Decay Constants of Heavy Pseudoscalar Mesons: Reconciling QCD Sum Rules and Lattice QCD

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QCD Sum Rules for Pseudoscalar Mesons

QCD sum rules [1] relate experimentally observable hadronic features to the fundamental QCD parameters, by evaluating vacuum expectation values of nonlocal products of appropriate interpolating quark currents at both QCD and hadron level. Wilson's operator product expansion (OPE) enables us to express any such nonlocal product as a series of local operators; the vacuum expectation values of the latter are referred to as vacuum condensates. Our, for obvious reasons persisting, ignorance about higher resonances is masked by assuming quark–hadron duality: beyond some effective threshold $s_{\rm eff}$ the perturbative QCD contributions are expected to counterbalance the ones of hadronic excitations and continuum. Application of Borel transformations, introducing so-called Borel parameters τ, suppresses heavier hadron states.

In order to raise the accuracy of sum-rule predictions for meson observables and to deduce reliable estimates of the systematic uncertainties involved [2], we suggested — with strong support by studies within quantum mechanics, where exact answers may be found by simply solving Schrödinger equations — some modifications of the QCD sum-rule method [3] centered around the idea to allow for a dependence of $s_{\rm eff}$ on the Borel parameter: $s_{\rm eff} \to s_{\rm eff}(\tau)$.

Our recent study of bottom mesons [4] starts from the two-point correlation functions of the pseudoscalar currents $j_5(x) \equiv (m_b + m_q)\,\bar{q}(x)\,\mathrm{i}\,\gamma_5\,b(x)$ of a heavy bottom quark of mass m_b and a light quark q of mass m_q, $q = u, d, s$:

$$\Pi(p^2) \equiv \mathrm{i} \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}\,p\,x} \left\langle 0 \left| \mathrm{T}\left(j_5(x)\, j_5^\dagger(0) \right) \right| 0 \right\rangle .$$

Application of the OPE recasts the correlator Π into the shape of a sum of a dispersion integral over a spectral density $\rho(s,\mu)$ that can be found as series in powers of the strong coupling $\alpha_\mathrm{s}(\mu)$ evaluated at renormalization scale μ,

$$\rho(s,\mu) = \rho_0(s,m_b) + \frac{\alpha_\mathrm{s}(\mu)}{\pi}\,\rho_2(s,m_b) + \frac{\alpha_\mathrm{s}^2(\mu)}{\pi^2}\,\rho_2(s,m_b,\mu) + \cdots ,$$

and nonperturbative corrections $\Pi_\mathrm{NP}(\tau,\mu)$, the lowest-order terms of which require the knowledge of the vacuum condensates $\langle\bar{q}q\rangle$, $\langle\bar{s}s\rangle$, and $\left\langle\frac{\alpha_\mathrm{s}}{\pi}\,GG\right\rangle$. Consequently, for bottom mesons $B_{(s)}$ of masses $M_{B_{(s)}}$ and decay constants $f_{B_{(s)}}$ defined by $\langle 0|j_5(0)|B_{(s)}\rangle = f_{B_{(s)}}\,M_{B_{(s)}}^2$, the sum rule sought is given by

$$f_{B_{(s)}}^2\,M_{B_{(s)}}^4 \exp\left(-M_{B_{(s)}}^2\,\tau\right) = \int\limits_{(m_b+m_q)^2}^{s_\mathrm{eff}(\tau)} \mathrm{d}s\, \mathrm{e}^{-s\,\tau}\,\rho(s,\mu) + \Pi_\mathrm{NP}(\tau,\mu) \equiv \widetilde{\Pi}(\tau, s_\mathrm{eff}(\tau)) .$$

