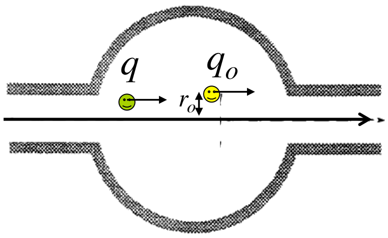


The study of the fields requires to solve the Maxwell equations in a given structure taking the beam current as source of fields. This is a quite complicated task for which it has been necessary to develop dedicated computer codes, which solve the e.m. problem in the frequency or in the time domain. There are several useful codes for the design of accelerator devices: **MAFIA, ABCI, URMEL,** etc...

Theoretical Analysis

The parasitic fields depends on the particular charge distribution of the beam. It is therefore desirable to know what is the effect of a single charge, i.e. **find the Green function *w***, in order to reconstruct the fields produced by any charge distribution.

Wake Potentials



$$\boldsymbol{F} = q\left[E_z\hat{z} + \left(E_x - vB_y\right)\hat{x} + \left(E_y + vB_x\right)\hat{y}\right] \equiv \boldsymbol{F}_{//} + \boldsymbol{F}_{\perp}$$

there can be two effects on the **test charge** :

1) a longitudinal force which changes its energy,

2) a transverse force which deflects its trajectory.

If we consider a device of length L:

the Energy Gain is: $$\boxed{U = \int_0^L F_z ds}$$

the Transverse Deflecting Kick is: $$\boxed{\boldsymbol{M} = \int_0^L \mathbf{F}_{\perp} ds}$$

These quantities, normalised to the charges, are called ***wake-potentials*** and are both function of the distance z.

Note that the integration is performed over a given path of the trajectory.

Longitudinal wake potential
[V/C]

$$w_{//} = -\frac{U}{q_o q}$$

Energy Loss

Transverse wake potential
[V/Cm]

$$\boldsymbol{w}_{\perp} = \frac{1}{r_o}\frac{\boldsymbol{M}}{q_o q}$$

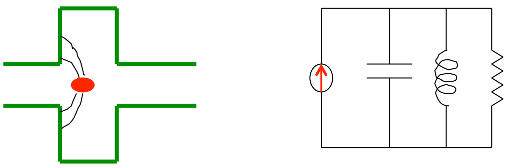
Transverse Kick

The sign minus in the longitudinal wake-potential means that the test charge loses energy when the wake is positive.

Positive transverse wake means that the transverse force is defocusing.

Longitudinal wake potential of a resonant HOM

When a charge crosses a resonant structure, it excites the fundamental mode and high order modes (HOM). Each mode can be treated as an electric RLC circuit loaded by an impulsive current.



Just after the charge passage, the capacitor is charged with a voltage $V_o=q/C$ and the electric field is $E_{so}= V_o/l_o$.

The time evolution of the electric field is governed by the same differential equation of the voltage

$$\ddot{V} + \frac{1}{RC}\dot{V} + \frac{1}{LC}V = \frac{1}{C}\dot{I}$$

The passage of the impulsive current charges only the capacitor, which changes its potential by an amount $V_c(0)$.

This potential will oscillate and decay producing a current flow in the resistor and inductance.

For t > 0 the potential satisfy the following equation and initial conditions:

$$\ddot{V} + \frac{1}{RC}\dot{V} + \frac{1}{LC}V = 0$$

$$V(t=0^+) = \frac{q}{C} \equiv V_0$$

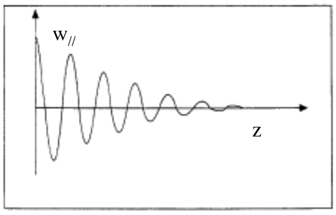
$$\dot{V}(t=0^+) = \frac{\dot{q}}{C} = \frac{I(0^+)}{C} = \frac{V_0}{RC}$$

$$V(t) = V_0 e^{-\gamma t}\left[\cos(\overline{\omega}t) - \frac{\gamma}{\overline{\omega}}\sin(\overline{\omega}t)\right]$$

$$\overline{\omega}^2 = \omega_r^2 - \gamma^2 \qquad 2\gamma = 1/RC \qquad \omega_r^2 = 1/LC$$

putting z = -ct (z is negative behind the charge),

$$w_{//}(z) = \frac{-V(z)}{q} = w_o e^{\gamma z/c}\left[\cos(\overline{\omega}z/c) + \frac{\gamma}{\overline{\omega}}\sin(\overline{\omega}z/c)\right]$$



