Integrand reduction at NLO and beyond

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EPSHEP 2013 – Stockholm, Sweden, 18-24 July

Outline

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Introduction and motivation

Motivation

- Understanding the basic analytic and algebraic structure of integrands and integrals of scattering amplitudes
- Exploration of methods for obtaining theoretical predictions in perturbative Quantum Field Theory at higher orders, required for experiments in high-energy physics

We developed a coherent framework for the integrand decomposition of Feynman integrals

- based on simple concepts of algebraic geometry
- applicable at all loops

Outline

Integrand reduction

- Generic ℓ-loop integral:

$$\mathcal{M}_n = \int d^d q_1 \ldots d^d q_\ell \;\; \mathcal{I}_{i_1 \ldots i_n}, \qquad \mathcal{I}_{i_1 \ldots i_n} \equiv \frac{\mathcal{N}_{i_1 \ldots i_n}}{D_{i_1} \ldots D_{i_n}}$$

  - the numerator $\mathcal{N}_{i_1 \ldots i_n}$ is polynomial in q_i
  - the denominators D_i are quadratic polynomials in q_i
- The integrand-reduction method leads to the decomposition:

$$\mathcal{I}_{i_1 \ldots i_n} = \frac{\Delta_{i_1 \cdots i_n}}{D_{i_1} \ldots D_{i_n}} + \ldots + \sum_{k=1}^{n} \frac{\Delta_{i_k}}{D_{i_k}} + \Delta_\emptyset$$

- The residues $\Delta_{i_1 \ldots i_k}$ are irreducible polynomials in q_i
  - universal topology-dependent parametric form
  - the coefficients of the parametrization are process-dependent

Polynomial division and residues

[Y, Zhang (2012), P. Mastrolia, E. Mirabella, G. Ossola, T. P. (2012)]

- Trade $(q_1, \ldots, q_\ell)$ with their coordinates $\mathbf{z} \equiv (z_1, \ldots, z_m)$
- Define the Ideal of polynomials

$$\mathcal{J} \equiv \langle D_{i_1}, \ldots, D_{i_n} \rangle = \left\{ p(\mathbf{z}) \; : \; p(\mathbf{z}) = \sum_j h_j(\mathbf{z}) D_j(\mathbf{z}), \; h_j \in P[z] \right\}$$

- Take a Gröbner basis $G_{\mathcal{J}}$ of $\mathcal{J}$

$$G_{\mathcal{J}} = \{g_1, \ldots, g_s\} \quad \text{such that} \quad \mathcal{J} = \langle g_1, \ldots, g_s \rangle$$

- Perform the multivariate polynomial division $\mathcal{N}_{i_1 \ldots i_n} / G_{\mathcal{J}}$

$$\mathcal{N}_{i_1 \cdots i_n}(z) = \sum_{k=1}^{n} \mathcal{N}_{i_1 \cdots i_{k-1} i_{k+1} \cdots i_n}(z) \, D_{i_k}(z) + \Delta_{i_1 \cdots i_n}(z)$$

- $\Delta_{i_1 \cdots i_n}$ is the remainder of the polynomial division, and can be identified with the residue

Recursive Relation for the integrand decomposition

[P. Mastrolia, E. Mirabella, G. Ossola, T. P. (2012)]

The recursive formula

$$\mathcal{N}_{i_1\cdots i_n} = \sum_{k=1}^{n} \mathcal{N}_{i_1\cdots i_{k-1} i_{k+1}\cdots i_n} D_{i_k} + \Delta_{i_1\cdots i_n}$$

$$\mathcal{I}_{i_1\cdots i_n} \equiv \frac{\mathcal{N}_{i_1\cdots i_n}}{D_{i_1}\cdots D_{i_n}} = \sum_{k} \mathcal{I}_{i_1\cdots i_{k-1} i_{k+1}\cdots i_n} + \frac{\Delta_{i_1\cdots i_n}}{D_{i_1}\cdots D_{i_n}}$$

- Fit-on-the-cut approach
  - from a generic $\mathcal{N}$, get the parametric form of the residues Δ
  - determine the coefficients sampling on the cuts (impose $D_i = 0$)
- Divide-and-Conquer approach
  - generate the $\mathcal{N}$ of the process
  - compute the residues by iterating the polynomial division algorithm

