Variations on AGT Correspondence

Giulio Bonelli (SISSA & ICTP)

January 22, 2013

with K. Maruyoshi, A. Tanzini, F. Yagi
Mainly based on arXiv:1208.0790

Outline

$\mathcal{N} = 2$ theories

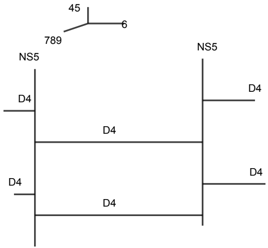
In 4d $\mathcal{N} = 2$ gauge theories have a non trivial RG-flow in the coupling constants and an exactly solvable moduli space of order parameters.

The IR effective dynamics of AF-theories can be computed from some spectral data (Seiberg-Witten curve & differential) defining an integrable system.

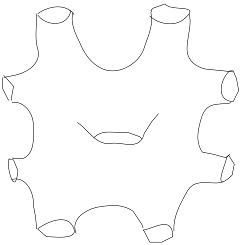
We concentrate on superconformal theories. AF ones can be obtained by scaling away dimensionfull parameters (see later).

The SW data materialize once the theory is geometrically engineered in M-theory via 5-branes.

Let's consider the example of $SU(2)$ with $N_f = 4$ (being super conformal, this looks more stable, but we will comment further on this)

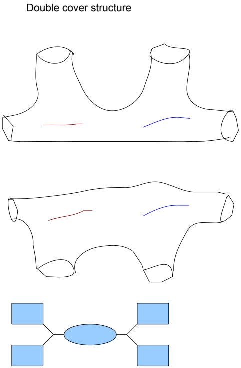


$\mathcal{N} = 2$ theories



This is the SW curve of the gauge theory!
With some more elaboration we can get also the SW differential for free!

$\mathcal{N} = 2$ theories



Actually the SW curve for $SU(2)$ and $N_f = 4$ is a double cover of the Riemann sphere with 4 punctures (which we denote as $C_{0,4}$).

$\mathcal{N} = 2$ theories

The M-theory geometry hosts the double covering in the total space $T^* C_{0,4}$. [That's a local K3!!!] The equation of the SW curve (in appropiate coordinates) is

$$x^2 = \phi_2(z)$$

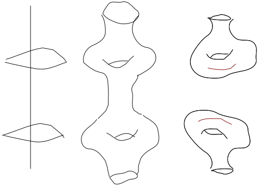
where x is a coordinate on the fiber and z on $C_{0,4}$.
The gauge theory data are encoded in the structure of the quadratic differential ϕ_2 and in the complex structure modulus of $C_{0,4}$.

- gauge coupling $\to$ complex structure
- masses $\to$ double pole residues of ϕ_2 at the punctures
- Coulomb modulus $\to$ remnant part of ϕ_2

The SW differential is simply $\lambda_{SW} = xdz = \sqrt{\phi_2(z)}dz$ and encodes all the geometry of the Coulomb branch (Prepotential).

$\mathcal{N} = 2$ theories

One can play again this game in the $SU(2)$ $\mathcal{N} = 2^*$ theory. (That's an elliptic quiver!)



The SW curve is

$$x^2 = M^2 \mathcal{P}(\tau, z) + u$$

where x is the fiber coordinate on $T^* C_{1,1}$ and the SW-differential is $\lambda_{SW} = xdz$.
(M =mass of the hyper in the adjoint, τ =gauge coupling and u = Coulomb modulus)
(The Weierstrass function $\mathcal{P}$ has a double pole on the torus!)