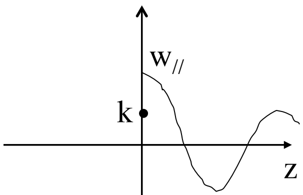
It is also useful to define the ***loss factor*** as the normalised energy lost by the **source charge q**

$$k = -\frac{U(z=0)}{q^2}$$

Although in general the loss factor is given by the longitudinal wake at z=0, for charges travelling with the light velocity the longitudinal wake potential is discontinuous at z=0

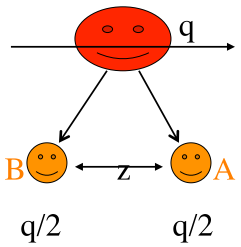
The exact relationship between *k* and *w(z=0)* is given by the ***beam loading theorem:***

$$k = \frac{w_{//}(z \to 0)}{2}$$



Causality requires that the longitudinal wake potential of a charge travelling with the velocity of light is discontinuous at the origin.

$$\boxed{U_o = -q^2 k}$$



$$U_A = -q_A^2 k = -\frac{q^2}{4} k$$

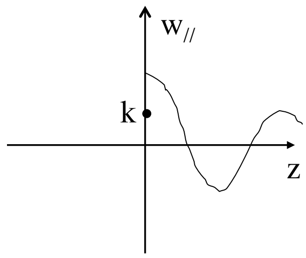
$$U_B = -q_B^2 k - q_A q_B w_{//}(z)$$
$$= -\frac{q^2}{4} k - \frac{q^2}{4} w_{//}(z)$$

$$\boxed{U_A + U_B = -\frac{q^2}{2} k - \frac{q^2}{4} w_{//}(z)}$$

$$\boxed{z \to 0 \quad U_o = -q^2 k = U_A + U_B}$$

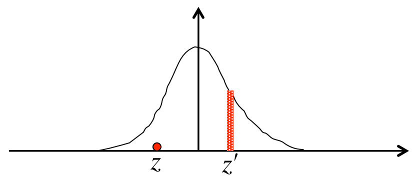
$$q^2 k = \frac{q^2}{2} k + \frac{q^2}{4} w_{//}(0)$$

$$\boxed{k = \frac{w_{//}(0)}{2}}$$



Wake potentials and energy loss of a bunched distribution

When we have a bunch with density λ(z), we may wander what is the amount of energy lost or gained by a single charge *e* in the beam



To this end we calculate the effect on the charge from the whole bunch by means of the convolution integral:

$$U(z) = -e \int_{-\infty}^{\infty} w_{//}(z - z') \lambda(z') dz'$$

Which allows to define the ***wake potential of a distribution***

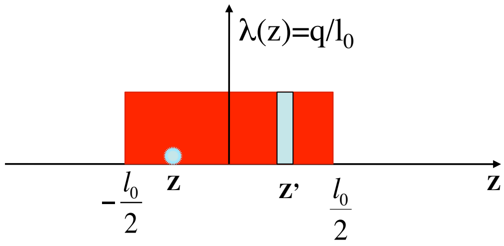
$$W_{//}(z) = -\frac{U(z)}{q q_o}$$

The total energy lost by the bunch is computed summing up the loss of all particles:

$$U_{bunch} = -\frac{1}{e} \int_{-\infty}^{\infty} U(z) \lambda(z) dz$$

Example: Energy spread and loss for a finite uniform beam due to a HOM

$$w_{//}(z) \approx w_0 \cos\left(\frac{\omega_r}{c} z\right) H(-z)$$



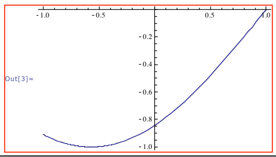
$$U(z) = -e \int_{-\infty}^{+\infty} w_{//}(z - z')\lambda(z')dz'$$

$$U(z) = -\frac{eqw_0}{l_0} \int_{z}^{\infty} \cos\left[\frac{\omega_r}{c}(z - z')\right] dz'$$

$$z - z' = x$$

$$U(z) = \frac{eqw_0}{l_0} \int_{0}^{(z-l_0/2)} \cos\left(\frac{\omega_r}{c} x\right) dx = \frac{eqw_0}{l_0} \left[\frac{\sin\left(\frac{\omega_r}{c} x\right)}{\left(\frac{\omega_r}{c}\right)} \right]_0^{(z-l_0/2)}$$

$$U(z) = -\frac{eqw_0}{2} \left[\frac{\sin\left[\frac{\omega_r}{c}\left(\frac{l_0}{2} - z\right)\right]}{\left(\frac{\omega_r}{c}\frac{l_0}{2}\right)} \right]$$