The dual correlator $\widetilde{\Pi}(\tau, s_\mathrm{eff}(\tau))$ fixes dual mass and dual decay constant by

$$M_\mathrm{dual}^2(\tau) \equiv -\frac{\mathrm{d}}{\mathrm{d}\tau}\log\widetilde{\Pi}(\tau, s_\mathrm{eff}(\tau)) , \qquad f_\mathrm{dual}^2(\tau) \equiv \frac{e^{M_{B_{(s)}}^2\,\tau}}{M_{B_{(s)}}^4}\,\widetilde{\Pi}(\tau, s_\mathrm{eff}(\tau)) .$$

Numerical values of the parameters required for the bottom-meson OPE [5]

Quantity	Numerical input value
$\overline{m}_d(2\ \mathrm{GeV})$	(3.5 ± 0.5) MeV
$\overline{m}_s(2\ \mathrm{GeV})$	(95 ± 5) MeV
$\alpha_\mathrm{s}(M_Z)$	0.1184 ± 0.0007
$\langle\bar{q}q\rangle(2\ \mathrm{GeV})$	$-[(269 \pm 17)\ \mathrm{MeV}]^3$
$\langle\bar{s}s\rangle(2\ \mathrm{GeV})$	$(0.8 \pm 0.3) \times \langle\bar{q}q\rangle(2\ \mathrm{GeV})$
$\left\langle\frac{\alpha_\mathrm{s}}{\pi}\,GG\right\rangle$	$(0.024 \pm 0.012)\ \mathrm{GeV}^4$

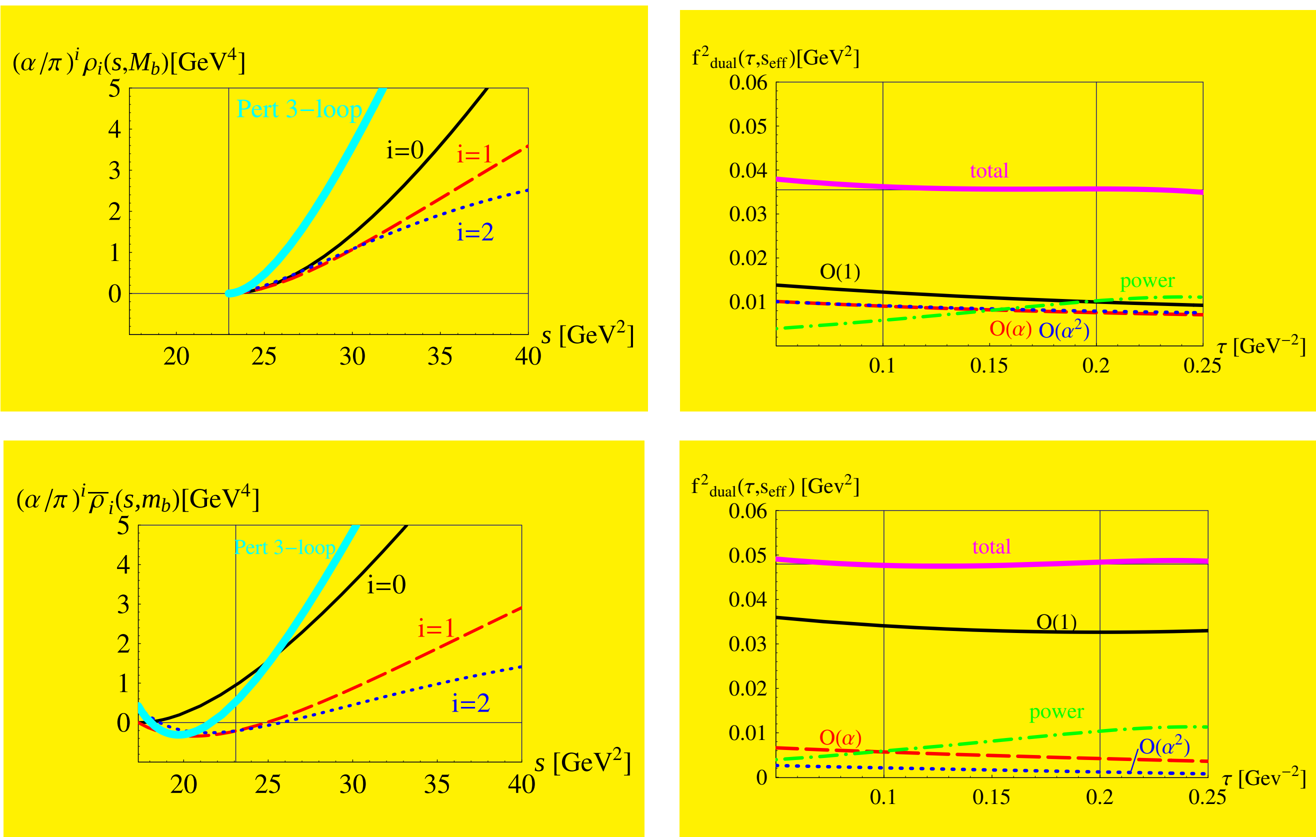
Advanced Hadron-Property Extraction [3]