$\mathcal{N} = 2$ theories

One can reverse the logic above and use M-theory 5-branes to engineer $\mathcal{N} = 2$ d=4 gauge theories by wrapping N of them on a two cycle in a K3 geometry, the local geometry being uniquely fixed to be that of $T^* C_{g,n}$.
For $SU(2)$ gauge theories one uses 2 M5-branes and the SW curve is a double cover $x^2 = \phi_2(z)$.
All the data of the gauge theory are encoded in the moduli space of the curve $C_{g,n}$ and in the quadratic differential ϕ_2.
Higher rank $SU(N)$ theories can be generated by wrapping N M5-branes over the cycle, the SW curve being

$$x^N + x^{N-2}\phi_2 + x^{N-3}\phi_3 + \cdots + \phi_N = 0 \quad \text{and} \quad \lambda_{SW} = xdz$$

where ϕ_j are $j-$ *differentials* on the curve with prescribed polar structure at the n punctures.

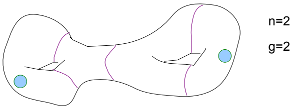
$\mathcal{N} = 2$ theories

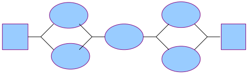
- If the curve is a punctured sphere, then one possibly get's linear quivers.
- If the curve is a punctured torus, then one possibly gets an elliptic quiver.
- If the genus is higher than one, one calls the quiver *generalized*.

To analyze the gauge theory spectrum one needs to shrink to a perturbative corner: decompose the curve $C_{g,n}$ in pair of pants and squeeze all necks. (Let's concentrate on the SU(2) case for the sake of simplicity) Then

- the theory in the sphere with 3 puncture is the building block: it has no gauge group, $SU(2)^3$ (manifest) flavour symmetry, four free hypers
- each neck is an $SU(2)$ gauge group: gluing is gauging a flavour $SU(2)$.

So, the theory engineered on $C_{g,n}$ has $SU(2)^n$ flavor symmetry and $SU(2)^{n+3g-3}$ gauge symmetry





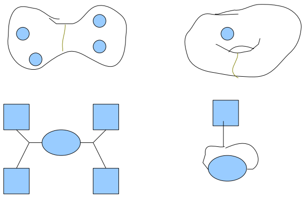
$\mathcal{N} = 2$ theories

Each pant decomposition (marking) defines a generalized quiver

The generic theory is non lagrangean!!!!

Large diffeos of $C_{g,n}$ make the S-duality group of the gauge theory

For example, the $N_f = 4$ and $\mathcal{N} = 2^*$ theories read



$\mathcal{N} = 2$ theories

Link to integrable system: why is all this working? The SW-curve can be interpreted as the spectral curve of the Hitchin system on $C_{g,n}$.

- Consider the internal part of the M5-branes geometry, that is the part sitting in $T^* C_{g,n}$
- BPS configurations are described by the reduction to $C_{g,n}$ of the self-dual connections

  $$F^+ = 0 \quad \rightarrow \quad D_{\bar{z}} \Phi_z = 0 \quad \text{and} \quad F_{z\bar{z}} = [\Phi_z, \Phi_{\bar{z}}]$$

  up to $SU(N)$ gauge transformations
- (modulo stability) it is like considering solutions to $D_{\bar{z}} \Phi_z = 0$ up to $SL(N, \mathbb{C})$ gauge transformations
- the eigenvalues of Φ_z are the displacements w.r.t. the base curve in $T^* C_{g,n}$ of the M5-brane strata
- therefore the SW-curve is identified with the spectral curve

  $$det(x\mathbf{1} - \Phi) = 0$$

  of the Hitchin system. For 2 M5-branes, we get holomorphic $SL(2, C)$ bundles over $C_{g,n}$.

AGT correspondence

[Suppose we knew the M5-brane theory exists] Consider then a system of N M5-branes on a product geometry $C_{g,n} \times M_4$ and some observable quantity independent on volume parameters $X\left(C_{g,n} \times M_4\right)$. This same object can be interpreted from the points of view of the two factors as

$$X^{CFT(M_4)}(C_{g,n}) = X^{4d-gauge(C_{g,n})}(M_4)$$

Now let's forget the M5-branes, and try to guess for $M_4 = \mathbb{R}^4$.

- on the right we might have a conformal 4d $\mathcal{N} = 2$ gauge theory with quiver structure dictated by the curve $C_{g,n}$.
- on the left a simple CFT_2 on $C_{g,n}$.

