Constructing Near-BPS Skyrmions with Constant Baryon Densities

L. Marleau, M.O. Beaudoin
Département de physique, génie physique et d'optique
Université Laval, Québec, Canada

Introduction

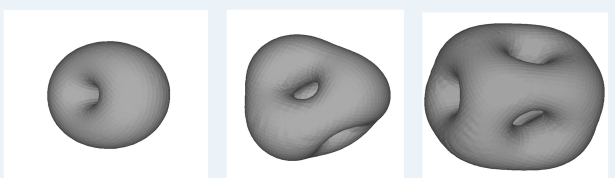
Despite the successes of the Standard Model, we still have no clear explanation on how quarks and gluons form nucleons and nuclei. The Skyrme Model provides an alternative. It is a low-energy QCD effective meson field theory where baryon emerges as topological solitons. However, the Skyrme Model seems unable to reproduce the small binding energy in nuclei although it remains a relatively accurate picture of the nucleons. This suggests that Skyrme-like models that nearly saturate the Bogomol'nyi bound may be more appropriate since their mass is roughly proportional to the baryon number.

We propose a Near-BPS Skyrme Model[5]. It consists of terms up to order six in derivatives of the pion fields, including the nonlinear and Skyrme terms which are assumed to be relatively small. Our special choice of mass term leads to well-behaved analytical BPS-type solutions with approximately constant baryon density configurations, as opposed to the usual shell-like configurations found in most extensions of the Skyrme Model. Fitting the 4 model parameters, we find a remarkable agreement for the binding energy per nucleon B/A with respect to experimental data.

Motivation

In the absence of a clear explanation on how quarks and gluons form nucleons and nuclei form the Standard Model, one has to rely on alternative approaches. One of the most attractive ideas in that regard is the Skyrme Model[1], an effective meson (pion) field theory motivated by low-energy QCD arguments such as a $1/N_c$ expansion or more recently holographic QCD[2]. In the Skyrme Model, the baryon and nuclei emerge as topological solitons (Skyrmions). This is somewhat odd since the model is usually constructed out of pion fields alone.

The Skyrme Model is relatively successful at describing pion and baryon physics with predictions that are an accurate to at least 30% (often to a few %) with regard to experimental data. However, it fails when it comes to multibaryon physics or more precisely nuclei. For instance, the binding energies are much too large especially for small nuclei (e.g. deuteron $\simeq 40\times$ observed value). Moreover, finding the lowest energy configurations is numerically challenging so that only small A solutions are known. They lead to approximately toroidal, tetrahedral, cubic configurations for $A = 1, 2, 3$ standard Skyrmions, respectively,



$A = 2$ $A = 3$ $A = 4$

This is in contradiction to the constant densities observed in nuclei. Further analysis of various potential (mass) terms, rotational deformations, higher order terms in derivatives or additional mesons (e.g. ω, ρ, ...) lead to similar configurations and binding energies

Given that the mass of the nuclei are closely proportional to that of the nucleon, $M_{\text{nuclei}} \approx A \cdot M_{\text{nucleon}}$, and that BPS-solitons follows this exact pattern, we propose a Skyrme-like model in a regime where the solutions are Near-BPS solitons. We further adjust the model so as to obtain a constant baryon density. This lead to a so-called **Near-BPS Model**

Near-BPS Model

In order to get closer to saturation of Bogomol'nyi bound without losing link with the Skyrme model, we consider an extension of the original Skyrme Model with the Lagrangian density

$$\mathcal{L}_{\text{NBPS}} = \underbrace{\mathcal{L}_0 + \mathcal{L}_6}_{\text{BPS-solitons}} + \underbrace{\mathcal{L}_2 + \mathcal{L}_4}_{\text{Skyrme}} \tag{1}$$

We then assume that $\mathcal{L}_0 + \mathcal{L}_6$ dominate and treat $\mathcal{L}_2$ and $\mathcal{L}_4$ are small as perturbations. The terms are respectively:

- $\mathcal{L}_0 = -\mu^2 V(U)$: the potential term (χSB term) which is responsible the pion mass.
- $\mathcal{L}_2 = -\alpha \text{Tr}[L_\mu L^\mu]$: the quadratic NLσ term.
  Here, $L_\mu = U^\dagger \partial_\mu U$ where the pion fields are introduced through the $SU(2)$ matrix $U = \phi_0 + i\tau_i\phi_i$ with $\phi_0^2 + \phi_i^2 = 1$.
- $\mathcal{L}_4 = \beta \text{Tr}\left([L_\mu, L_\nu]^2\right)$: the quartic Skyrme term (necessary to stabilize soliton in the Skyrme Model).
- $\mathcal{L}_6 = -\frac{3}{2}\frac{\lambda^2}{16^2}\text{Tr}\left([L_\mu, L_\nu]\left[L^\nu, L^\lambda\right][L_\lambda, L_\mu]\right)$: the sextic term which remains **quadratic** in time derivatives.

Finite energy solutions for the Skyrme fields are characterizes by a conserved topological charge which Skyrme identified as the baryon number or mass number A in the context of nuclei

$$A = \int d^3r \mathcal{B}^0 = -\frac{\epsilon^{ijk}}{24\pi^2}\int d^3r \text{Tr}\left(L_i L_j L_k\right). \tag{2}$$

When $\alpha, \beta = 0$, the solutions are BPS-solitons so their masses are exactly proportional to A.

This allows us to choose an axially symmetric solution for U in the limit where $\alpha = \beta = 0$

$$U = \cos F(r) + i\hat{\mathbf{n}} \cdot \tau \sin F(r) \tag{3}$$

where $\hat{\mathbf{n}}$ is the unit vector can be written in terms of the spherical coordinates r, θ, and ϕ.

$$\hat{\mathbf{n}} = (\sin\theta\cos A\varphi, \sin\theta\sin A\varphi, \cos\theta). \tag{4}$$

Method

The model has evolve over the last few years:

1- Adam et al **[ASW]**[3]: consists of $\mathcal{L}_0 + \mathcal{L}_6$ alone ($\alpha = \beta = 0$) with usual mass term $V(U) = -\frac{1}{2}\text{Tr}[I - U] = -\frac{1}{2}\text{Tr}[U_-]$ where $U_\pm = (2I \pm U \pm U^\dagger)/4$ The solutions are compactons. They saturate the Bogomol'nyi bound leading to zero binding energies and unstable nuclei.

2- Near-BPS Model **[BoM]**[4]**:** The choice of potential $V_{\text{BoM}}(U) = -\frac{1}{2}\text{Tr}\left[U_+ U_-^3\right]$ gives baryon density with unsatisfactory shell-like configuration.

3- Other Near-BPS Model **[BHM]**[4]**:** With $V_{\text{BHM}}(U) = -\frac{4}{9}\frac{\text{Tr}\left[U_+ U_-^3\right]}{\ln\left(\frac{1}{2}\text{Tr}[U_-]\right)}$, the baryon density is gaussian-like

We are interested in a model that can reproduce the constant baryon density in nuclei. We propose an "Improved the Near-BPS Model [BeM]".

An Improved the Near-BPS Model [BeM][5]:

Our approach proceeds through of the following steps:

Step 1: Introduce an appropriate $V(U)$, here

$$V_{\text{BeM}}(U) = \frac{112}{45}\text{Tr}\left[U_+ U_-^3\right]\frac{(1 - (14/5)\ln(\text{Tr}[U_-]/2))}{1 - \sqrt{1 - (14/5)\ln(\text{Tr}[U_-]/2)}} \tag{5}$$

in order to generate constant baryon density.

Step 2: Find the **analytical solution** **($\alpha = \beta = 0$)** where $x = ar$ with $a = \mu/(18A\lambda)$

$$F_{\text{BeM}}(x) = \pi - 2\arccos[\exp\left(-x^2 - \frac{7}{5}x^4\right)] \tag{6}$$

The proposed model with $V_{\text{BeM}}(U)$ leads to a constant baryon density.

