Strong coupling α_s from inclusive hadronic decay width of tau lepton

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Outline

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Introduction

- The inclusive hadronic decay width of the τ lepton provides a very clean way to determine α_s at low energies.
- The perturbative QCD contribution is known to $O(\alpha_s^4)$.
- The nonperturbative corrections are predicted to be small.
- The main uncertainty originates from the treatment of higher-order corrections and improvement of the perturbative series through renormalization group method.

QCD description

- The R ratio for the τ decays is defined as:

$$R_{\tau,V/A} \equiv \frac{\Gamma[\tau^- \to \mathrm{hadrons}\,\nu_\tau]}{\Gamma[\tau^- \to e^-\overline{\nu}_e\nu_\tau]}. \tag{1}$$

- We are interested in the τ decay rate into light u and d quarks, which proceeds either through a vector or an axialvector current.
- R_τ can also be expressed in the form

$$R_{\tau,V/A} = \frac{N_c}{2}\, S_{\mathrm{EW}}\, |V_{ud}|^2 \left[1 + \delta^{(0)} + \delta'_{\mathrm{EW}} + \sum_{D\geq 2} \delta^{(D)}_{ud}\right]. \tag{2}$$

Braaten-Narison-Pich

- $S_{\mathrm{EW}} = 1.0198 \pm 0.0006$ Marciano and Sirlin 1988
  $\delta'_{\mathrm{EW}} = 0.0010 \pm 0.0010$ Braaten and Li 1990

QCD description

- Our main interest is in the perturbative corrections $\delta^{(0)}$ which can be written

$$\delta^{(0)} = \frac{1}{2\pi i} \oint\limits_{|s|=M_\tau^2} \frac{ds}{s} \left(1 - \frac{s}{M_\tau^2}\right)^3 \left(1 + \frac{s}{M_\tau^2}\right) \hat{D}_{\rm pert}(a, L), \qquad (3)$$

  where $a \equiv a(\mu^2) \equiv \alpha_s(\mu^2)/\pi$ and $L \equiv \ln \frac{-s}{\mu^2}$ and $\hat{D}_{\rm pert}(a, L)$, is the Adler function series.

- A natural approach is to expand $\alpha_s(s)$ in a power series in $\alpha_s(M_\tau^2)$ and truncate it where the first unknown β_i coefficient appears and put $\mu^2 = M_\tau^2$.
  This is called '**Fixed-Order Perturbation Theory**' (FOPT).

$$\hat{D}_{FOPT}(s) = \sum_{n=1}^{\infty} a^n \sum_{k=1}^{n} k\, c_{n,k}\, L^{k-1} \ . \qquad (4)$$

QCD description

- A different approach would be to keep the full solution of the RGE and perform a numerical integration and choose $\mu^2 = -s$. This is called '**Contour Improved Perturbation Theory**'.
  Pivovarov 1991, Le Diberder and Pich 1992

$$\hat{D}_{\mathrm{CIPT}}(\alpha_s(-s)/\pi, 0) = \sum_{n=1}^{\infty} c_{n,1} \left(\frac{\alpha_s(-s)}{\pi} \right)^n . \tag{5}$$

- In the expansion above, the leading known coefficients $c_{n,1}$ are

$$c_{1,1} = 1,\ c_{2,1} = 1.640,\ c_{3,1} = 6.371,\ c_{4,1} = 49.076,$$

  Baikov, Chetyrkin and Kuhn 2008
  $c_{5,1} = 283$ estimeted, Beneke and Jamin 2008.