[in view of the previous analysis] to each puncture we associate matter: mass parameters are continuous implies that the CFT has to be non rational! The easiest (interacting) is Liouville field theory!

AGT correspondence

Indeed, it has been checked in a remarkable paper by Alday-Gaiotto-Tachikawa that an equality holds between

$$\mathcal{B}^{CFT}_{C_{g,n}}(\Delta, \alpha, c, \Omega) = Z^{quiver-gauge-th.}_{Nek}(m, a, \epsilon_1, \epsilon_2, \tau)$$

where

- (conformal dimensions Δ) $\equiv$ [hypers masses m]
- (intermediate momenta α) $\equiv$ [Cartan moduli a]
- (central charge c) $\equiv$ [$1 - 6Q^2$ where $Q = \sqrt{\frac{\epsilon_1}{\epsilon_2}} + \sqrt{\frac{\epsilon_2}{\epsilon_1}}$]
- (period matrix Ω) $\equiv$ [gauge couplings τ]

There are analytic proofs for $\mathcal{N} = 2^*$ and $N_f = 4$, the general case for the instanton part has been solved by Maulik-Okounkov but not the identification of the three point functions in general.
Actually, there is more than that! And much more to check....

AGT correspondence

There's a full correspondence of the Liouville correlation functions with the full partition function of the $\mathcal{N} = 2$ gauge theory on S^4 in the Omega background of Pestun.
One compares (with positive result!) [consider $N_f = 4$ for simplicity]

$$< \prod_{i=1}^{4} e^{\alpha_i \varphi(z_i)} >_{Liouville} = Z_{full}^{N_f=4}\left(S^4\right).$$

- The *CFT* side evaluates to conformal blocks $\times$ three-point functions integrated over the intermediate momentum

$$\int da |C_{1,2,a} C_{a,3,4}|^2 |\mathcal{B}_{C_{0,4}}(\Delta, a, c, q)|^2$$

- The gauge theory side has two contributions from the north and south patches (localized instantons), glued at the equator of S^4

$$\int da \left[a^2 Z^N_{pert} Z^S_{pert}\right] \left[Z^N_{inst} Z^S_{inst}\right]$$

and one can check that the perturbative part of the gauge theory also matches the three point function term!
(So, the match is not with an abstract CFT, but indeed with Liouville)

AGT correspondence

The AGT correspondence has been extended to the higher rank cases

The Nekrasov partition function of the $SU(N)$ gauge theory corresponds to the A_{N-1} Toda CFT along th above lines.

Since higher Toda CFTs are not fully solved yet, one has many partial checks, but not a full proof.

One can consider the classical limit and realize the classical Toda field theory as the manifestly unitary form of the Hitchin system at fixed SW-curve.

On top of it, one can match the gauge theory moduli space with Hitchin's one $SL(N,\mathbb{C})$ holomorphic bundles over $C_{g,n}$.

The SW-curve $x^N + x^{N-2}\phi_2 + x^{N-3}\phi_3 + \cdots + \phi_N = 0$ is defined by holomorphic i-differentials ϕ_i with a certain polar structure at the punctures. This implies $\mathcal{W}_{N-1}$ classical symmetry.

One can match also the conformal anomaly! Reduce the N M5-brane anomaly eight-form polynomial to the product geometry $C \times \mathbb{R}_{\epsilon_1,\epsilon_2}$ and match the Toda central charge with the gauge theory relevant anomaly (a and c coeff.)

More checks later...