Parasitic loss

$$U_{bunch} = \frac{1}{e} \int_{-\infty}^{+\infty} U(z')\lambda(z')dz' \approx \frac{-q^2 w_0}{2l_0\left(\frac{\omega_r}{c}\frac{l_0}{2}\right)} \int_{-\frac{l_0}{2}}^{\frac{l_0}{2}} \sin\left[\frac{\omega_r}{c}\left(\frac{l_0}{2} - z'\right)\right] dz'$$

$$U_{bunch} = \frac{-q^2 w_0 c}{\omega_r l_0^2} \left| \frac{-\cos\left[\frac{\omega_r}{c}\left(\frac{l_0}{2} - z'\right)\right]}{-\frac{\omega_r}{c}} \right|_{-\frac{l_0}{2}}^{\frac{l_0}{2}}$$

$$U_{bunch} = -\frac{q^2 w_0 c^2}{\omega_r^2 l_0^2}\left[1 - \cos\left(\frac{\omega_r l_0}{c}\right)\right] = -\frac{2q^2 w_0 c^2}{\omega_r^2 l_0^2} \sin^2\left(\frac{\omega_r l_0}{2c}\right)$$

$$U_{bunch} = -\frac{q^2 w_0}{2} \frac{\sin^2\left(\frac{\omega_r l_0}{2c}\right)}{\left(\frac{\omega_r l_0}{2c}\right)^2}$$

$$\lim_{l_0 \to 0}\left(U_{bunch}\right) = -\frac{q^2 w_0}{2}$$

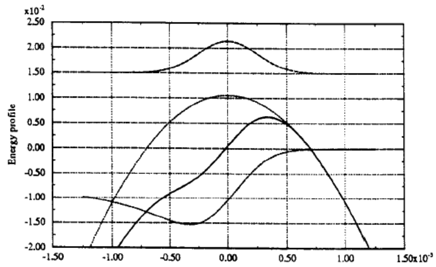


Fig. 5 Energy profile within the bunch sitting on the crest of the rf wave

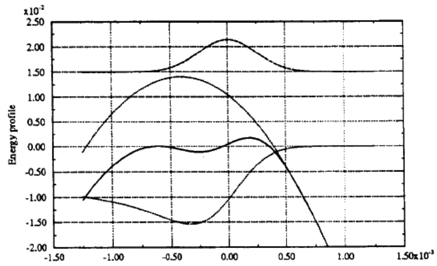


Fig. 6 Energy profile within the bunch after optimization of the rf phase

Coupling Impedance

The wake potentials are used for to study the beam dynamics in the time domain ($s=vt$). If we take the equation of motion in the frequency domain, we need the Fourier transform of the wake potentials. Since these quantities have Ohms units are called ***coupling impedances***:

Longitudinal impedance (Ω)

$$Z_{//}(\omega) = \frac{1}{v}\int_{-\infty}^{\infty} w_{//}(z) e^{-i\frac{\omega z}{v}} dz$$

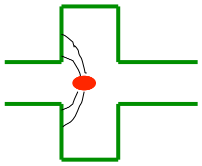
Transverse impedance (Ω/m)

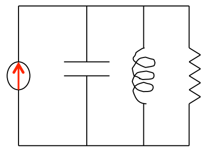
$$\boldsymbol{Z}_{\perp}(\omega) = \frac{i}{v}\int_{-\infty}^{\infty} \boldsymbol{w}_{\perp}(z) e^{-i\frac{\omega z}{v}} dz$$

Z_R is responsible for the energy losses

Z_j defines the phase between the beam response & exciting wake potential

Impedance and wake potential of a resonant mode





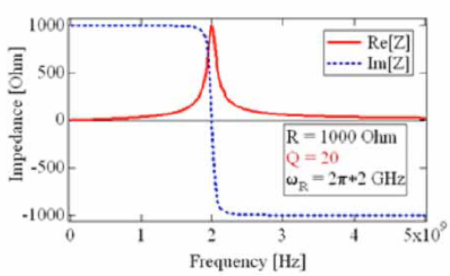
$$Z(\omega) = \frac{R}{1 + jQ\left(\frac{\omega}{\omega_R} - \frac{\omega_R}{\omega}\right)}, \qquad \omega_R = \frac{1}{\sqrt{LC}}, \quad Q = R\sqrt{\frac{C}{L}}$$