- The β-function was calculated to four loops in the $\overline{\mathrm{MS}}$-renormalization scheme, the known coefficients are

$$\beta_0 = 9/4,\ \beta_1 = 4,\ \beta_2 = 10.0599,\ \beta_3 = 47.228.$$

  Larin, Ritbergen and Vermaseren 1997 and Czakon 2005

Outline

Renormalization Group Summed Perturbation Theory

- We use a method based on the explicit summation of all renormalization-group accessible logarithms.

$$\hat{D}_{RGSPT}(aL) = a(c_{1,1} + 2c_{2,2}aL + 3c_{3,3}a^2L^2 + \cdots) + a^2(c_{2,1} + 2c_{3,2}aL + 3c_{4,3}a^2L^2 + \cdots)$$
$$+ a^3(c_{3,1} + 2c_{4,2}aL + 3c_{5,3}a^2L^2 + \cdots) + \cdots = \sum_{n=1}^{\infty} a^n D_n(aL). \qquad (6)$$

Maxwell and A. Mirjalili 2000
Ahmady, Chishtie, Elias, Fariborz, Fattahi, McKeon, Sherry, Steele 2002, 03

$$D_n(aL) \equiv \sum_{k=n}^{\infty} (k-n+1) c_{k,k-n+1} (aL)^{k-n}. \qquad (7)$$

- The Adler function defined by (4) is scale independent

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \left\{ \hat{D}_{\mathrm{FOPT}}(aL) \right\} = 0. \qquad (8)$$

$$\beta(a) \frac{\partial \hat{D}_{\mathrm{FOPT}}}{\partial a} - \frac{\partial \hat{D}_{\mathrm{FOPT}}}{\partial L} = 0. \qquad (9)$$

- We derive following RGE equation

$$
\begin{aligned}
0 = & -\sum_{n=1}^{\infty}\sum_{k=2}^{n} k(k-1)c_{n,k}a^n L^{k-2} \\
& -\left(\beta_0 a^2 + \beta_1 a^3 + \beta_2 a^4 + \ldots + \beta_l a^{l+2} + \ldots\right) \times \sum_{n=1}^{\infty}\sum_{k=1}^{n} nkc_{n,k}a^{n-1}L^{k-1}. \quad (10)
\end{aligned}
$$

- By extracting the aggregate coefficient of $a^n L^{n-p}$ one obtains the recursion formula ($n \geq p$)

$$
0 = (n-p+2)c_{n,n-p+2} + \sum_{\ell=0}^{p-2}(n-\ell-1)\beta_\ell c_{n-\ell-1,n-p+1}. \quad (11)
$$

- Multiplying both sides of (11) by $(n-p+1)(aL)^{n-p}$ and summing from $n = p$ to ∞, we obtain a set of first-order linear differential equation for the functions defined in (7), written as

$$
\frac{\mathrm{d}D_n}{\mathrm{d}(aL)} + \sum_{\ell=0}^{n-1}\beta_\ell\left((aL)\frac{\mathrm{d}}{\mathrm{d}(aL)} + n - \ell\right)D_{n-\ell} = 0, \quad (12)
$$

for $n \geq 1$, with the initial conditions $D_n(0) = c_{n,1}$ which follow from (7). The solution of the above Eq (12) can be found iteratively in an analytical closed form.

- The first two solutions are

$$D_1(aL) = \frac{c_{1,1}}{y}, \; D_2(aL) = \frac{c_{2,1}}{y^2} - \frac{\beta_1 c_{1,1} \ln y}{\beta_0 w^2}, \; y = 1 + \beta_0 aL. \tag{13}$$

- The RGSPT expansion of the Adler function is

$$\hat{D}_{\mathrm{RGSPT}}(aL) = \sum_{n=1}^{N} a^n D_n(aL), \tag{14}$$

$$\delta^{(0)}_{\mathrm{RGSPT}} = \sum_{n=1}^{\infty} a(M_\tau^2)^n d_n \,, \tag{15}$$

where

$$d_n = \frac{1}{2\pi i} \oint_{|s|=M_\tau^2} \frac{ds}{s} \left(1 - \frac{s}{M_\tau^2}\right)^3 \left(1 + \frac{s}{M_\tau^2}\right) D_n(aL). \tag{16}$$

	$\delta^{(0)}_{\rm FOPT}$	$\delta^{(0)}_{\rm CIPT}$	$\delta^{(0)}_{\rm RGSPT}$
$n = 1$	0.1082	0.1479	0.1455
$n = 2$	0.1691	0.1776	0.1797
$n = 3$	0.2025	0.1898	0.1931
$n = 4$	0.2199	0.1984	0.2024
$n = 5$	0.2287	0.2022	0.2056

Table: Predictions of $\delta^{(0)}$ by the standard FOPT, CIPT and the RGSPT, for various truncation orders n using $\alpha_s = 0.34$.