A check of AGT for arbitrary $C_{g,n}$: generalized matrix models

The correspondence among the conformal blocks and the Nekrasov partition function can be checked by using the Coulomb gas realization of the conformal blocks.
On a side we know that in the $\epsilon_i \to 0$ limit, the Nekrasov partition function reproduces the SW prepotential, that is it encodes the data of the SW curve.

$$Z_{Nek} = e^{-\frac{1}{\epsilon_1\epsilon_2}[\mathcal{F}+\mathcal{O}(\epsilon_1,\epsilon_2)]]}$$

This means that the conformal blocks should encode the SW data too in the classical limit (large momenta and large background charge).

$$\mathcal{B}^{Coulonb\ \ gas}_{C_{g,n}} =< \prod_{k=1}^{n} e^{\alpha_k\varphi(w_k)} >^{Background\ \ charge}_{C_g}$$

can be written as a generalized matrix model
The building block is the prime form on C_g: that's an holomorphic bidifferential $E(z,w)(dz)^{1/2}(dw)^{1/2}$ with vanishing A-periods and local singularity $E(z,w) \sim \frac{1}{z-w}$.
(It is the Green function of the $\bar{\partial}$-operator on C_g.)

A check of AGT for arbitrary $C_{g,n}$: generalized matrix models

The relevant integral form of the conformal block for the chiral boson with background charge insertions is

$$Z_N^{\mathcal{C}_{g,n}}(\mathrm{w}, \mathrm{m}, \mathrm{p}, \mathrm{v}) \equiv$$

$$\int \prod_{i=1}^{N} dz_i \left[\omega(z_i)^{1+b^2} \prod_{i,l} E(z_i, \xi_l)^{-1-b^2} \right] \prod_{1 \le i < j \le N} E(z_i, z_j)^{-2b^2} \prod_i E(z_i, z^*)^{2b \sum_k m_k / g_s}$$

$$\times \exp\left(\frac{b}{g_s} \sum_{i=1}^{N} \left(\sum_{k=1}^{n} 2m_k \log \frac{E(z_i, w_k)}{E(z_i, z^*)} + 4\pi \sum_{a=1}^{g} p_a \int^{z_i} \omega_a \right) \right),$$

The large N limit amounts to take $g_s \to 0$ keeping $g_s N$, b, m_k and p_a finite. In the large N (background charge) and momenta (scaling limit), we can compute the spectral curve of the singular part of the generalized resolvent

$$R(z) \equiv \int_{\sum_\alpha C_\alpha} \rho(z') d_z \log\left(\frac{E(z, z')}{E(z, z^*)} \right), \quad \nu_\alpha = b g_s N_\alpha = \frac{1}{2\pi i} \oint_{C_\alpha} R(z).$$

expanding $R = \frac{1}{2} dW + R_{sing}$, one finds that $R_{sing}^2 = \phi_2$ which equals the SW-curve.
[This is also expected by the mirror topological string!]

Generalized M. M. and quantization of Hitchin systems

The integrals we just discussed define a generalization of matrix models:

(diagonalized) Hermitean m.m. $\rightarrow$ β-deformed generalized m.m.

$$\int \prod_i dz_i \prod_{ij} (z_i - z_j) e^{\sum_i V(z_i)} \quad \rightarrow \quad \int \prod_i dz_i \mu(z_i) \prod_{ij} [E(z_i - z_j)]^\beta e^{\sum_i V(z_i)}$$

The scaling limit describes the classical integrable system (Hitchin system).
We can study an intermediate limit $\epsilon_2 \rightarrow 0$ with N and all other parameters finite. (Nekrasov – Shatashvili limit).
In the NS limit the β-deformed generalized matrix model describes the wave-function of the corresponding quantum Hitchin system.
The loop equations for the generalized resolvent $\Psi(z) = < R(z) >_{gMM}$ can be written in a closed form in the NS limit

$$\left[-\epsilon_1^2 \partial_z^2 + \phi_2^{(\epsilon_1)} \right] \Psi = 0$$

where $\phi_2^{(\epsilon_1)} = \phi_2 + \mathcal{O}(\epsilon_1)$.
It is a deformation/quantization of the SW curve $x^2 + \phi_2 = 0$.

Generalized M. M. and quantization of Hitchin systems

The above differential equation is the quantization of the associated Hitchin system. This has been checked directly in the $g = 0, 1$ cases. (Gaudin models and elliptic Calogero-Moser models resp.). [Higher genus cases are just more involved technically, but interesting since they define higher genus integrable systems and their quantization].