From integrands to integrals

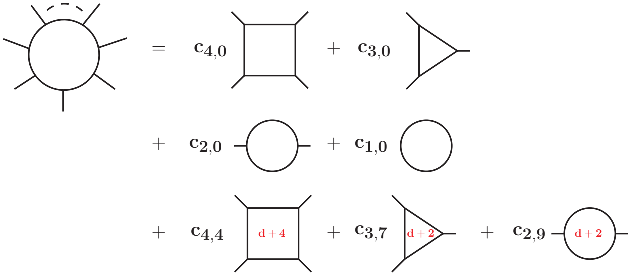
- By integrating the integrand decomposition

$$\mathcal{M}_n = \int d^d q_1 \ldots d^d q_\ell \left(\frac{\Delta_{i_1 \cdots i_n}}{D_{i_1} \ldots D_{i_n}} + \ldots + \sum_{k=1}^{n} \frac{\Delta_{i_k}}{D_{i_k}} + \Delta_\emptyset \right)$$

  - some terms vanish and do not contribute to the amplitude $\Rightarrow$ spurious terms
  - non-vanishing terms give Master Integrals (MIs)
- The amplitude is a linear combination of MIs

Outline

One-loop decomposition



- Start from the most general one-loop amplitude in $d = 4 - 2\epsilon$
- Apply the recursive formula for the integrand decomposition
- Drop the spurious terms

$\Rightarrow$ Get the most general integral decomposition (well knwon result)

Integrand Reduction via Laurent series expansion

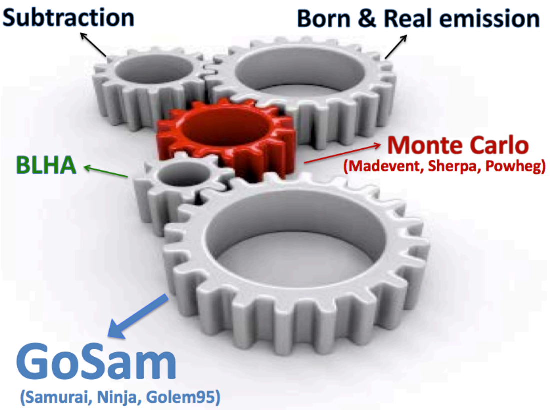
P. Mastrolia, E. Mirabella, T. P. (2012)

- The one loop integrand decomposition

$$\mathcal{I}_{i_1\cdots i_n} = \frac{\mathcal{N}_{i_1\cdots i_n}}{D_{i_1}\cdots D_{i_n}} = \sum_{j_1\cdots j_5} \frac{\Delta_{j_1j_2j_3j_4j_5}}{D_{j_1}D_{j_2}D_{j_3}D_{j_4}D_{j_5}} + \sum_{j_1j_2j_3j_4} \frac{\Delta_{j_1j_2j_3j_4}}{D_{j_1}D_{j_2}D_{j_3}D_{j_4}} + \sum_{j_1j_2j_3} \frac{\Delta_{j_1j_2j_3}}{D_{j_1}D_{j_2}D_{j_3}} + \sum_{j_1j_2} \frac{\Delta_{j_1j_2}}{D_{j_1}D_{j_2}} + \sum_{j_1} \frac{\Delta_{j_1}}{D_{j_1}}$$

- The integrand reduction via Laurent expansion
  - fits residues by taking their asymptotic expansions on the cuts
  - requires the computation of fewer coefficients
  - subtractions at the coefficient level
- Semi-numerical implementation in the C++ library NINJA
  - Laurent expansions via a simplified polynomial-division algorithm
  - interfaced with the package GOSAM
  - is a faster and more stable integrand-reduction algorithm

From amplitudes to observables with GoSam

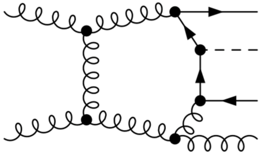


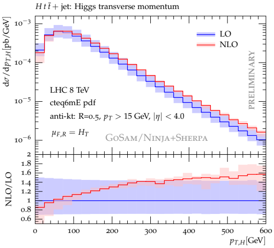
The GoSam collaboration:

G. Cullen, H. van Deurzen, N. Greiner, G. Heinrich, G. Luisoni, P. Mastrolia, E. Mirabella, G. Ossola, J. Reichel , J. Schlenk, J. F. von Soden-Fraunhofen, T. Reiter, F. Tramontano, T. P.