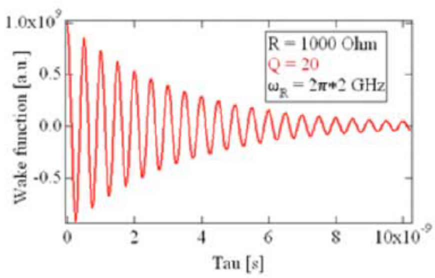
$$W(\tau) = \begin{cases} 0 & \tau < 0 \\ \frac{e^{-\omega_R \tau/2Q}}{C}\left[\cos\left(\omega_R \tau \sqrt{1 - 1/4Q^2}\right) - \frac{\sin\left(\omega_R \tau \sqrt{1 - 1/4Q^2}\right)}{\sqrt{4Q^2 - 1}}\right] & \tau > 0 \end{cases}$$

Narrow-band modes are characterized by moderate Q & narrow spectrum

==> Associated wake lasts for a relatively long time

==> Capable of exciting multi-bunch instabilities



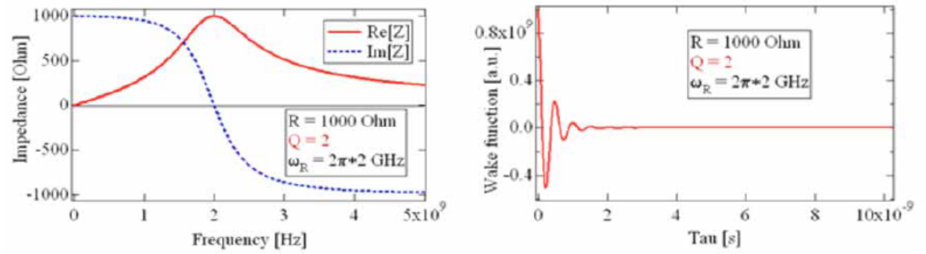


Narrow band impedances are usually higher order modes of high Q accelerating structures

Broad-band impedance modes have a low Q and a broader spectrum.

==> The associated wake last for a relatively short time

==> Important only for single bunch instabilities



Broad band impedances raise from irregularities or variations in the environment of the beam

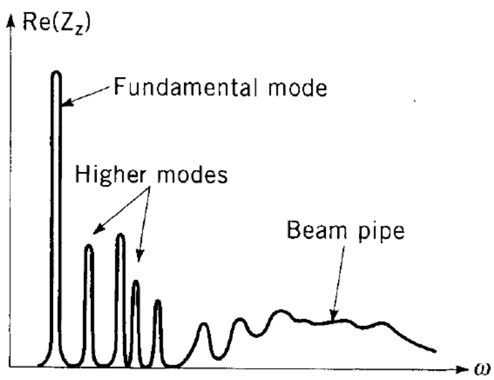
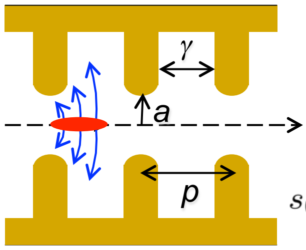
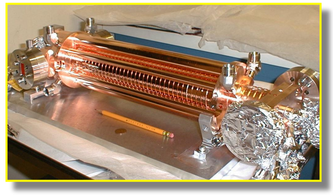


Figure 11.10 Typical frequency spectrum of the real part of an accelerator cavity impedance. [From T. Weiland and R. Wanzenberg, Frontiers of Particle Beams: Intensity Limitations, Proc. Joint U.S.–CERN School on Particle Accelerators at Hilton Head Island, *Lecture Notes Phys.* **400**, M. Dienes, M. Month, and S. Turner (Eds.), 39–79 (1992).]