For example for $\mathcal{N} = 2^*$ one finds

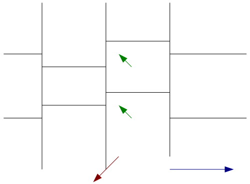
$$\left[-\epsilon_1^2 \partial_z^2 + M(M+\epsilon_1)\mathcal{P}(z) - u\right]\Psi = 0$$

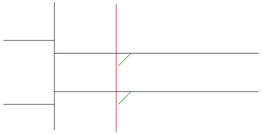
In general

- the above equation can be seen as the null state equation for a resulting null state insertion which in the generalized M.M. generates the resolvent.
- the quantum SW-curve is solved in the semiclassical expansion by $\Psi = e^{\frac{1}{\epsilon_1}\int^z x_{SW}(z)+\dots}$.
- it describes the quantum geometry behind instanton counting in the NS limit.

mini-AGT: vortex counting and degenerate fields

Classical limit (ungauging) with surface operators.
$SU(N) \times SU(N)$ 4d superconf. $\rightarrow SU(N)$ 2d $\mathcal{N} = (2,2)$ with $N_f = N$.
Instanton counting $\rightarrow$ Vortex counting.





mini-AGT: vortex counting and degenerate fields

When we reduce $4d$ to $2d$, the super conformality condition becomes the saturation of the flavor/color locking in the vortex solution.

The Vortex moduli space is a Lagrangean submanifold of the ADHM Instanton moduli space

One might compare the vortex counting to which the instanton counting reduces with the string engineered Nekrasov partition function.

The corresponding toric diagram (over which one has to evaluate the string theory contribution) is the strip with b.c. corresponding to the surface operators. This gives the 5d theory on a further twisted S^1 of radius β.

The result is given in closed form by a suitable combination of basic-Hypergeometric functions (where $q = e^{-\beta\hbar}$) – ${}_N\Phi_{N-1}$ – and matches the (equivariant) vortex counting.

In the 4d limit, the combination of Hypergeometrics ${}_NF_{N-1}$ reproduces the degenerate field conformal blocks of Toda theory.

[There's a correspondence among fusion channels of degenerate fields in the CFT , boundary conditions for the topological string on the strip and particular sectors in the Vortex counting.]

It is an higher rank check of the AGT conjecture.

AGT on ALE spaces

One can generalize the AGT correspondence by

$$\mathbb{R}^4 \to M_4$$

and ask

- if it still holds
- how does the M_4 geometry affects the CFT_2 (the other half dictionary!!!!)

Only partial answers by now

It was noted that a $\mathbb{Z}_2$ quotient of the Nekrasov instanton counting for the pure $SU(2)$ gauge theory reveals to compute the conformal blocks of the NS sector of $\mathcal{N} = 1$ superCFT in 2d in the Wittaker limit.

Actually the relevant gauge theory computation is that of the theory on *ALE* space and corresponds to the SuperLiouville $\mathcal{N} = 1$ CFT in 2d for the $\mathbb{C}^2/\mathbb{Z}_2$ case.

AGT on ALE spaces

More specifically one can show that

- Instanton counting on *resolved* $\mathbb{Z}_2$ quotients matches with the conformal blocks in the NS sector of $\mathcal{N} = 1$ sCFT in 2d. Explicit check for $N_f = 2$, $\mathcal{N} = 2^*$ and a linear $SU(2) \times SU(2)$ quiver.
- The exact three point functions of the $\mathcal{N} = 1$ sLiouville theory match the perturbative parts of the full partition function in the Ω background on $\mathbb{C}^2/\mathbb{Z}_2$.

Higher rank , i.e. $SU(N)$, gauge theories can be computed, as well as higher rank quotients, i.e. $\mathbb{C}^2/\mathbb{Z}_p$ and there are some proposals for their duals....

