Four-Loop On-shell Integrals: MS–onshell relation and $g - 2$

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DESY
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Outline

1. Introduction
2. Technicalities
3. $\overline{\text{MS}}$ – on-shell relation for quark masses in QCD
4. Lepton anomalous magnetic moment
5. Conclusions and Outlook
At four-loop level two classes of integrals have been studied extensively: **massive tadpoles** and **massless propagators**.

Both classes have many phenomenological applications.

We are now ready for the treatment of a new class of integrals: **on-shell integrals**!

This new class of integrals has many phenomenological applications, too!

In the following: Calculation organized by number of massless fermion loops.

Note: Only $n_1^3$ and $n_1^2$ part finished!
Application I: $\overline{\text{MS}}$ – on-shell relation

- Fundamental relation between different renormalization schemes.
- Last missing renormalization constant at four loops in QCD.
- Improved precision needed e.g. for the measurement of the top quark mass (the PS mass) at a linear collider
  - Aim: top-mass measurement with $\Delta M_t \approx 100 \text{ MeV}$
  - three-loop correction in the $\overline{\text{MS}}$–on-shell relation $\approx 300 \text{ MeV}$
Application II: lepton anomalous magnetic moment

- Best experimentally measured and theoretically predicted quantity

\[
\begin{align*}
\left| a_e \right|_{\text{exp}} &= 0.00115965218073(28) \\
\left| a_e \right|_{\text{theo}} &= 0.00115965218178(6)(4)(3)(77)
\end{align*}
\]
Application II: lepton anomalous magnetic moment

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\end{align*}
\]

\[
\begin{align*}
a_\mu|_{\text{exp}} &= 1.16592080(54)(33)[63] \cdot 10^{-3} \\
a_\mu|_{\text{theo}} &= 1.16591790(65) \cdot 10^{-3} \quad 3.2\sigma \text{ diff.}
\end{align*}
\]

- QED contributions known numerically up to 5 loops but starting from four loops not checked by an independent calculation

Technicalities

Why discussing $\overline{\text{MS}}$ – on-shell relation and $g - 2$ together?
Why discussing \( \overline{\text{MS}} \) – on-shell relation and \( g - 2 \) together?

Both lead to the same type of topologies / integrals: on-shell integrals!
Order of Complexity

\[ Z_m : \Sigma(q^2, M^2)|_{q^2=M^2} \]
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\[ Z_m : \Sigma(q^2, M^2) \bigg|_{q^2=M^2} \]

\[ Z_2 : \frac{d}{dq^2} \Sigma(q^2, M^2) \bigg|_{q^2=M^2} \]
Order of Complexity

\[ Z_m : \Sigma(q^2, M^2) \mid q^2 = M^2 \]

\[ Z_2 : \frac{d}{dq^2} \Sigma(q^2, M^2) \mid q^2 = M^2 \]

\[ g - 2 : \frac{d}{dp^2} \Gamma(p^2, q^2 = M^2, M^2) \mid p^2 = 0 \]
Setup

- common setup for both calculations
- tools used include
  - qgraf
    Generation of Feynman diagrams
  - q2e, exp
    Expansion / Mapping to topologies
  - FORM
    Algebra
  - CRUSHER, FIRE
    Reduction to master integrals
  - FIESTA
    Calculation of master integrals
- two independent calculations
Master Integrals: simple

Expressible through Gamma functions for arbitrary dimension $D!$
Master Integrals: difficult

- Calculated analytically in expansion in $\epsilon = (4 - D)/2$ using the DRA (dimensional recurrence and analyticity) method and checked using FIESTA!
- Calculated up to $O(\epsilon^3)$
Only $n^3_\ell$ and $n^2_\ell$ part
Previous works

The $\overline{\text{MS}}$ – on-shell relation has been studied extensively

- two loop
- three loop
  - numerical
  - analytical
- large $\beta_0$ approximation

[Broadhurst, Grafe, Gray, Schilcher 1990]

[Chetyrkin, Steinhauser 1999]

[Melnikov, van Rittbergen 2000; PM, Mihaila, Piclum, Steinhauser]

