Instantaneous Bethe–Salpeter Approach to Pseudoscalar Mesons

Wolfgang Lucha

Institute for High Energy Physics, Austrian Academy of Sciences, Vienna, Austria

Incentive: twofold nature of pion and kaon

In principle, bound states of the basic degrees of freedom of a quantum field theory emerge as solutions of the homogeneous Bethe–Salpeter equation [1]; the natural targets of the latter are quantum electro- and chromodynamics.

However, conceptual reasons (like the problem of interpretation of time-like excitations) and practical issues (like the proper embedding of interactions) provide reasonable grounds to discard relativistic covariance and to seek for manageable three-dimensional reductions of the Bethe–Salpeter formalism. Prominent cornerstones along such path of nonrelativistic reduction are the Salpeter equation [2] and the reduced Salpeter equation [3]; see, e.g., Ref. [4]. For both these relations, powerful solution techniques have been devised [5].

Surprisingly or not, upon solving the Salpeter equation for interactions that should entail only stable solutions, one obtains also unstable ones [5,6]. This prompts us to embark on a systematic, ideally analytic study of this puzzle: first partial results exist for reduced Salpeter equations and more general [7] instantaneous Bethe–Salpeter equations [8], and even Salpeter equations [9].

The analysis of instabilities is greatly facilitated by analytic knowledge of at least a few rigorous solutions to compare with. A simple novel approach [10], starting from states of zero mass, provides not only such solutions but also a Salpeter treatment of light pseudoscalar mesons as quark–antiquark bound states which accounts for their (almost) masslessness due to (explicitly and) spontaneously broken global symmetries of quantum chromodynamics [11].

Full [2] and reduced [3] Salpeter equations

Assuming the bound-state constituents $i = 1, 2$ to interact instantaneously and to propagate as free particles of effective masses m_i, the (homogeneous) Bethe–Salpeter equation reduces to the Salpeter equation; see, e.g., Ref. [4]. For fermion–antifermion states, it reads in their center-of-momentum frame

$$\phi(\boldsymbol{p}) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \sum_\Gamma V_\Gamma(\boldsymbol{p}, \boldsymbol{q}) \left(\frac{\Lambda_1^+(\boldsymbol{p})\,\gamma_0\,\Gamma\,\phi(\boldsymbol{q})\,\Gamma\,\Lambda_2^-(\boldsymbol{p})\,\gamma_0}{M - \sqrt{\boldsymbol{p}^2 + m_1^2} - \sqrt{\boldsymbol{p}^2 + m_2^2}} \right.$$

$$\left. - \frac{\Lambda_1^-(\boldsymbol{p})\,\gamma_0\,\Gamma\,\phi(\boldsymbol{q})\,\Gamma\,\Lambda_2^+(\boldsymbol{p})\,\gamma_0}{M + \sqrt{\boldsymbol{p}^2 + m_1^2} + \sqrt{\boldsymbol{p}^2 + m_2^2}} \right) ,$$

with the projectors for positive and negative energies of particle i defined by

$$\Lambda_i^\pm(\boldsymbol{p}) \equiv \frac{\sqrt{\boldsymbol{p}^2 + m_i^2} \pm \gamma_0\,(\boldsymbol{\gamma}\cdot\boldsymbol{p} + m_i)}{2\sqrt{\boldsymbol{p}^2 + m_i^2}} , \qquad i = 1, 2 ;$$

the interaction terms involve Dirac matrices Γ reflecting the Lorentz nature of any constituent's effective coupling and related scalar functions $V_\Gamma(\boldsymbol{p}, \boldsymbol{q})$. For any state, the related solution $\phi(\boldsymbol{p})$, its Salpeter amplitude, encodes the distribution of the relative momentum of this bound state's constituents, $\boldsymbol{p}$. Ignoring negative-energy contributions gives the reduced Salpeter equation

$$\left[M - \sqrt{\boldsymbol{p}^2 + m_1^2} - \sqrt{\boldsymbol{p}^2 + m_2^2} \right] \phi(\boldsymbol{p})$$

$$= \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \sum_\Gamma V_\Gamma(\boldsymbol{p}, \boldsymbol{q})\,\Lambda_1^+(\boldsymbol{p})\,\gamma_0\,\Gamma\,\phi(\boldsymbol{q})\,\Gamma\,\Lambda_2^-(\boldsymbol{p})\,\gamma_0 .$$