Toric singularities and resolution

General toric singularities are $\mathbb{C}^2/\Gamma_{p,q}$, where $\Gamma_{p,q}$ is the $\mathbb{Z}_p$ action

$$(z_1, z_2) \to (\omega z_1, \omega^q z_2)$$

where $\omega = e^{2\pi i/p}$, $p > q$ and coprime.
The Hirzebruch-Jung resolution of the above singularity $X_{p,q}$ is prescribed as follows. Let $p/q = [e_1, \ldots, e_{L-1}] \equiv e_1 - \frac{1}{e_2 - \frac{1}{e_3 - \ldots}}$ be the continuous fraction expansion of the ratio p/q in terms of the finite sequence of positive integers $\{e_1, \ldots, e_{L-1}\}$. The minimal HJ resolution divisors are $L-1$ rational curves with intersection matrix

$$\mathcal{I}_{p,q} = \begin{pmatrix} -e_1 & 1 & 0 & \ldots & 0 \\ 1 & -e_2 & \ddots & & \\ 0 & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & 1 \\ 0 & & & 1 & -e_{L-1} \end{pmatrix}$$

The resolved geometry is spanned by L copies of $\mathbb{C}^2$ patched together.
Examples:
— the resolved $\Gamma_{p,1}$ is the $\mathcal{O}_{\mathbb{P}^1}(-p)$ space ($L = 2$, $e_1 = p$)
— the resolved $\Gamma_{p,p-1}$ is ALE A_p ($L = p$ and $e_\ell = 2$ since $p/(p-1) = 2 - 1/[(p-1)/(p-2)]$).

Nekrasov partition function on toric singularities

Using equivariant localization and the geometry of the resolved space, we computed

$$Z_{\text{full}}^{X_{p,q}}(\vec{a},\epsilon_1,\epsilon_2)=\sum_{\{\vec{k}^{(\ell)}\}}\prod_{\ell=0}^{L-1}Z_{\text{full}}^{\mathbb{C}^2}(\epsilon_1^{(\ell)},\epsilon_2^{(\ell)},\vec{a}^{(\ell)})\xi_\ell^{c_1^{(\ell)}},$$

where $\vec{a}=\{a_\alpha\},\alpha=1,\dots,N$ are the vev's of the scalar field of the $\mathcal{N}=2$ vector multiplet,

$$a_\alpha^{(\ell)}=a_\alpha+k_\alpha^{(\ell+1)}\epsilon_1^{(\ell)}+k_\alpha^{(\ell)}\epsilon_2^{(\ell)}$$

(with $k^{(0)}=k^{(L)}=0$), $c_1^{(\ell)}=\sum_{\alpha=1}^N\sum_{m=1}^{L-1}C_{\ell m}k_\alpha^{(m)}$ and the local equivariant weights $(\epsilon_1^{(\ell)},\epsilon_2^{(\ell)})$ are appropriate linear combinations of ϵ_1 and ϵ_2 dictated by the dual fan. This can be decomposed as classical, one-loop and instanton parts as

$$Z_{\text{full}}^{X_{p,q}}(\epsilon_1,\epsilon_2,\vec{a},\{\xi_\ell\})=Z_{\text{cl}}^{X_{p,q}}(\epsilon_1,\epsilon_2,\vec{a})\sum_{I_1=0}^{p-1}\cdots\sum_{I_N=0}^{p-1}Z_{1-\text{loop}}^{X_{p,q}}\left(\epsilon_1,\epsilon_2,\vec{a},\vec{I}\right)Z_{\text{inst}}^{X_{p,q}}\left(\epsilon_1,\epsilon_2,\vec{a},\vec{I},\{\xi_\ell\}\right)$$

CFT_2 counterpart

The blow-up formula for the full partition function implies a CFT_2 symmetry algebra given by

$$\mathcal{A}_N(X_{p,q}) \equiv \oplus_{\ell=0}^{L-1}(\mathcal{H} \oplus {}^{\ell}W_N)$$

where the central charge of ${}^{\ell}W_N$

$$c_\ell = (N-1)\left(1 + Q_\ell^2 N(N+1)\right) \ , \quad Q_\ell^2 = \frac{(\epsilon_1^{(\ell)})^2 + (\epsilon_2^{(\ell)})^2}{\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}} + 2$$