The perturbative expansion of the spectral density $\rho(s,\mu)$ is sensitive to the renormalization scheme adopted for defining the heavy-quark mass, i.e., m_b in our case. Usually, $\rho(s,\mu)$ is formulated in terms of the corresponding pole mass M_b. However, the pole-mass series for $\rho(s,\mu)$ converges rather poorly. This is cured by substituting M_b by its expression in terms of the associated $\overline{\rm MS}$ running quark mass $\overline{m}_b(\nu)$ at some renormalization scale ν which needs not be identical to the scale μ chosen for the truncated expansion of $\rho(s,\mu)$:

$$M_b = \overline{m}_b(\nu)\left(1+\frac{\alpha_{\rm s}(\nu)}{\pi}\,r_1+\frac{\alpha_{\rm s}^2(\nu)}{\pi^2}\,r_2+\cdots\right),$$

with known expansion terms $r_{1,2}$ [6]. In contrast to its pole-mass expression, the $\overline{\rm MS}$-mass expansion of $\rho(s,\mu)$ exhibits a clear-cut hierarchical ordering. The respective extractions of $f_{\rm dual}$ differ significantly, which is disastrous for the Borel-stability argument for the wide-spread belief in sum-rule findings.

Perturbative terms $\rho_i(s,m_b)$ of the spectral density $\rho(s)$ (left column) and corresponding contributions to the dual decay constant $f_{\rm dual}$ (right column) in pole-mass (top row) and $\overline{\rm MS}$-mass (bottom row) renormalization scheme



For the extraction of hadronic features, we developed [3] a simple algorithm:

- The admissible τ range is determined by requiring, at its lower end, the ground-state contribution to be sufficiently large and, at the upper end, the contribution of nonperturbative corrections to be reasonably small.
- The threshold function $s_{\rm eff}(\tau)$ is found by adopting a power-law Ansatz