[Beneke, Braun]
$z_m^{OS} = \frac{\bar{m}(\mu)}{M} = 1 + \ldots + \left( \frac{\alpha_s(\mu)}{\pi} \right)^4 \delta Z_m^{(4)} + O(\alpha_s^5)$

$\delta Z_m^{(4)} = \delta Z_m^{(40)} + n_1 \delta Z_m^{(41)} + n_2^2 \delta Z_m^{(42)} + n_3^3 \delta Z_m^{(43)}$
Results: analytical

\[ z_m^{\text{OS}} = \frac{\bar{m}(\mu)}{M} = 1 + \ldots + \left( \frac{\alpha_s(\mu)}{\pi} \right)^4 \delta z_m^{(4)} + O(\alpha_s^5) \]

\[ \delta z_m^{(4)} = \delta z_m^{(40)} + n_l \delta z_m^{(41)} + n_l^2 \delta z_m^{(42)} + n_l^3 \delta z_m^{(43)} \]

\[ \delta z_m^{(43)} = C_F T^3 \left( \frac{\ell_M^4}{144} + \frac{13 \ell_M^3}{216} + \left( \frac{89}{432} + \frac{\pi^2}{36} \right) \ell_M^2 \right. \]

\[ + \ell_M \left( \frac{\zeta_3}{3} + \frac{1301}{3888} + \frac{13\pi^2}{108} \right) \]

\[ + \frac{317\zeta_3}{432} + \frac{71\pi^4}{4320} + \frac{89\pi^2}{648} + \frac{42979}{186624} \) \), \ell_M = \log \frac{\mu^2}{M^2} \]

\[ \delta z_m^{(42)} = \ldots \]
Results: Numerics

\[
\begin{align*}
z_m^{\text{OS}} &= 1 - A_s 1.333 + A_s^2 (-14.229 - 0.104 n_h + 1.041 n_l) \\
&\quad + A_s^3 (-197.816 - 0.827 n_h - 0.064 n_h^2 \\
&\quad \quad + 26.946 n_l - 0.022 n_h n_l - 0.653 n_l^2) \\
&\quad + A_s^4 (-43.465 n_l^2 - 0.017 n_h n_l^2 + 0.678 n_l^3 + \ldots) + \mathcal{O}(A_s^5),
\end{align*}
\]

with \(n_l = 5, \ n_h = 1\):

\[
\begin{align*}
z_m^{\text{OS}} &= 1 - A_s 1.333 + A_s^2 (-14.332 + 5.207 n_l) \\
&\quad + A_s^3 (-198.707 + 134.619 n_l - 16.317 n_l^2) \\
&\quad + A_s^4 (-1087.060 n_l^2 + 84.768 n_l^3 + \ldots) + \mathcal{O}(A_s^5).
\end{align*}
\]
large $\beta_0$ approximation

\[
\frac{M_q}{\bar{m}_q(m_q)} \bigg|_{\text{large-}\beta_0} = 1 + a_s 1.333 + a_s^2 (17.186 - 1.041 n_l)
\]
\[
+ a_s^3 (177.695 - 21.539 n_l + 0.653 n_l^2)
\]
\[
+ a_s^4 (3046.294 - 553.872 n_l + 33.568 n_l^2 - 0.678 n_l^3)
\]

[Beneke, Braun 1995]
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Definition: anomalous magnetic moment $a_{\mu}$

\[
(g - 2) = (-ie)\bar{u}(p_2) \left\{ \gamma^{\mu} F_E(q^2) + i \frac{\sigma^{\mu\nu} q^\nu}{2m} F_M(q^2) \right\} u(p_1)
\]

\[
a_{\mu} = F_M(0)
\]
Definition: anomalous magnetic moment $a_\mu$

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g - 2 = \left\{ \gamma^\mu F_E(q^2) + i \frac{\sigma^{\mu\nu} q^\nu}{2m} F_M(q^2) \right\} u(p_1)
$$

$$
a_\mu = F_M(0)
$$

Important!