This proposal passes several tests:

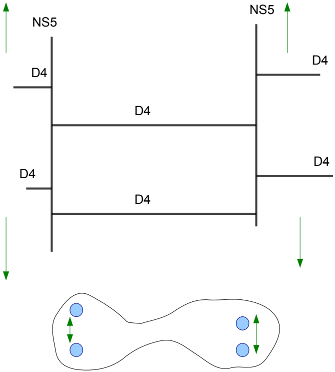
- M-theory compactification reproduces the small ϵs lim. and central charges
- Abelian case and counting of point-like instantons (Hilbert scheme of points) agree: for $\Sigma = T^2$ ($\mathcal{N} = 4$), and $q = p - 1$ character expansion for $\hat{su}(p)_1$.
- A_{p-1} case is well understood: $\mathbb{Z}_p$ parafermionic W_N algebra plus p copies of Heisenberg algebrae

Caveat: the non hyperkäler case is not done out of Kac-Moody's because Mac Kay correspondence doesn't hold anymore.
Problem: full interpretation as a CFT_2 in the general $X_{p,q}$ case.

Wild Quiver Gauge Theories

To extend from conformal to asymptotically free gauge theories : integrate out some massive d.o.f.

Geometric description: example $N_f = 4$ to pure SYM $SU(2)$.



Wild Quiver Gauge Theories

(After redefining the dynamical scale) $\phi_2^{N_f=4} \to \phi_2^{pure}$ The resulting SW curve is

$$x^2 = \frac{\Lambda^2}{z^3} + \frac{u}{z^2} + \frac{\Lambda^2}{z}$$

The corresponding CFT dual is now the (normalization of the) two point function of the operators corresponding to the cubic poles. This is a Wittaker states such that

$$L_1|G_1 >= \Lambda^2|G_1 > \quad \text{and} \quad L_n|G_1 >= 0 \quad \text{for } n > 1$$

The state is a perturbation in Λ of a primary normalized state of weight $\Delta = \alpha(Q - \alpha)$
One can show that

$$< G_1|G_1 >= Z_{Nek}^{pure}$$

after we identify the parameters as above.
The news here are that

- The AGT correspondence still holds in the AF cases too.
- Higher singularities should be allowed to the quadratic differential. More specifically: primary fields insertions correspond to lagrangian matter, higher ones generically to strongly coupled sectors.

Wild Quiver Gauge Theories

The pure $SU(2)$ curve, can be seen as the gauging of two Riemann Spheres with a cubic puncture and a regular one! The SW curve for each sphere comes from ungauging, that is $\Lambda \to 0$ with $z = \Lambda^2 w$ and $x = \Lambda^{-2}\tilde{x}$ [fixed by the $T^*\mathbb{P}^1$ geometry]

$$x^2 = \frac{\Lambda^2}{z^3} + \frac{u}{z^2} + \frac{\Lambda^2}{z} \quad \to \quad \tilde{x}^2 = \frac{1}{w^3} + \frac{u}{w^2} + \frac{\Lambda^2}{w} \quad \underset{\Lambda\to 0}{\to} \quad \tilde{x}^2 = \frac{1}{w^3} + \frac{u}{w^2}$$

which is the SW curve associated to the state $|G_1>$.
So, we learn that in other to extend to AF theories one has to extend the order of the poles of the quadratic differential to higher ones.
The underlying Hitchin field is therefore allowed to have higher singularities too.
The counting of the moduli is always the counting of $SL(N,\mathbb{C})$ holomorphic bundles over the M-theory curve with specified singularities and matches the gauge theory moduli and couplings.
This situation is called *wild ramifications*. The associated quiver gauge theory, wild ramified.
The curve above, is the first of a series, named by Cecotti and Vafa D_n-theories, which arise as strongly coupled limits of $SU(n-1)$ gauge theories with $N_f = 2$. These have an $SU(2)$ flavor symmetry.