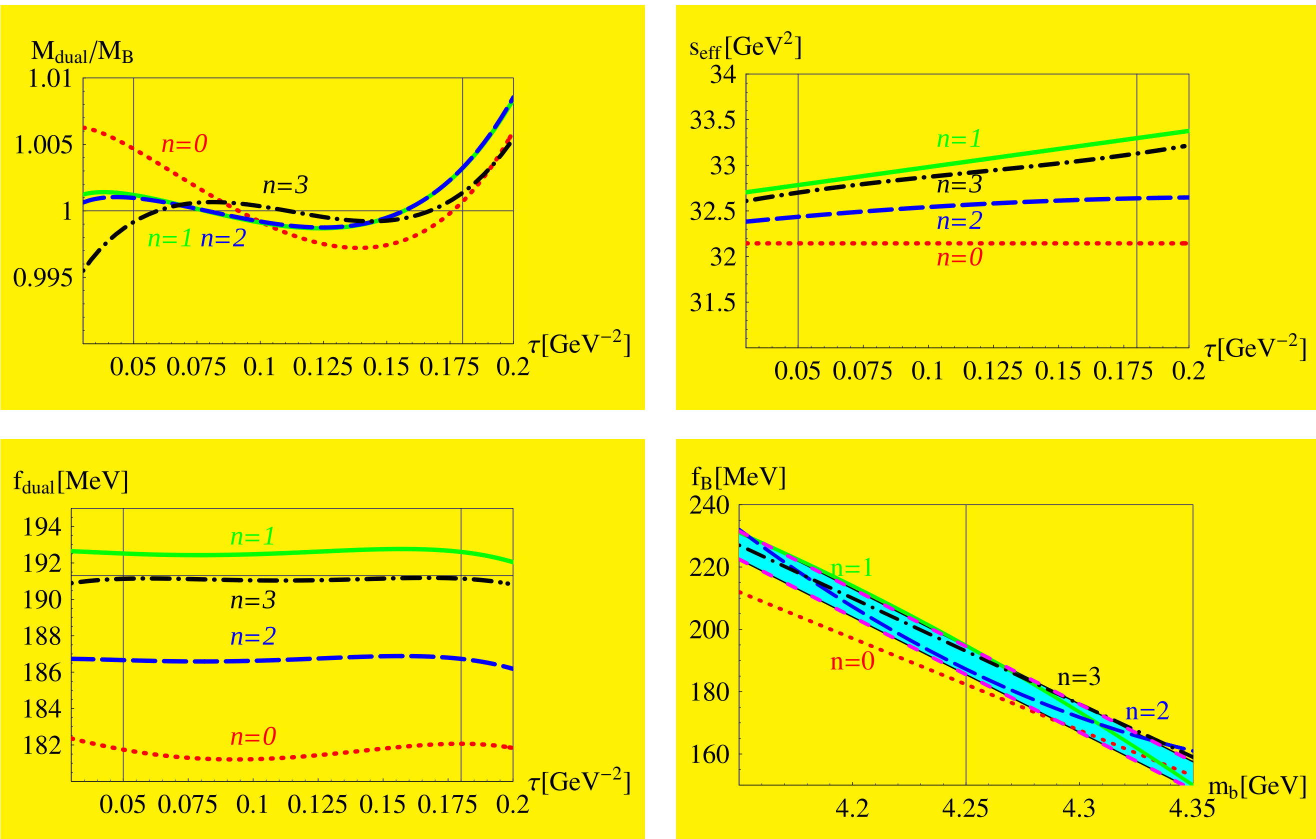
$$s_{\rm eff}^{(n)}(\tau) = \sum_{j=0}^{n} s_j\,\tau^j\ ,$$

  with expansion coefficients s_j determined by minimizing the expression

$$\chi^2 \equiv \frac{1}{N}\sum_{i=1}^{N}\left[M_{\rm dual}^2(\tau_i) - M_B^2\right]^2$$

  over a set of N equidistant discrete points τ_i in the allowable range of τ.
- The spread of results for $n = 1, 2, 3$ yields their intrinsic sum-rule error.

Sum-rule description of the B meson for polynomial dependence on τ of the effective threshold $s_{\rm eff}(\tau)$: dual mass $M_{\rm dual}(\tau)$ (top left), effective threshold $s_{\rm eff}(\tau)$ vs. Borel variable τ (top right), dual decay constant $f_{\rm dual}(\tau)$ (bottom left), and b-quark mass dependence of the decay constant f_B (bottom right)



Bottom-Meson Decay Constants $f_{B_{(s)}}$ [4]