For simplicity, let all interactions between bound-state constituents respect spherical symmetry and thus be describable, for a specific Lorentz structure $\Gamma \otimes \Gamma$, in configuration space by related central potentials $V_\Gamma(r)$, $r \equiv |\boldsymbol{x}|$, or in momentum space by the $L = 0, 1, \ldots$ Fourier–Bessel transforms defined in terms of spherical Bessel functions of the first kind $j_n(z)$, $n = 0, \pm 1, \ldots,$

$$V_L(p, q) \equiv \frac{2}{\pi} \int_0^\infty \mathrm{d}r\, r^2\, j_L(p\,r)\, j_L(q\,r)\, V_\Gamma(r) , \qquad p \equiv |\boldsymbol{p}| , \qquad q \equiv |\boldsymbol{q}| .$$

This simplifies each Salpeter equation to a set of coupled radial equations [5] for bound-state mass eigenvalues M and radial Salpeter components $\varphi_j(p)$.

Rigorous interaction–solution relation [10]

We seek exact analytic solutions of homogeneous Bethe–Salpeter equations with instantaneous interactions. Constructing, for a given potential $V(r)$, a single rigorous solution is tantamount to determining, for a chosen solution, that potential $V(r)$ for which the bound-state equation yields this solution: The relation between the properties of the bound state and the interactions experienced by its constituents is established. In order to extract $V(r)$ from some Salpeter equation in momentum-space representation, we have to cast this bound-state equation by application of Fourier–Bessel transformations into configuration-space formulation; needless to say, this will be achievable only under very favourable circumstances. As first step, we focus to reduced Salpeter equations and to systems requiring the least conceivable number of Salpeter components: bound states of spin sum 0. Their Salpeter amplitude involves just two independent components, which, moreover, by the generic structure of reduced Salpeter equations become identical, i.e., a single $\varphi(p)$. The remaining task then is to single out Lorentz structures $\Gamma\otimes\Gamma$ that allow, for suitable $\varphi(p)$ ansatzes, the reduced Salpeter equation to be transformed to configuration space, where the corresponding $V(r)$ can be easily read off. In the examples presented below (with all dimensional quantities in units of adequate powers of mass) a non-zero bound-state mass may be absorbed by $V(r)$; thus, let $M = 0$ and, for notational convenience, $m_1 = m_2 \equiv m \geq 0$.

The simplest reduced Salpeter equations are those for $m = 0$; among these, upon introduction of a parameter $\eta = 1, 2$, the ones for the Dirac structures $\Gamma\otimes\Gamma = \gamma_\mu\otimes\gamma^\mu$ ($\eta = 2$) and $\Gamma\otimes\Gamma = \frac{1}{2}\left(\gamma_\mu \otimes \gamma^\mu + \gamma_5 \otimes \gamma_5 - 1 \otimes 1\right)$ ($\eta = 1$) can be subsumed under a common form involving just the function $V_0(p, q)$:

$$2\,p\,\varphi(p) + \eta \int_0^\infty \mathrm{d}q\, q^2\, V_0(p, q)\, \varphi(q) = M\, \varphi(p)\ .$$

A rather obvious first idea for $\varphi(p)$ is the exponential $\varphi(p) \propto \exp(-p)$; this yields a potential which remains finite for $r \to \infty$ and thus is not confining:

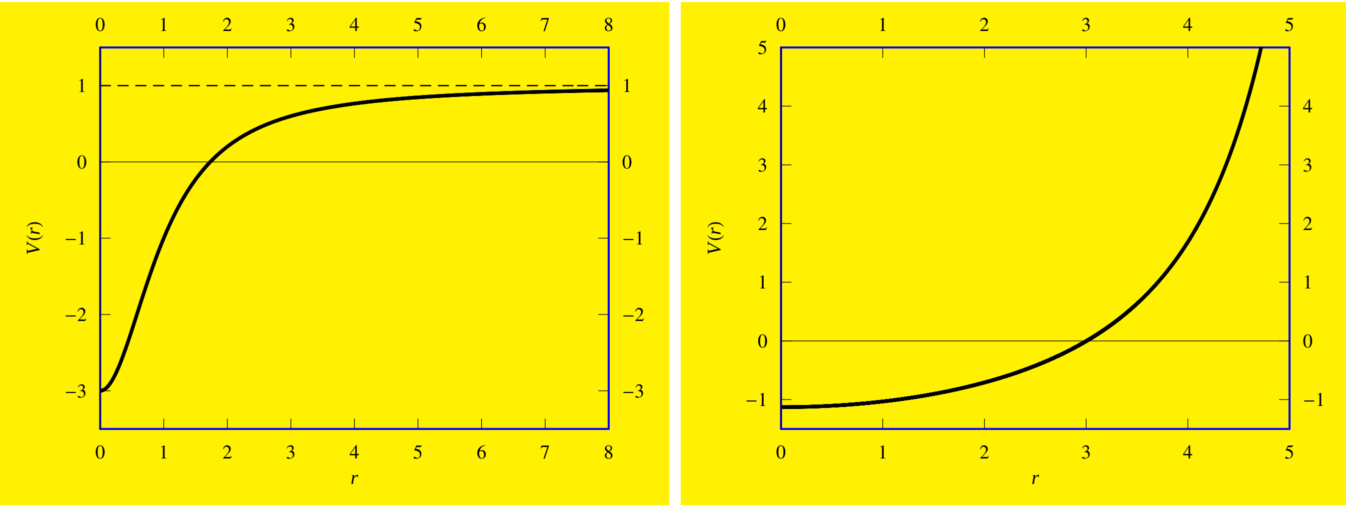
$$V(r) = \frac{2}{\eta}\left(1 - \frac{4}{r^2 + 1}\right).$$

The experimental feature of colour confinement, the nonobservability of the coloured degrees of freedom of quantum chromodynamics as isolated or free particles, should be reflected by any Bethe–Salpeter description of hadrons.

A potential that, for $r \to \infty$, rises beyond bounds [$V(r) \to \infty$ for $r \to \infty$] and thus may be labelled as confining requires an ansatz $\varphi(p)$ which is more concentrated near $r = 0$, such as a Gaussian $\varphi(p) \propto \exp(-p^2)$ which yields a potential that involves the "imaginary error function" $\operatorname{erfi} z \equiv -\mathrm{i}\,\operatorname{erf}(\mathrm{i}\,z)$:

$$V(r) = \frac{1}{\eta}\left[\left(r - \frac{2}{r}\right)\operatorname{erfi}\left(\frac{r}{2}\right) - \frac{2}{\sqrt{\pi}}\exp\left(\frac{r^2}{4}\right)\right], \qquad V(0) = -\frac{4}{\eta\,\sqrt{\pi}} \ .$$

$V(r)$ for $\Gamma\otimes\Gamma = \gamma_\mu\otimes\gamma^\mu$, from exponential (left) and Gaussian (right) $\varphi(p)$:



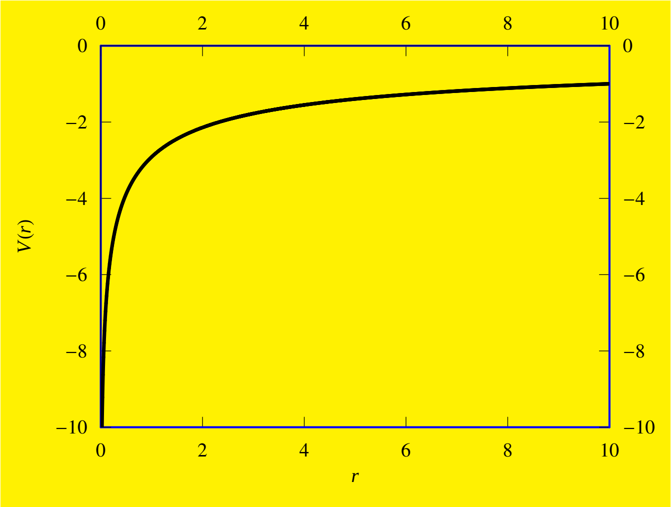
It is scarcely surprising that bound-state constituents of nonvanishing mass $m \gneqq 0$ call for somewhat more careful or sophisticated selection of tentative solutions $\varphi(p)$. The square root in the relativistic free energy suggests to try rational functions such as $\varphi(p) \propto (p^2+1)^{-2}$. The reduced Salpeter equation for, e.g., the Lorentz structure $\Gamma\otimes\Gamma = \frac{1}{2}\,(\gamma_\mu \otimes \gamma^\mu + \gamma_5 \otimes \gamma_5 - 1 \otimes 1)$ reads

$$2\sqrt{p^2 + m^2}\,\varphi(p) + \int_0^\infty \mathrm{d}q\,q^2\,V_0(p,q)\,\varphi(q) = M\,\varphi(p) \ .$$