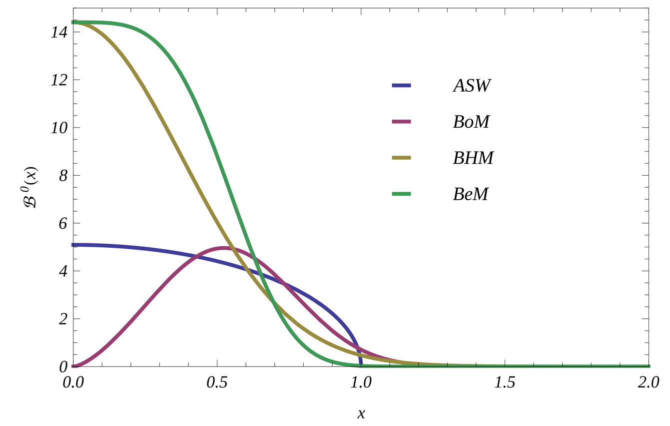


Figure : Baryon densities $\mathcal{B}^0(x)$ for various models: compacton [ASW], shell-like configuration [BoM], gaussian-like density [BHM] and the constant baryon density Near-BPS model [BeM].

Step 3: Relax the constraint α, $\beta = 0$ and include $\mathcal{L}_2 + \mathcal{L}_4$ as small perturbations. The static energy then reads

$$E_{\text{s}} = 15.93\ \mu\lambda\ A + \frac{8\pi\alpha}{a}\left(2.6880 + 0.4850\left(1 + A^2\right)\right) + 64\pi\beta a\left(5.138\left(1 + A^2\right) + 1.882A^2\right)$$

Step 4: Add the rotational energy

$$E_{\text{r}} = \frac{1}{2}\left[\frac{j(j+1)}{V_{11}} + \frac{i(i+1)}{U_{11}} + \left(\frac{1}{U_{33}} - \frac{1}{U_{11}} - \frac{A^2}{V_{11}}\right)\kappa^2\right]$$

where U_{33}, U_{11}, V_{11} are (iso)rotational moments of inertia.

Step 5: Add Coulomb energy E_{C} from charge density $\rho(\mathbf{x}) = J^0_{EM} \equiv \frac{1}{2}\mathcal{B}^0(\mathbf{x}) + i_3\frac{\mathcal{U}_{33}(\mathbf{x})}{U_{33}}$.

Step 6: Add isospin breaking term E_{I} to account for proton-neutron mass difference using $E_{\text{I}} = a_I i_3$ with $a_I = (E^n_{\text{C}} - E^p_{\text{C}}) - \Delta M^{\text{expt}}_{n-p}$

Step 7: The values of the parameters μ, α, β and λ remain to be fixed. This is done by fitting these 4 parameters using our result for the nuclear mass:

$$E_{\text{t}} = E_{\text{s}} + E_{\text{r}} + E_{\text{C}} + E_{\text{I}} \tag{7}$$

Once these parameters are fixed the masses and binding energies per nucleon can be predicted for all nuclei.

Results

We first consider the case where $\alpha = \beta \sim 0$ which provides a good estimate for the values of μ, α, β, and λ required in the 4-parameter model (1) and corresponds to the limit where the minimization of the static energy leads to the exact analytical BPS solution in (6). For simplicity, we choose the mass of the nucleon and that of a nucleus X with no (iso)rotational energy; the best choice turns out to be the nuclei H and ^{40}Ca (Fit I). A second fit is performed using the masses of 140 most stable isotopes (Fit II). Finally, we optimize for the binding energy per nucleon B/A with the same set of 140 most stable isotopes (Fit III).

	Fit I	Fit II	Fit III
μ (10^4 MeV^2)	1.232	1.0226	1.3264
α (10^{-3} MeV^2)	0	1.48244	0.00016580
β (10^{-8} MeV^0)	0	1.20427	0.00091319
λ (10^{-3} MeV^{-1})	4.741	5.7037	4.3991

The masses of the nuclei including static, (iso)rotational, Coulomb, and isospin breaking contributions are then computed using Eq. (7) which results in predictions that are accurate to at least 0.8% for the masses, even for heavier nuclei, and within 10% for binding energy per nucleon B/A. The predictions remain surprisingly good compared to that of the Skyrme model which overestimates the B/A by at least an order of magnitude.

Results...

The results are presented in the next figure which displays the general behavior of B/A as a function of the baryon number for our model [BeM].