Wild Quiver Gauge Theories

By gauging the diagonal $SU(2)$ of the D_n and the D_m theories, one can define the theory corresponding to the SW-curve

$$x^2 = \phi_2^{A_{n,m}}$$

with a quadratic differential $\phi_2^{A_{n,m}}$ with

- a pole of order $n+2$ at $z=0$
- a pole of order $m+2$ at $z=\infty$

These are called $A_{n,m}$ theories and are generically non-lagrangean. (The first ones are anyway, $A_{1,1}$ is pure $SU(2)$, $A_{1,2}$ is $N_f = 1$, $A_{2,2}$ is $N_f = 2$)
We can apply the AGT correspondence to compute the Nekrasov partition function of the $A_{n,m}$ theories if we compute the corresponding Wittaker states, that is the states corresponding to the operator $\Phi_n(z)$ with OPE with the stress-energy tensor

$$T(w)\Phi_n(z) \sim \frac{\Lambda^2 \Phi_n(w)}{(w-z)^{n+2}} + \text{l.s.t.} + \frac{\Delta + \mathcal{O}(\Lambda)}{(w-z)^2} + \text{l.s.t.}$$

where the l.s.t.'s and $\mathcal{O}(\Lambda)$ contain all the moduli of the strongly coupled theory.

Wild Quiver Gauge Theories

The closed form of the Wittaker state $|G_n>$ for the D_n theory can be explicitly computed in terms of the Shapovalov matrix of the reference primary state

$$Q_\Delta(Y, Y') =< \Delta|\mathcal{L}_Y \mathcal{L}_{-Y'}|\Delta >,$$

the moduli and the dynamical scale Λ in the Verma module of $|\Delta>$.
The Nekrasov partition function can be computed from the CFT correspondent construction as

$$Z_{Nek}^{A_{n,m}} =< G_n|G_m >$$

extending to the non-lagrangean cases the computations for the low (n, m)'s.
Of course, by picking the $\epsilon_i \to 0$ limit one can obtain the prepotential of the theory

$$\mathcal{F}^{A_{n,m}} = -\lim_{\epsilon_i \to 0} \epsilon_1 \epsilon_2 \ln Z_{Nek}^{A_{n,m}}.$$

As a check, the correct quadratic differential

$$-\lim_{\epsilon_i \to 0} \epsilon_1 \epsilon_2 < G_n|T(z)|G_m > / < G_n|G_m >\sim \phi^{A_{n,m}}(z)$$

is reproduced in the classical limit.
Generically $\Psi_{C_{g,hp}}(z) =< \Phi_{2,1}(z)|C_{g,hp} >$ is the wave function of the corresponding quantum Hitchin system with wild ramifications.

Conclusions

RECAP:

- AGT correspondence has been checked and generalized in several ways
- In doing that, one reinterprets the gauge theory data in the language of integrable systems and their quantization (Quantum Hitchin System). Here the links to *generalized* matrix models [and topological strings] are important (mirror pictures)
- Extension to toric singularities: incomplete dictionary!
- Extension to asymptotically free theories implies considering a well defined generalization of the above named *Wild* quiver gauge theories (Quantum Hitchin System with Wild Ramifications)
- M-theory engineering sheds some light on the nature of the correspondence

Open Issues

- Prove AGT in the general case (a Phys proof!)
- Generalize AGT to more general 4d space-times (other than $X_{p,q}$) [give the full 4d-cft dictionary!]
- Understand the geometry behind the $|G_n>$ Wittaker states and generalize to more generic irregular states
- What's the role of integrable hierarchies in all that? (Which hierarchy?)
- Who is the (quantum) integrable system corresponding to $X_{p,q}$? And more in general?
- • Extend these techniques along breaking $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$: is it possible?

Thank you!