In contrast to the situation encountered in the charmed-meson system [7], a straightforward application of the above prescription to the bottom mesons reveals that the sum-rule prediction for f_B is very sensitive to the numerical value of the heavy-quark mass. Regarding this finding not as a disadvantage but as a serendipity, we invert the logic by choosing $m_b \equiv \overline{m}_b(\overline{m}_b)$ such that the average of lattice results for f_B, $f_B = (191.5 \pm 7.3)$ MeV, is reproduced: $m_b = (4.247 \pm 0.034)$ GeV. Then, the dual decay constants $f_{B_{(s)}}^{\rm dual}$ depend on m_b and the quark condensates employed, if everything else is kept fixed, like

$$f_B^{\rm dual}(m_b, \mu = \nu = m_b, \langle \bar{q}q \rangle) = \left(192.0 - 37\,\frac{m_b - 4.247\ \text{GeV}}{0.1\ \text{GeV}} + 4\,\frac{|\langle \bar{q}q \rangle|^{1/3} - 0.269\ \text{GeV}}{0.01\ \text{GeV}} \pm 3_{\rm syst}\right) \text{MeV}\ ,$$

$$f_{B_s}^{\rm dual}(m_b, \mu = \nu = m_b, \langle \bar{s}s \rangle) = \left(228.0 - 43\,\frac{m_b - 4.247\ \text{GeV}}{0.1\ \text{GeV}} + 3.5\,\frac{|\langle \bar{s}s \rangle|^{1/3} - 0.248\ \text{GeV}}{0.01\ \text{GeV}} \pm 4_{\rm syst}\right) \text{MeV}\ .$$

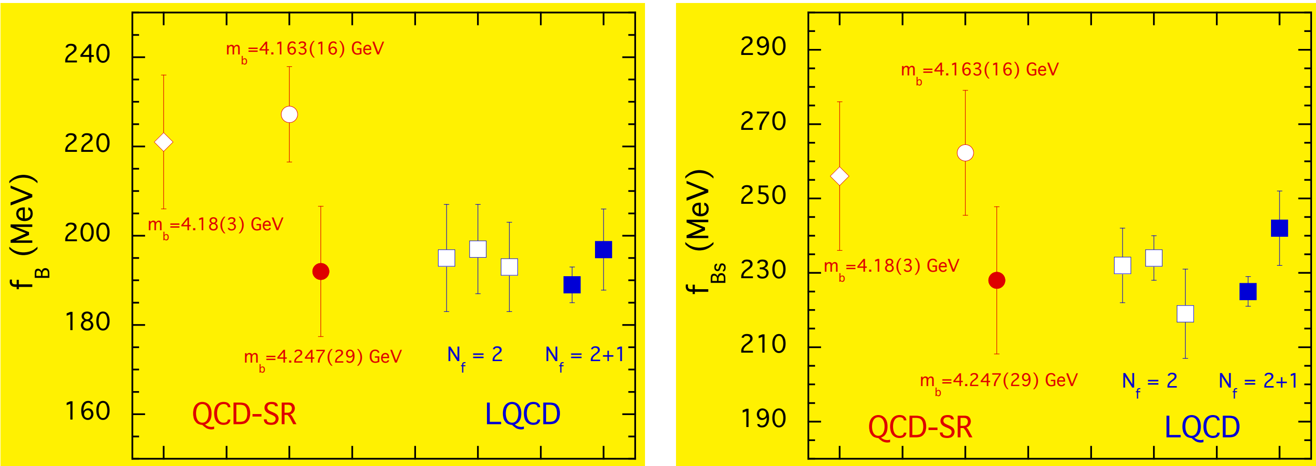
Some recent lattice-QCD determinations of the decay constants f_B and f_{B_s}

Collaboration [8]	Number of flavours	f_B [MeV]	f_{B_s} [MeV]	f_{B_s}/f_B
ETM	2	195 ± 12	232 ± 10	1.19 ± 0.05
	2	197 ± 10	234 ± 6	1.19 ± 0.05
ALPHA	2	193 ± 10	219 ± 12	1.13 ± 0.09
HPQCD	$2+1$	191 ± 9	228 ± 10	1.188 ± 0.018
	$2+1$	189 ± 4	225 ± 4	—
FNAL/MILC	$2+1$	196.9 ± 9.1	242 ± 10	1.229 ± 0.026
Our averages [4]		191.5 ± 7.3	228.8 ± 6.9	1.198 ± 0.030