For $n = 4$, the difference between the results of the RGSPT and the standard FOPT is 0.01754, and the difference from the RGSPT and CIPT is 0.0039, which confirms that the new expansion gives results close to those of the CIPT.

Adler function in the complex s-plane

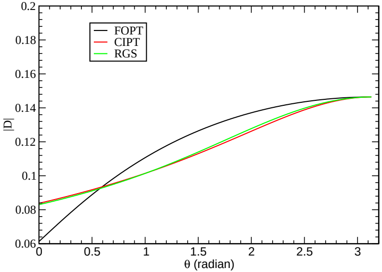

Figure: Adler function expansions, summed up to the order $N = 5$, along the circle $s = M_\tau^2 \exp(i\theta)$.

Outline

Higher order behaviour of RGSPT expansion

- The coefficients $c_{n,1}$ display a factorial growth, *i.e.* the series has a vanishing radius of convergence.
- We consider a model proposed by Beneke & Jamin (2008) which predicts coefficients $c_{n,1}$ for $n > 5$.

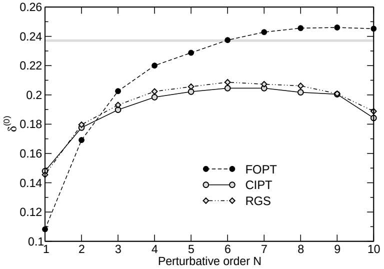

Figure: In the figure we show the dependence on the perturbative order of $\delta^{(0)}$ in FOPT, CIPT and RGSPT in the BJ model. The gray band is the true value obtained from Borel integral in this model.

Outline

Determination of α_s from RGSPT expansion

- We use as input the recent phenomenological value of the pure perturbative correction to the hadronic τ width

$$\delta^{(0)}_{\rm phen} = 0.2037 \pm 0.0040_{exp} \pm 0.0037_{\rm PC}. \qquad (17)$$

Beneke & Jamin 2011, Workshop on Precision Measurements of α_s 2011

- With the above phenomenological value of $\delta^{(0)}$ and a conservative choice $c_{5,1} = 283 \pm 283$ for the next coefficient and the next terms in the expansion of the β function, $\beta_4 = \pm\beta_3^2/\beta_2$, we obtain

$$\alpha_s(M_\tau^2) = 0.3378 \pm 0.0046_{\rm exp} \pm 0.0042_{\rm PC}\ {}^{+0.0062}_{-0.0072}(c_{5,1}) \\ {}^{+0.0005}_{-0.0004}(\text{scale}) \pm^{+0.000085}_{-0.000082}(\beta_4). \qquad (18)$$

- Combining errors in quadrature

$$\alpha_s(M_\tau^2) = 0.338 \pm 0.010. \qquad (19)$$

$$\alpha_s(M_\tau^2) = 0.320^{+0.012}_{-0.007} \qquad \text{FOPT}$$
$$\alpha_s(M_\tau^2) = 0.342 \pm 0.012 \qquad \text{CIPT} \qquad (20)$$

Outline

RGS Non-Power Perturbation Theory

- We improve the convergence of the RGSPT expansion by the analytical continuation in the Borel plane. Caprini & Fischer 1999, 2000, 2009, 2011
- The method was applied to FOPT and CIPT by Caprini and Fischer in the past.
- The method cannot be applied in the α_s plane but can be applied to the Borel transform, $B(u)$ of the Adler function in the u plane.
- The Taylor exapnsion of the Borel transform, $B(u)$ converges only in the disk $|u| < 1$, limited by the nearest singularity at $u = -1$.