Application: $pp \to t\bar{t}H + jet$ with GoSam + Ninja

H. van Deurzen, G. Luisoni, P. Mastrolia, E. Mirabella, G. Ossola, T. P. (work in progess)





- Interfaced with the Monte Carlo Sherpa
- Benchmarks:

sub-process	# diagrams	seconds/event
$q\bar{q} \to t\bar{t}Hg$	320	0.2
$gg \to t\bar{t}Hg$	1575	2.5

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Extension to higher loops

- The integrand-level approach to scattering amplitudes one-loop
  - can be used to compute any amplitude in any QFT
  - has been implemented in several codes, some of which public [SAMURAI, CUTTOOLS, NGLUONS]
  - has produced (and is still producing) results for LHC [GOSAM (see G. Ossola's talk), BLACKHAT, MADLOOP, NJETS, FORMCALC, OPENLOOP . . .]
- At two or higher loops
  - no general recipe is available
  - the standard and most successful approach is the Integration By Parts (IBP) method, but it becomes difficult for high multiplicities

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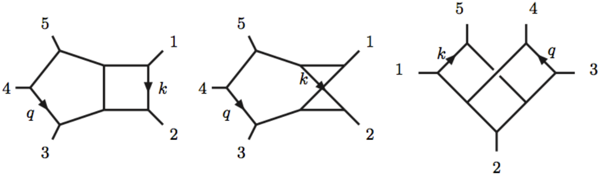
- . . . we are moving the first steps in this direction

$\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA amplitudes

P. Mastrolia, G. Ossola (2011); P. Mastrolia, E. Mirabella, G. Ossola, T. P. (2012)



- Examples in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA amplitudes ($d = 4$)
  - generation of the integrand
    - graph based [Carrasco, Johansson (2011)]
    - unitarity based [U. Schubert (Diplomarbeit)]
  - fit-on-the-cut approach for the reduction
- Results:

$\mathcal{N} = 4$ linear combination of 8 and 7-denominators MIs
$\mathcal{N} = 8$ linear combination of 8, 7 and 6-denominators MIs

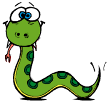
Divide-and-Conquer approach

P. Mastrolia, E. Mirabella, G. Ossola, T. P. (2013)

The divide-and-conquer approach to the integrand reduction

- does not require the knowledge of the solutions of the cut
- can always be used to perform the reduction in a finite number of purely algebraic operations
- has been automated in a PYTHON package which uses MACAULAY2 and FORM for algebraic operations

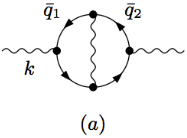
MACAULAY2 $\Leftarrow$ PYTHON $\Rightarrow$ FORM



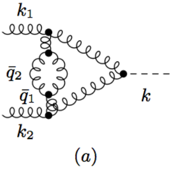
- also works in special cases where the fit-on-the-cut approach is not applicable (e.g. in presence of double denominators)

Examples of divide-and-conquer approach

- Photon self-energy in massive QED, $(4-2\epsilon)$-dimensions



- Diagrams entering $gg \to H$, in $(4-2\epsilon)$-dimensions



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Conclusions

- We developed a general framework for the reduction at the integrand level
  - can be applied to any amplitude in any QFT
  - is valid at every loop order
- At one-loop
  - naturally reproduces the OPP result
  - allows to compute the amplitude without performing any (new) integration
  - leads to well established and successful techniques
- At higher loops
  - it gives a recursive formula for the integrand decomposition
  - generates the form of the residue for every cut
- The divide-and-conquer approach
  - can be used to implement the whole reduction of any integrand with purely algebraic operations
  - has been automated in a python package

THANK YOU
FOR THE ATTENTION