Finally, a bootstrap analysis provides the error due to the OPE parameters:

$$f_B = (192.0 \pm 14.3_{\rm OPE} \pm 3.0_{\rm syst})\ {\rm MeV}\ ,$$
$$f_{B_s} = (228.0 \pm 19.4_{\rm OPE} \pm 4_{\rm syst})\ {\rm MeV}\ ,$$
$$f_{B_s}/f_B = 1.184 \pm 0.023_{\rm OPE} \pm 0.007_{\rm syst}\ .$$

Sum-rule (QCD-SR) vs. lattice (LQCD) results for f_B (left) and f_{B_s} (right)



References

[1] M.A. Shifman, A.I. Vainshtein & V.I. Zakharov, Nucl. Phys. B **147** (1979) 385.

[2] W. Lucha, D. Melikhov & S. Simula, Phys. Rev. D **76** (2007) 036002, arXiv:0705.0470 [hep-ph]; Phys. Lett. B **657** (2007) 148, arXiv:0709.1584 [hep-ph]; Phys. Atom. Nucl. **71** (2008) 1461; Phys. Lett. B **671** (2009) 445, arXiv:0810.1920 [hep-ph]; D. Melikhov, Phys. Lett. B **671** (2009) 450, arXiv:0810.4497 [hep-ph].

[3] W. Lucha, D. Melikhov & S. Simula, Phys. Rev. D **79** (2009) 096011, arXiv:0902.4202 [hep-ph]; J. Phys. G **37** (2010) 035003, arXiv:0905.0963 [hep-ph]; Phys. Lett. B **687** (2010) 48, arXiv: 0912.5017 [hep-ph]; Phys. Atom. Nucl. **73** (2010) 1770, arXiv:1003.1463 [hep-ph]; W. Lucha, D. Melikhov, H. Sazdjian & S. Simula, Phys. Rev. D **80** (2009) 114028, arXiv:0910.3164 [hep-ph].

[4] W. Lucha, D. Melikhov & S. Simula, arXiv:1305.7099 [hep-ph].

[5] J. Beringer *et al.* (Particle Data Group), Phys. Rev. D **86** (2012) 010001.

[6] M. Jamin and B.O. Lange, Phys. Rev. D **65** (2002) 056005, arXiv:hep-ph/0108135.

[7] W. Lucha, D. Melikhov & S. Simula, J. Phys. G **38** (2011) 105002, arXiv:1008.2698 [hep-ph]; Phys. Lett. B **701** (2011) 82, arXiv:1101.5986 [hep-ph].

[8] ETM Coll.: P. Dimopoulos *et al.*, JHEP **1201** (2012) 046, arXiv:1107.1441 [hep-lat]; N. Carrasco *et al.*, PoS(**Lattice 2012**)104 (2012), arXiv:1211.0568 [hep-lat]; PoS(**ICHEP2012**)428 (2012), arXiv:1212.0301 [hep-ph]; ALPHA Coll.: F. Bernardoni *et al.*, Nucl. Phys. Proc. Suppl. **234** (2013) 181, arXiv:1210.6524 [hep-lat]; HPQCD Coll.: H. Na *et al.*, Phys. Rev. D **86** (2012) 034506, arXiv:1202.4914 [hep-lat]; C. McNeile *et al.*, Phys. Rev. D **85** (2012) 031503, arXiv: 1110.4510 [hep-lat]; Fermilab Lattice and MILC Coll.: A. Bazavov *et al.*, Phys. Rev. D **85** (2012) 114506, arXiv:1112.3051 [hep-lat].