Longitudinal Wakefields of RF Structures

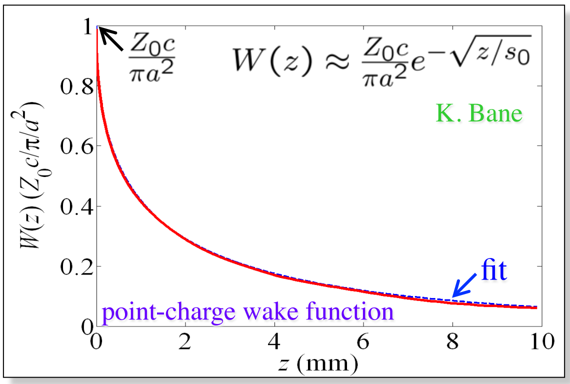


SLAC *S*-band:
$a \approx 11.6$ mm
$\gamma \approx 29.2$ mm
$p \approx 35.0$ mm

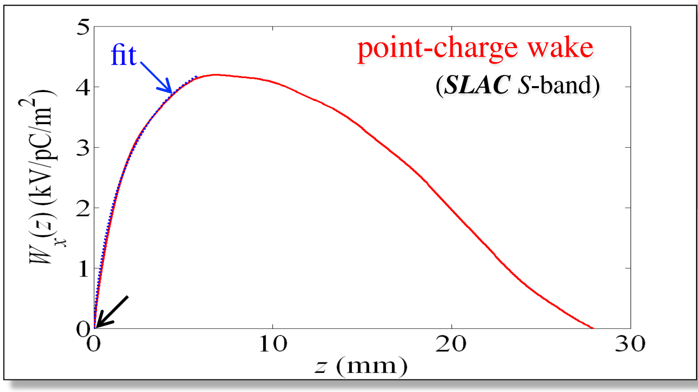


$$s_0 \approx 0.41 \frac{a^{1.8} g^{1.6}}{p^{2.4}}$$

$W_{//} \propto \omega^2$



Transverse Wakefields



SLAC *S*-band
$s_1 \approx 0.56$ mm
$a \approx 11.6$ mm
$A \approx 1.13$
$z <$ ~6 mm

$W_{\perp} \propto \omega^3$

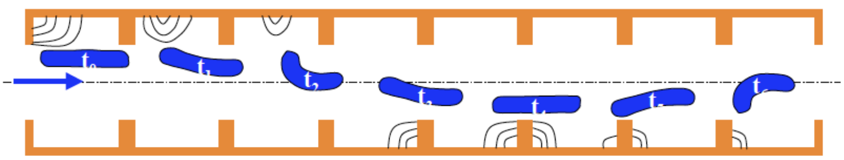
transverse point-charge wakefield function and short-range fit:

$$W_x(z) \approx A \frac{4 Z_0 c s_1}{\pi a^4} \left[1 - \left(1 + \sqrt{z/s_1}\right) e^{-\sqrt{z/s_1}} \right] \quad , \; z < 6 \text{ mm}$$

Short Range Wake Fields Effects in Linear Accelerators

Beam Break Up

A beam injected off-center in a LINAC, because of the focusing quadrupoles, execute betatron oscillations. The displacement produces a transverse wake field in all the devices crossed during the flight, which deflects the trailing charges.



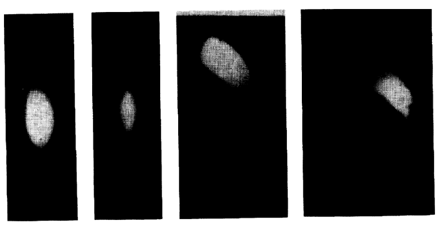
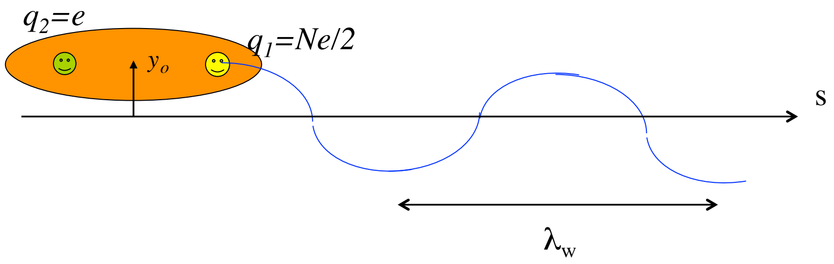


Figure 3.4. Four transverse beam profiles observed at the end of the SLAC linac are shown when the beam was carefully injected and injected with 0.2, 0.5, and 1 mm offsets. The beam sizes σ_x and σ_y are about 120 μm. (Courtesy John Seeman, 1991.)

In order to understand the effect, we consider a simple model with only two charges $q_1=Ne/2$ (leading = half bunch) and $q_2=e$ (trailing = single charge).



the leading charge executes free betatron oscillations:

$$y_1(s) = \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right); \quad \frac{\omega_y}{c} = \frac{2\pi}{\lambda_w}$$

the trailing charge, at a distance z behind, over a length L_w experiences a deflecting force proportional to the displacement y_1, and dependent on the distance z:

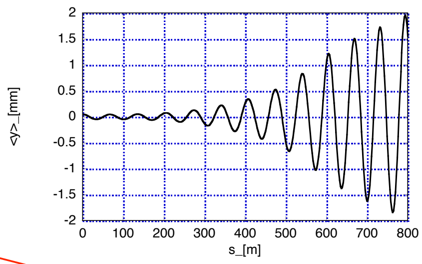
$$\left\langle F_y^{wake}(z, y_1) \right\rangle = \frac{Ne^2}{2L_w} w_\perp(z) y_1(s)$$

This force drives the motion of the trailing charge:

$$y_2'' + \left(\frac{\omega_y}{c}\right)^2 y_2 = \frac{Ne^2 w_\perp(z)}{2\beta^2 E_o L_w} \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right)$$

This is the typical equation of a resonator driven at the resonant frequency. The solution is given by the superposition of the "**free**" oscillation and a "**driven**" oscillation which, being driven at the resonant frequency, grows linearly with s.