If $m = 1$, the potential corresponding to $\varphi(p) \propto (p^2+1)^{-2}$ makes use of the modified Bessel function of order zero $K_0(z)$; for real $z \to \infty$, $K_0(z)$ decays faster than exponential. Thus, $V(r)$ is not confining (but singular at $r = 0$):

$$V(r) = -\frac{8}{\pi}\,K_0(r)\exp r \ , \qquad V(r) \xrightarrow[r\to 0]{} \frac{8}{\pi}\ln r \ , \qquad V(r) \xrightarrow[r\to\infty]{} 0 \ .$$

$V(r)$ for $\Gamma\otimes\Gamma=\frac{1}{2}\,(\gamma_\mu\otimes\gamma^\mu+\gamma_5\otimes\gamma_5-1\otimes 1)$, from rational $\varphi(p)\propto(p^2+1)^{-2}$:



If $m>1$, analytic continuation plus contour integration immediately imply

$$V(r)=-2\sqrt{m^2-1}-\frac{2}{r\sqrt{m^2-1}}+\frac{8}{\pi\,r}\int_m^\infty \mathrm{d}\rho\,\rho\exp[(1-\rho)\,r]\,\frac{\sqrt{\rho^2-m^2}}{(\rho^2-1)^2}\ ,$$

$$V(r)\xrightarrow[r\to\infty]{}-2\sqrt{m^2-1}\xrightarrow[m\to 1]{}0\ .$$

Here, for $r\to 0$ the two r-dependent portions of $V(r)$ conspire to develop a merely logarithmic singularity; so $V(r)$ behaves similarly to that for $m=1$.

For $\Gamma\otimes\Gamma=\gamma^0\otimes\gamma^0$, even for $m=0$ both $V_0(p,q)$ and $V_1(p,q)$ enter into the reduced Salpeter equation, which renders rather useless the application of a Fourier–Bessel transformation with unique L to such bound-state equation:

$$2\,p\,\varphi(p)+\frac{1}{2}\int_0^\infty \mathrm{d}q\,q^2\,[V_0(p,q)+V_1(p,q)]\,\varphi(q)=M\,\varphi(p)\ .$$

However, a closer inspection of how some reduced Salpeter equation may be derived from its full-Salpeter counterpart provides the clue how to continue: In the $\gamma^0\otimes\gamma^0$ reduced Salpeter equation, set $V_1(p,q)$ equal to $V_0(p,q)$ to get the resulting $V_0(r)$, set $V_0(p,q)$ equal to $V_1(p,q)$ to get the associated $V_1(r)$, and then determine that linear combination of these $V_0(r)$ and $V_1(r)$, if any, that fits to the desired solution. An exponential $\varphi(p)\propto\exp(-p)$ thus gives

$$V(r)=1-\frac{8}{r^2+1}\ .$$

Ultimate target: pseudoscalar mesons [11]

In general, any full Salpeter equation is represented by an equivalent system of more than one coupled equations; even for bound states of spin sum 0 the set consists of two equations. One notable exception to this limitation forms again the Dirac structure $\Gamma\otimes\Gamma = \frac{1}{2}\,(\gamma_\mu\otimes\gamma^\mu + \gamma_5\otimes\gamma_5 - 1\otimes 1)$; there one of these relations contains no interactions at all and is thus purely algebraic:

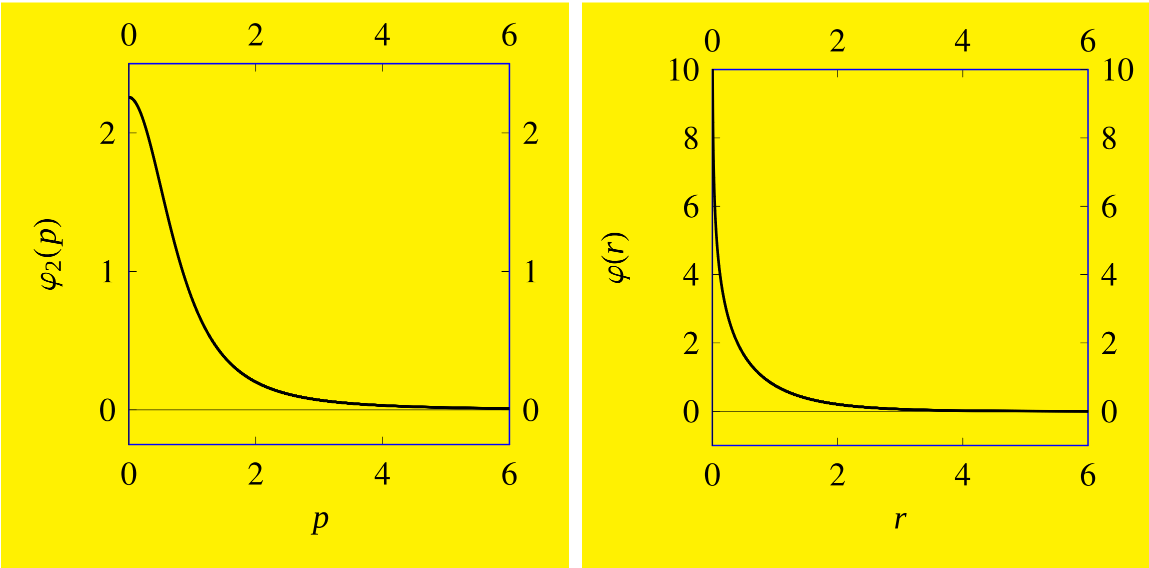
$$2\,\sqrt{p^2+m^2}\,\varphi_2(p) + 2\int_0^\infty \mathrm{d}q\,q^2\,V_0(p,q)\,\varphi_2(q) = M\,\varphi_1(p)\ ,$$

$$2\,\sqrt{p^2+m^2}\,\varphi_1(p) = M\,\varphi_2(p)\ .$$

For $M=0$, this problem is identical to that for reduced Salpeter equations; so, also all arising relations between interactions and solutions are the same. This serendipity enables us to treat the light pseudoscalar mesons $\mathrm{P}=\pi,\mathrm{K}$ as bound states of quarks and still respect their (pseudo) Goldstone nature.

In the chiral limit of QCD with only spontaneously broken chiral symmetry, pseudoscalar-meson Bethe–Salpeter amplitudes fall off (for large Euclidean momenta) like the inverse fourth power of the quarks' relative 4-momentum [12]. Analytic accessibility motivates us to model this, at finite momenta, by $(p_0^2+\boldsymbol{p}^2+1)^{-2}$. The Salpeter amplitude is found by integration over p_0. This implies $\varphi_2(p)\propto(p^2+1)^{-3/2}$, with Fourier–Bessel transform $\varphi(r)\propto K_0(r)$.

QCD-inspired ansatz for the nonvanishing Salpeter component for massless pseudoscalar mesons P in momentum (left) and configuration (right) space:



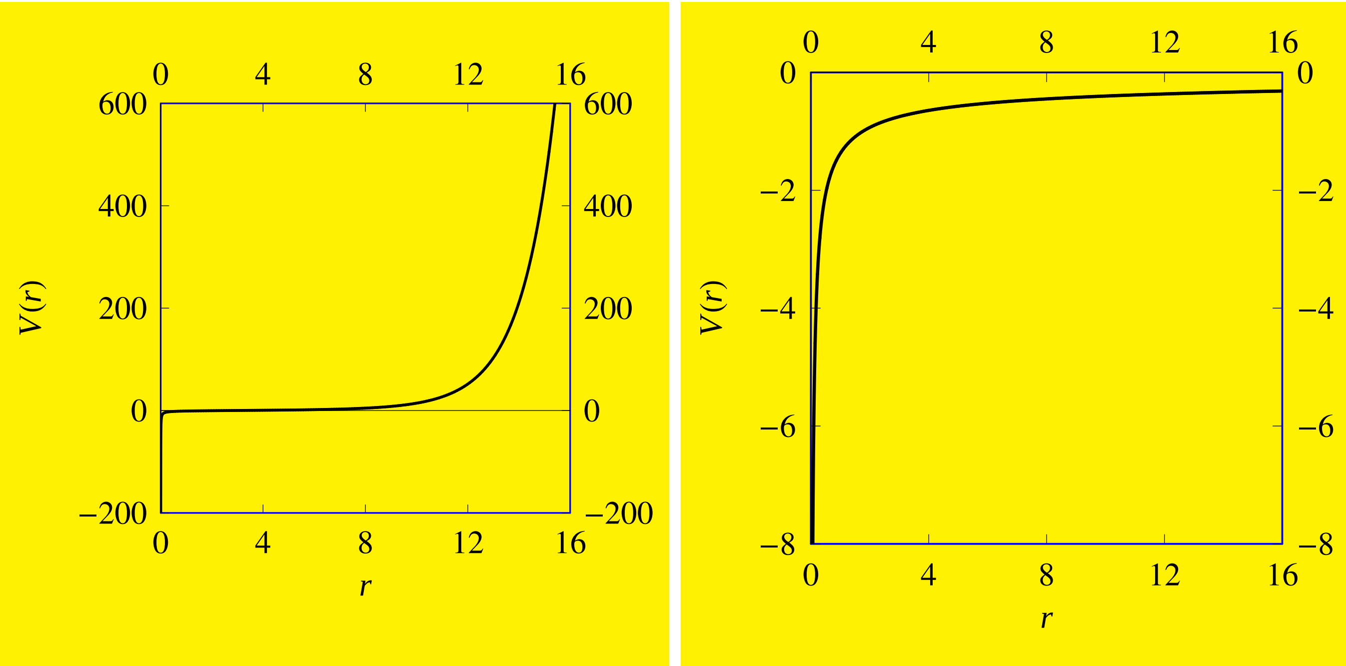
If $m = 0$, $V(r)$ involves modified Bessel ($I_{0,1}$) and Struve ($\mathbf{L}_{-1,0}$) functions:

$$V(r) = \frac{\pi}{2} \frac{r\,\mathbf{L}_{-1}(r) + \mathbf{L}_0(r) - I_0(r) - r\,I_1(r)}{r\,K_0(r)} \begin{cases} \xrightarrow[r\to 0]{} \dfrac{\pi}{2\,r \ln r} \to -\infty \,, \\ \xrightarrow[r\to\infty]{} \dfrac{\sqrt{8}\exp r}{\sqrt{\pi\,r^7}} \to \infty \,. \end{cases}$$

For completeness, let's also consider $m \neq 0$. For $m = 1$, we find analytically

$$V(r) = -\frac{\pi}{2} \frac{\exp(-r)}{r\,K_0(r)} \begin{cases} \xrightarrow[r\to 0]{} \dfrac{\pi}{2\,r \ln r} \to -\infty \,, \\ \xrightarrow[r\to\infty]{} -\sqrt{\dfrac{\pi}{2\,r}} \to 0 \,. \end{cases}$$

$V(r)$ for $\Gamma\otimes\Gamma = \frac{1}{2}(\gamma_\mu\otimes\gamma^\mu + \gamma_5\otimes\gamma_5 - 1\otimes 1)$ kernel from $\varphi_2(p) \propto (p^2+1)^{-3/2}$ for a pseudoscalar bound state of massless (left) and massive (right) quarks:



For arbitrary $m \neq 0, 1$, seeking numerical solutions is our method of choice.