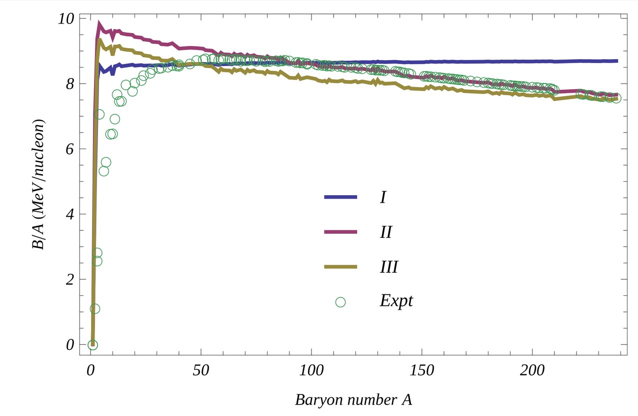


Figure : Binding energy per nucleon B/A as a function of the baryon number for model [BeM] using the parameters of Fits I, II, III, compared with experimental values.

The mass of each nucleus gets a contribution from static, (iso)rotational, Coulomb, and isospin breaking energies. It is interesting to compare how these contributions affect binding energy per nucleon B/A. While they emerge from a low-energy QCD effective meson field theory, the results are similar to that of the more empirical Bethe-Weizsäcker mass formula.

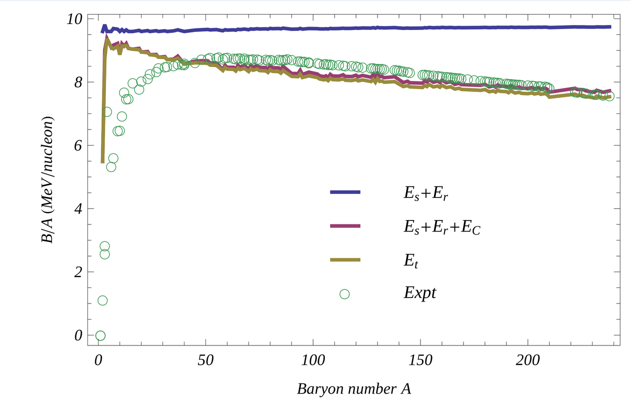


Figure : Contribution to the binding energy per nucleon B/A coming from the static, (iso)rotational, Coulomb, and isospin breaking for Fits III, compared with experimental values.

The predicted masses (Fit III) are accurate to at least $\pm 0.4\%$ with respect to the experimental values. Further analysis required:

- The size of nuclei or root mean square radius for the charge density is, as observed, proportional to $A^{\frac{1}{3}}$ but is slightly too large: $\langle r^2_{\text{em}}\rangle^{\frac{1}{2}} = (1.91\text{ fm})\,A^{\frac{1}{3}}$ (Expt$=$ $(1.23\text{ fm})\,A^{\frac{1}{3}}$).
- The rotational energy is small especially for large nucleus. It would imply for example that the mass difference between the Δ and the nucleon is $M_\Delta - M_N \sim 50$ MeV
- The model assumes that the parameters α, β are small in the Near-BPS approach. Since the pion decay constant is $F_\pi = 4\sqrt{\alpha} \ll 186$ MeV $m_\pi = \mu\sqrt{2/\alpha} \gg 138$ MeV , the link to soft-pion physics is yet unclear
- Several aspects of the model (e.g. magnetic moments, vibrational and rotational excitations....) remain to be studied.

Conclusion

The model includes both the NLσ terms and the Skyrme term in order to retain some of the successes of the Skyrme Model which is not the case for some pure BPS Skyrmion. Since the potential and the term of order 6 in derivatives dominate, it is possible to approximate the lowest energy solution by an analytical axial form. This allows computing most of the relevant physical quantities contrary to the original Skyrme Model where it remains a challenge to even find the lowest energy solution.

We have demonstrated that it is possible to construct constant baryon and charge densities for all A. Although it remains a prototype model, it leads to a remarkably accurate description of B/A and other properties of the nuclei. More generally, it clearly supports the idea that nuclei could be Near-BPS Skyrmions.

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email: lmarleau@phy.ulaval.ca