$$B(u) = \sum_{n=0}^{\infty} c_{n+1,1} \frac{u^n}{\beta_0^n \, n!} \tag{21}$$

- The region of convergence can be enlarged if the series in powers of u is replaced by a series in powers of an "optimal" variable $\widetilde{w}(u)$ that conformally maps the holomorphy domain of $B(u)$, *i.e.* the u-plane with cut along $u \geq 2$ and $u \leq -1$, onto the unit disk $|w| < 1$.
- This also accelerates the convergence rate at all points in the holomorphy domain. Ciulli & Fischer 1961, Caprini & Fischer 2011

RGS Non-Power Perturbation Theory

- We introduce the Borel transform of the RGSPT expansion of the Adler function

$$B_{\mathrm{RGSPT}}(u,y) = B(u) + \sum_{n=0}^{\infty} \frac{u^n}{\beta_0^n\, n!} \sum_{j=1}^{n} c_{j,1} d_{n+1,j}(y), \tag{22}$$

where $y = 1 + \beta_0 a L$.

- We consider the functions

$$\widetilde{w}_{lm}(u) = \frac{\sqrt{1+u/l} - \sqrt{1-u/m}}{\sqrt{1+u/l} + \sqrt{1-u/m}}, \quad l \geq 1, m \geq 2 \tag{23}$$

where l, m are positive integers satisfying $l \geq 1$ and $m \geq 2$. The function $\widetilde{w}_{lm}(u)$ maps the u-plane cut along $u \leq -l$ and $u \geq m$ onto the disk $|w_{lm}| < 1$ in the plane $w_{lm} \equiv \widetilde{w}_{lm}(u)$.

- We define further the class of compensating factors of the simple form

$$S_{lm}(u) = \left(1 - \frac{\widetilde{w}_{lm}(u)}{\widetilde{w}_{lm}(-1)}\right)^{\gamma_1^{(l)}} \left(1 - \frac{\widetilde{w}_{lm}(u)}{\widetilde{w}_{lm}(2)}\right)^{\gamma_2^{(m)}}, \tag{24}$$

RGS Non-Power Perturbation Theory

- where the exponents are

$$\gamma_1^{(l)} = \gamma_1(1+\delta_{l1}), \quad \gamma_2^{(m)} = \gamma_2(1+\delta_{m2}),$$
$$\gamma_1 = 1.21, \qquad \gamma_2 = 2.58\,, \tag{25}$$

  are chosen such that $S_{lm}(u)$ cancel the dominant singularities on the real axis in the u-plane.
- We further expand the product $S_{lm}(u)B_{\mathrm{RGSPT}}(u,y)$ in powers of the variable $\widetilde{w}_{lm}(u)$, as

$$S_{lm}(u)B_{\mathrm{RGSPT}}(u,y) = \sum_{n\geq 0} c_{n,\mathrm{RGSPT}}^{(lm)}(y)\,(\widetilde{w}_{lm}(u))^n. \tag{26}$$

- We are led to the class of RGSNPPT expansions

$$\widehat{D}_{\mathrm{RGSNPPT}}(s) = \sum_{n\geq 0} c_{n,\mathrm{RGSPT}}^{(lm)}(y)\,\mathcal{W}_{n,\mathrm{RGSPT}}^{(lm)}(s), \tag{27}$$

  where

$$\mathcal{W}_{n,\mathrm{RGSPT}}^{(lm)}(s) = \frac{1}{\beta_0}\mathrm{PV}\int_0^\infty \exp\left(\frac{-u}{\beta_0\tilde{a}_s(-s)}\right)\frac{(\widetilde{w}_{lm}(u))^n}{S_{lm}(u)}\,\mathrm{d}u, \tag{28}$$

  and the coefficients $c_{n,\mathrm{RGS}}^{(lm)}(y)$ are defined by the expansion (26).
- The coupling, $\widetilde{a}_s(-s)$, entering in the Laplace-Borel integral is the one-loop solution of the RGE, a novel feature given by RGSPT.