- The calculation is performed with a **massive** muon and a **massless** electron
- Therefore only correct up to power correction
- Logarithmic contributions can be obtained through renormalization
Previous works

- **analytical results**
  - one loop: $a_{\mu}^{(1)} = \frac{1}{2}$
  - two loop
  - three loop
  - four loops: only partial results, mainly contributions due to corrections to the vacuum polarization function of the photon

- **numerical results**
  - four loop
  - five loop
Diagrams

Only $n_1^3$ and $n_1^2$ part
Diagrams

Only \( n_3 \) and \( n_2 \) part

- calculated for massless electrons!
- light-by-light contribution not finite in this approximation!
Results

\[
a_{\mu}^{(43)} = \frac{1}{54} L_{\mu e}^3 - \frac{25}{108} L_{\mu e}^2 + \left( \frac{317}{324} + \frac{\pi^2}{27} \right) L_{\mu e} - \frac{2\zeta_3}{9} - \frac{25\pi^2}{162} - \frac{8609}{5832} \\
\approx 7.19666
\]

[Laporta; Aguilar, Greynat, De Rafael]
Results

\[ a^{(43)}_{\mu} = \frac{1}{54} L^3_{\mu e} - \frac{25}{108} L^2_{\mu e} + \left( \frac{317}{324} + \frac{\pi^2}{27} \right) L_{\mu e} - \frac{2\zeta_3}{9} - \frac{25\pi^2}{162} - \frac{8609}{5832} \]
\[ \approx 7.19666, \]

[159x264]g

\[ a^{(42)}_{\mu} = a^{(42)a}_{\mu} + n_h a^{(42)b}_{\mu} \]

\[ a^{(42)a}_{\mu} = L^2_{\mu e} \left[ \pi^2 \left( \frac{5}{36} - \frac{\log 2}{6} \right) + \frac{\zeta_3}{4} - \frac{13}{24} \right] + \ldots \approx -3.62427, \]

\[ a^{(42)a}_{\mu} \Bigg|_{\text{num}} = -3.64204(112), \]

[Laporta; Aguilar, Greynat, De Rafael]

[Aoyama, Hayakawa, Kinoshita, Nio 2012]
Results

\[ a_{\mu}^{(43)} = \frac{1}{54} L_{\mu e}^3 - \frac{25}{108} L_{\mu e}^2 + \left( \frac{317}{324} + \frac{\pi^2}{27} \right) L_{\mu e} - \frac{2\zeta_3}{9} - \frac{25\pi^2}{162} - \frac{8609}{5832} \]
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\[ a_{\mu}^{(42)a} = L_{\mu e}^2 \left[ \pi^2 \left( \frac{5}{36} - \frac{\log 2}{6} \right) + \frac{\zeta_3}{4} - \frac{13}{24} \right] + \ldots \approx -3.62427 \]

\[ a_{\mu}^{(42)a}\big|_{\text{num}} = -3.64204(112) \]

\[ a_{\mu}^{(42)b} = \left( \frac{119}{108} - \frac{\pi^2}{9} \right) L_{\mu e}^2 + \left( \frac{\pi^2}{27} - \frac{61}{162} \right) L_{\mu e} - \frac{4\pi^4}{45} + \frac{13\pi^2}{27} + \frac{7627}{1944} \]
\[ \approx 0.49405 \]

[Laporta; Aguilar,Greynat,De Rafael]

[Aoyama,Hayakawa,Kinoshita,Nio 2012]
Results: $n_l$ part

- Decompose $a_{\mu}^{(41)}$ further

\[ a_{\mu}^{(41)} = a_{\mu}^{(41)a} + n_h a_{\mu}^{(41)b} + n_h^2 a_{\mu}^{(41)c} \]

- Preliminary result for $a_{\mu}^{(41)b}$ and $a_{\mu}^{(41)c}$

\[ a_{\mu}^{(41)b} = -1.06(5) \]
\[ a_{\mu}^{(41)c} = 0.0280 \]

Compare with [Aoyama, Hayakawa, Kinoshita, Nio 2012]

\[ a_{\mu}^{(41)b} = -1.046 \]
\[ a_{\mu}^{(41)c} = 0.0280 \]

- $n_l n_h^0$ part calculated but further cross checks necessary!
First steps towards the calculation of $g - 2$ and the $\overline{\text{MS}}$–on-shell relation at four loops.

Setup for the full calculation established

Main obstacle: Calculation of $\mathcal{O}(600)$ master integrals

- Numerical solution for needed master integrals
- Analytical solution of needed master integrals still missing