$$y_2(s) = \hat{y}_2 \cos\left(\frac{\omega_y}{c}s\right) + y_2^{driven}$$



$$y_2^{driven} = \frac{cNe^2 w_\perp(z)}{4\omega_y E_o L_w} s\hat{y}_1 \sin\left(\frac{\omega_y}{c}s\right)$$

continuos growth

At the end of the LINAC of length L_L, the oscillation amplitude is grown by :

$$\boxed{\left(\frac{\Delta\hat{y}_2}{\hat{y}_2}\right)_{\max} = \frac{cNe^2 w_\perp(z) L_L}{4\omega_y E_o L_w}}$$

Balakin-Novokhatsky-Smirnov Damping

The BBU instability is quite harmful and hard to take under control even at high energy with a strong focusing, and after a careful injection and steering.

A simple method to cure it has been proposed observing that the strong oscillation amplitude of the bunch tail is mainly due to the **"resonant" driving**.

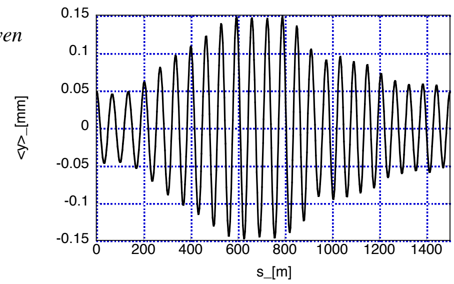
If the tail and the head move with a different frequency, this effect can be significantly removed.

Let us assume that the tail oscillate with a frequency $\omega_y+\Delta\omega_y$, the equation of motion reads:

$$\boxed{y_2'' + \left(\frac{\omega_y + \Delta\omega_y}{c}\right)^2 y_2 = \frac{Ne^2 w_\perp(z)}{2\beta^2 E_o L_w} \hat{y}_1 \cos\left(\frac{\omega_y}{c}s\right)}$$

the solution of which is:

$$y_2(s) = \hat{y}_2 \cos\left(\frac{\omega_y + \Delta\omega_y}{c} s\right) + y_2^{driven}$$



$$y_2^{driven}(s) = \frac{c^2 N e^2 w_\perp(z)}{4\omega_y \Delta\omega_y E_o L_w} \hat{y}_1 \left[\cos\left(\frac{\omega_y + \Delta\omega_y}{c} s\right) - \cos\left(\frac{\omega_y}{c} s\right)\right] \qquad \Delta\omega_y << \omega_y$$

by a suitable choice of $\Delta\omega_y$, it is possible to fully depress the oscillations of the tail.

$$\boxed{y_2(s) = \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right)}$$

$$\boxed{\Delta\omega_y = \frac{c^2 N e^2 w_\perp(z)}{4\omega_y E_o L_w}}$$

$$\boxed{\hat{y}_2 = \hat{y}_1}$$

Exploit the **energy spread** across the bunch which, because of the chromaticity, induces a spread in the betatron frequency. An energy spread correlated with the position is attainable with the external accelerating voltage, or with the wake fields.

$$\frac{\Delta\omega_y}{\omega_y} = -\frac{\Delta\gamma}{\gamma}$$

More general model including charge distribution and acceleration

$$\frac{\partial}{\partial s}\left[\gamma(s)\frac{\partial y(z,s)}{\partial s}\right]+k_y^2(s)\gamma(s)y(z,s)=-\frac{e^2N_p}{m_0c^2L_w}\int_z^\infty y(s,z')w_\perp(z'-z)\lambda(z')dz'$$

$$y(L_L)=y_m\sqrt{\frac{\gamma_i}{6\pi\gamma_f}}\eta^{-1/6}\exp\left[\frac{3\sqrt{3}}{4}\eta^{1/3}\right]\cos\left[k_yL_L-\frac{3}{4}\eta^{1/3}+\frac{\pi}{12}\right]$$

$$\eta=\frac{e^2N_p}{k_y(dE_0/ds)}\frac{w_{\perp 0}}{L_w}\ln\left(\frac{\gamma_f}{\gamma_i}\right)$$

A. W. Chao - *Physics of collective beam instabilities in high energy accelerators* - Wiley, NY 1993

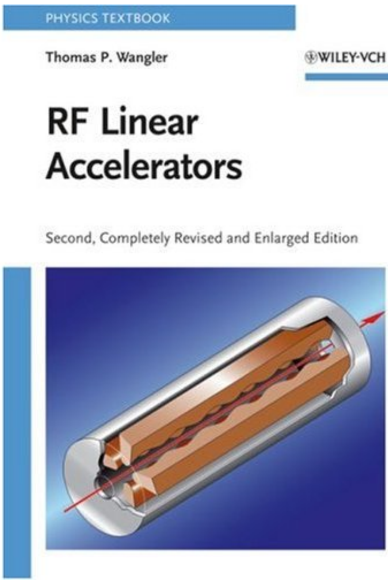
A. Mosnier - *Instabilities il Linacs* - CAS (Advanced) - 1994