Outline

The convergence of RGSNPPT expansions

- The difference $\delta^{(0)} - \delta^{(0)}_{\rm exact}$ for the model $B_{\rm BJ}$ proposed in BJ model for $\alpha_s(M_\tau^2) = 0.34$ with the standard CIPT, FOPT and RGSPT expansions, and the new RGSNPPT expansions for various conformal mappings w_{lm}, truncated at order N. Exact value $\delta^{(0)}_{\rm exact} = 0.2371$

N	CIPT	FOPT	RGSPT	RGSNPPT w_{12}	RGSNPPT w_{13}	RGSNPPT $w_{1\infty}$	RGSNPPT w_{23}
2	-0.0595	-0.0679	-0.0574	-0.0347	-0.0239	-0.0417	-0.0177
3	-0.0473	-0.0345	-0.0440	-0.0333	-0.0301	-0.0349	-0.0303
4	-0.0388	-0.0171	-0.0347	-0.0089	-0.0142	-0.0067	-0.0132
5	-0.0349	-0.0083	-0.0315	-0.0070	-0.0086	-0.0058	-0.0070
6	-0.0325	-0.0043	-0.0284	-0.0073	-0.0071	-0.0064	-0.0072
7	-0.0325	-0.0029	-0.0298	-0.0059	-0.0057	-0.0056	-0.0044
8	-0.0354	-0.0018	-0.0309	-0.0041	-0.0035	-0.0041	-0.0011
9	-0.0367	-0.0004	-0.0363	-0.0023	-0.0019	-0.0028	-0.0010
10	-0.0529	0.0019	-0.0483	0.0014	-0.0012	-0.0020	0.0004
11	-0.0409	0.0031	-0.0458	0.0036	-0.0008	-0.0016	-0.0009
12	-0.1248	0.0065	-0.1335	0.0031	-0.0006	-0.0015	0.0005
13	0.0258	0.0037	0.0534	0.0026	-0.0004	-0.0015	-0.0005
14	-0.5286	0.0204	-0.7850	0.0018	-0.0003	-0.0015	-0.0011
15	0.8640	-0.0201	1.7734	0.0006	-0.0002	-0.0015	0.0044
16	-3.5991	0.1447	-7.7043	0.0001	$-7 \cdot 10^{-6}$	-0.0015	-0.0131
17	9.3560	-0.4252	24.8586	-0.0004	$4 \cdot 10^{-6}$	-0.0014	0.0238
18	-31.76	1.907	-94.26	-0.0013	-0.0001	-0.0013	-0.0310

Outline

Determination of α_s from RGSNPPT expansions

- We obtain with RGSNPPT expansions

$$\alpha_s(M_\tau^2) = 0.3189 \pm 0.0034_{\rm exp} \pm 0.0031_{\rm PC} \ {}^{+0.0138}_{-0.0105}({\rm c}_{5,1}) \ \pm 0.0010_{\beta_4}, \qquad (29)$$

after combining the errors in quadrature,

$$\alpha_s(M_\tau^2) = 0.3189 \ {}^{+0.0145}_{-0.0115} \,. \qquad (30)$$

- By evolving to the scale of M_Z our prediction reads

$$\alpha_s(M_Z^2) = 0.1184 \ {}^{+0.0018}_{-0.0015} \,, \qquad (31)$$

Outline

Summary

- This work is motivated by the well-known discrepancy between the predictions of $\alpha_s(M_\tau^2)$ from the standard fixed-order and CIPT expansions.
- The main result is that the summation of leading logarithms provides a systematic expansion with good convergence properties in the complex plane.
- The results of the new RGSPT expansion are similar to those obtained by the CIPT expansion.
- The divergent character of the perturbative series is tamed by analytic continuation in the Borel plane.
- The RGSNPPT expansions lead to prediction for α_s which is similar to standard FOPT (Beneke & Jamin 2008) and CINPPT (Caprini & Fischer 2011).