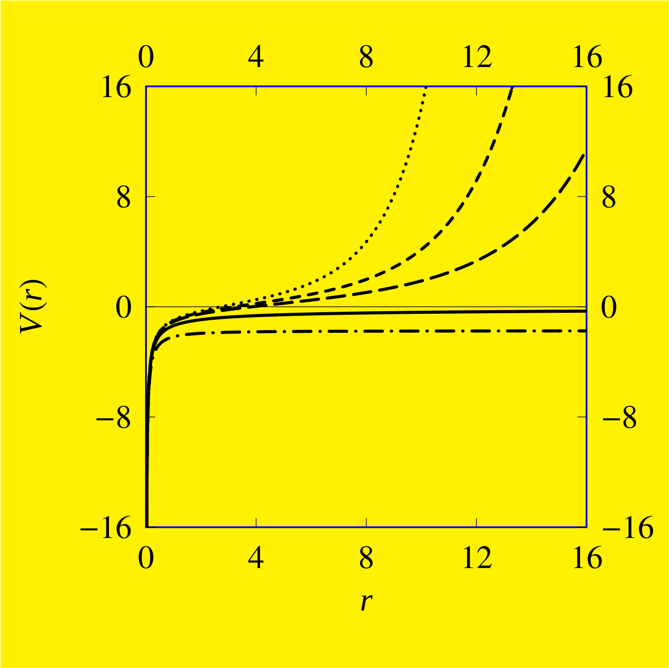
References

[1] H.A. Bethe & E.E. Salpeter, Phys. Rev. **82** (1951) 309; M. Gell-Mann & F. Low, Phys. Rev. **84** (1951) 350; E.E. Salpeter & H.A. Bethe, Phys. Rev. **84** (1951) 1232.

[2] E.E. Salpeter, Phys. Rev. **87** (1952) 328.

[3] A.B. Henriques, B.H. Kellett & R.G. Moorhouse, Phys. Lett. B **64** (1976) 85; S. Jacobs, M.G. Olsson & C.J. Suchyta III, Phys. Rev. D **35** (1987) 2448; A. Gara, B. Durand, L. Durand & L.J. Nickisch, Phys. Rev. D **40** (1989) 843; A. Gara, B. Durand & L. Durand, Phys. Rev. D **42** (1990) 1651; **43** (1991) 2447(E); W. Lucha, H. Rupprecht & F.F. Schöberl, Phys. Rev. D **45** (1992) 385.

$V(r)$ for $\Gamma\otimes\Gamma=\frac{1}{2}(\gamma_\mu\otimes\gamma^\mu+\gamma_5\otimes\gamma_5-1\otimes1)$ kernel from $\varphi_2(p)\propto(p^2+1)^{-3/2}$ for a pseudoscalar bound state of quarks of mass $m=0$ (dotted), $m=0.35$ (short-dashed), $m=1$ (solid), $m=0.5$ (long-dashed), $m=2$ (dotdashed); masses $m<1$ yield confining, masses $m\geq1$ yield not confining potentials:



[4] W. Lucha, F.F. Schöberl & D. Gromes, Phys. Rep. **200** (1991) 127; W. Lucha & F.F. Schöberl, Int. J. Mod. Phys. A **14** (1999) 2309; Fizika B **8** (1999) 193.

[5] J.F. Lagaë, Phys. Rev. D **45** (1992) 305; M.G. Olsson, S. Veseli & K. Williams, Phys. Rev. D **52** (1995) 5141; **53** (1996) 504.

[6] A. Archvadze *et al.*, Nucl. Phys. A **581** (1995) 460; J. Parramore & J. Piekarewicz, Nucl. Phys. A **585** (1995) 705; J. Parramore, H.C. Jean & J. Piekarewicz, Phys. Rev. C **53** (1996) 2449; T. Babutsidze, T. Kopaleishvili & A. Rusetsky, Phys. Lett. B **426** (1998) 139; Phys. Rev. C **59** (1999) 976; M. Uzzo & F. Gross, Phys. Rev. C **59** (1999) 1009; T. Kopaleishvili, Phys. Part. Nucl. **32** (2001) 560; T. Babutsidze, T. Kopaleishvili & D. Kurashvili, GESJ: Phys. **39** (2004) 20.

[7] W. Lucha & F.F. Schöberl, J. Phys. G **31** (2005) 1133; AIP Conf. Proc. **892** (2007) 524; Z.F. Li, W. Lucha & F.F. Schöberl, Mod. Phys. Lett. A **21** (2006) 1657.

[8] W. Lucha & F.F. Schöberl, AIP Conf. Proc. **964** (2007) 318; Frascati Phys. Ser. **46** (2007) 1539; Z.F. Li, W. Lucha & F.F. Schöberl, Phys. Rev. D **76** (2007) 125028; J. Phys. G **35** (2008) 115002.

[9] W. Lucha, PoS(Confinement8)164 (2009); AIP Conf. Proc. **1317** (2010) 122; **1343** (2011) 625.

[10] W. Lucha & F.F. Schöberl, Phys. Rev. D **87** (2013) 016009.

[11] W. Lucha & F.F. Schöberl, in preparation.

[12] P. Maris, C.D. Roberts & P.C. Tandy, Phys. Lett. B **420** (1998) 267; P. Maris & C.D. Roberts, Phys. Rev. C **56** (1997) 3369.