L. Palumbo, V. Vaccaro, M. Zobov- *Wakes fields and Impedance* - CAS (Advanced) - 1994

G. V. Stupakov - *Wake and Impedance* - SLAC-PUB-8683

K.L.F. Bane, A. Mosnier, A. Novokhatsky, K. Yokoya-*Calculations of the Short Range Longitudinal Wakefields in the NLC Linac* - EPAC' 98

M. Ferrario, M. Migliorati, L. Palumbo - *Wake Fields and Instabilities in Linacs* - CAS (Advanced) - 2005


PHYSICS TEXTBOOK
Thomas P. Wangler
WILEY-VCH
RF Linear Accelerators
Second, Completely Revised and Enlarged Edition

THE END

Instabilities : driven oscillators

Consider an harmonic oscillator with natural frequency ω, with an external excitation at frequency Ω

$$\ddot{x} + \omega^2 x = A\cos(\Omega t)$$

General solution:

$$x(t) = x^{free}(t) + x^{driven}(t)$$
$$\cos(\Omega t) \Rightarrow e^{i\Omega t}$$
$$x^{free}(t) = \tilde{x}_m^f e^{i\omega t}$$
$$x^{driven}(t) = \tilde{x}_m^d e^{i\Omega t}$$

substitution in the diff. equation:

$$(\omega^2 - \Omega^2)\tilde{x}_m^d e^{i\Omega t} = Ae^{i\Omega t}$$
$$x^{driven}(t) = \frac{A}{(\omega^2 - \Omega^2)} e^{i\Omega t}$$

The general solution has to satisfy the initial condition at t=0. In our case we assume that the oscillator is at rest for t=0:

$$x^{free}(t=0) = -x^{driven}(t=0)$$
$$\tilde{x}_m^f = -\frac{A}{\omega^2 - \Omega^2}$$

thus we get:

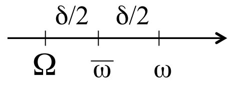
$$x(t) = \frac{A}{\omega^2 - \Omega^2}\left[e^{i\Omega t} - e^{i\omega t}\right]$$

taking only the real part:

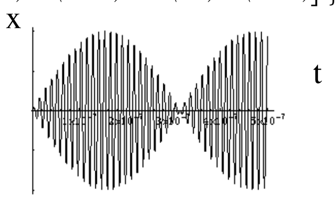
$$\boxed{x(t) = \frac{A}{\omega^2 - \Omega^2}\left[\cos(\Omega t) - \cos(\omega t)\right]}$$

This expression is suitable for deriving the response of the oscillator driven at resonance or at frequency very close:

$$\omega - \Omega = \delta$$
$$\overline{\omega} = (\omega + \Omega)/2$$
$$\omega = \overline{\omega} + \delta/2$$
$$\Omega = \overline{\omega} - \delta/2$$

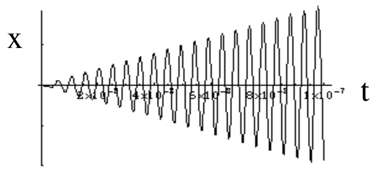


$$x(t) = \frac{A}{2\overline{\omega}\delta}\left\{ \left[cos(\overline{\omega}t)cos(\delta t/2) + sin(\overline{\omega}t)sin(\delta t/2)\right] - \left[cos(\overline{\omega}t)cos(\delta t/2) + sin(\overline{\omega}t)sin(\delta t/2)\right] \right\}$$



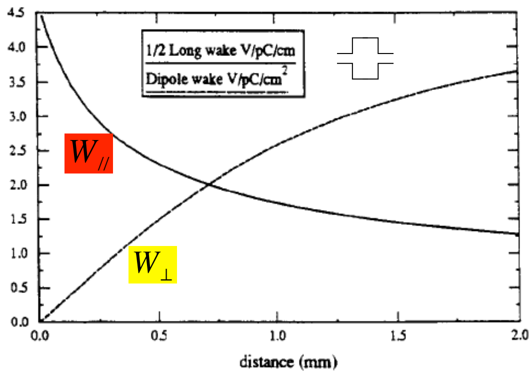
$$\boxed{x(t) = \frac{A}{\overline{\omega}\delta}\, sin(\overline{\omega}t)\, sin\left(\frac{\delta t}{2}\right) \equiv \frac{At}{2\overline{\omega}} sin(\overline{\omega}t) \frac{sin\left(\frac{\delta t}{2}\right)}{\frac{\delta t}{2}}}$$

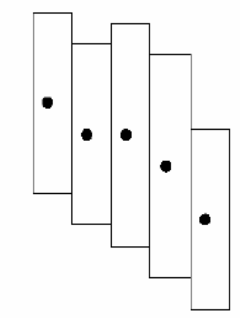
$$\lim_{\delta \to 0} x(t) = \frac{At}{2\overline{\omega}} \sin(\overline{\omega}t)$$



Relationship between transverse and longitudinal forces :
"**Panofsky-Wenzel theorem**".

$$\boxed{\begin{aligned} \nabla_{\perp} F_{//} &= \frac{\partial}{\partial z} \boldsymbol{F}_{\perp} \\ \nabla_{\perp} w_{//} &= \frac{\partial}{\partial z} \boldsymbol{w}_{\perp} \end{aligned}}$$

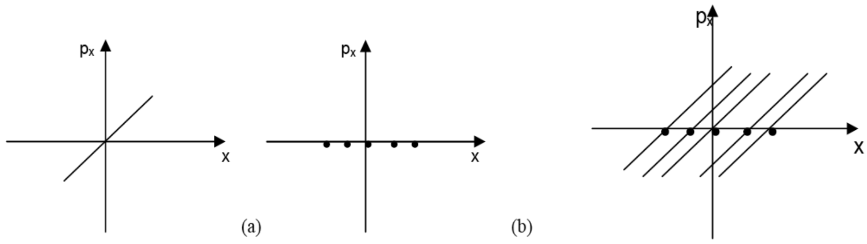




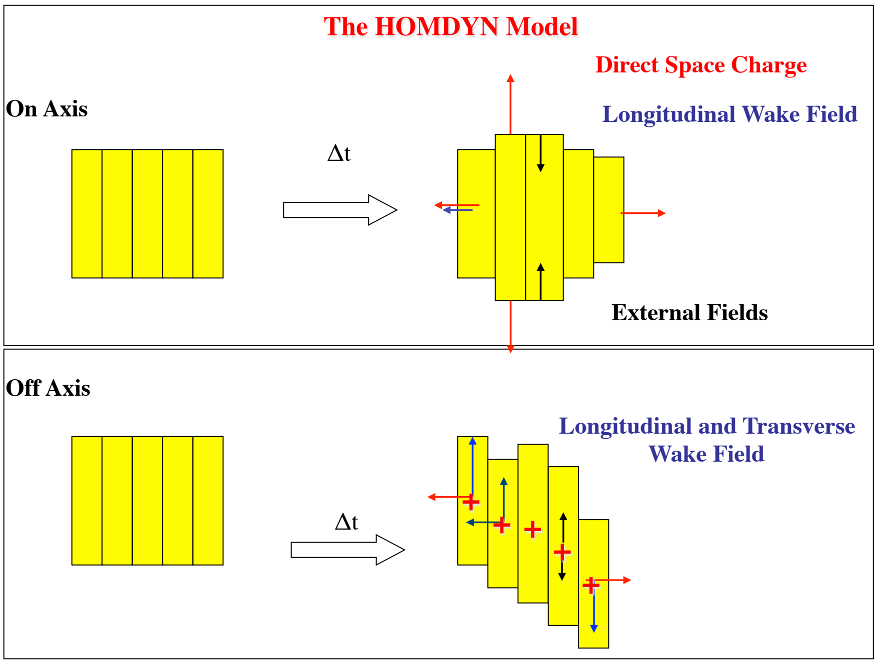
$$\varepsilon_{nx} = \langle\gamma\rangle\sqrt{\left\langle\left(x-\langle x\rangle\right)^2\right\rangle\left\langle\left(x'-\langle x'\rangle\right)^2\right\rangle-\left\langle\left(x-\langle x\rangle\right)\left(x'-\langle x'\rangle\right)\right\rangle^2}$$

$$x = x_c + \delta x$$

$$\varepsilon_{nx} = \langle\gamma\rangle\sqrt{\varepsilon_{\delta x}^2 + \varepsilon_{x_c}^2 + \varepsilon_{x_c\delta x}^2}$$



M.Ferrario, V.Fusco, M.Migliorati, L. Palumbo, Int. Journal of Modern Physics A, Vol 22, No. 3 (2007) 4214-4234



http://www.lnf.infn.it/acceleratori/sparc/hsparxino5.zip